An. Şt. Univ. Ovidius Constanţa Vol. 19(1), 2011, 211–222

A GENERALIZATION OF THE n-WEAK AMENABILITY OF BANACH ALGEBRAS

O.T. Mewomo and G. Akinbo

Abstract

Let A be a Banach algebra and $\varphi : A \to A$ be a continuous homomorphism. We generalize the notion of n-weak amenability of A to that of $(\varphi) - n$-weak amenability for $n \in \mathbb{N}$. We give conditions under which the module extension Banach algebra and second dual of A are $(\varphi) - n$-weakly amenable.

1 Introduction

In [4], Bodaghi, Gordji and Medghalchi generalized the concept of weak amenability of Banach algebras to that of (φ, ψ)-weak amenability. They determined the relations between weak amenability and (φ, ψ)-weak amenability of a Banach algebra A.

Also, in [7], Dales, Ghahramani, and Gronbaek introduced the concept of n-weak amenability for Banach algebras for $n \in \mathbb{N}$. They determined some relations between m- and n-weak amenability for general Banach algebras and for Banach algebras in various classes, and proved that, for each $n \in \mathbb{N}$, $(n + 2)$- weak amenability always implies n-weak amenability. Let A be a weakly amenable Banach algebra. Then it is also proved in [7] that in the case where A is an ideal in its second dual $(A'', \Box)$, A is necessarily $(2m-1)$-weakly amenable for each $m \in \mathbb{N}$. The authors of [7] asked the following questions:
(i) Is a weakly amenable Banach algebra necessarily 3-weakly amenable? (ii) Is a 2-weakly amenable Banach algebra necessarily 4-weakly amenable? A

Key Words: Banach algebra, continuous homomorphism, (φ)-derivation, $(\varphi) - n-$weak amenability.
Mathematics Subject Classification: 46H20, 46H10, 46H25
Received: April, 2010
Accepted: December, 2010

counter-example resolving question (i) was given by Zhang in [18], but it seems that question (ii) is still open.

It is also shown in Corollary 5.4 of [7] that for certain Banach space E the Banach algebra $\mathcal{N}(E)$ of nuclear operators on E is n-weakly amenable if and only if n is odd.

Let $L^1(G)$ be the group algebra of a locally compact group G. It is proved in Theorem 4.1 of [7] that each group algebra is n-weakly amenable whenever n is odd, and it is conjectured that $L^1(G)$ is n-weakly amenable for each $n \in \mathbb{N}$; this is true whenever G is amenable, and it is true when G is a free group [12].

A class of Banach algebras that was not considered in [3] is the Banach algebras on semigroups. In [13] Mewomo considered this class of Banach algebras by examining the n-weak amenability of some semigroup algebras, and give an easier example of a Banach algebra which is n-weakly amenable if n is odd.

In this paper, we shall extend the notion of (φ, ψ)-weak amenability to that of $(\varphi) - n-$ weak amenability of Banach algebras.

2 Preliminaries

First, we recall some standard notions; for further details, see [6] and [17].

Let A be an algebra and let X be an A-bimodule. A *derivation* from A to X is a linear map $D : A \to X$ such that

$$D(ab) = Da \cdot b + a \cdot Db \quad (a, b \in A).$$

For example, $\delta_x : a \mapsto a \cdot x - x \cdot a$ is a derivation; derivations of this form are the *inner derivations*.

Let A be a Banach algebra, and let X be an A-bimodule. Then X is a Banach A-bimodule if X is a Banach space and if there is a constant $k > 0$ such that

$$\|a \cdot x\| \leq k \|a\| \|x\|, \quad \|x \cdot a\| \leq k \|a\| \|x\| \quad (a \in A, x \in X).$$

By renorming X, we can suppose that $k = 1$. For example, A itself is Banach A-bimodule, and X', the dual space of a Banach A-bimodule X, is a Banach A-bimodule with respect to the module operations defined by

$$\langle x, a \cdot \lambda \rangle = \langle x \cdot a, \lambda \rangle, \quad \langle x, \lambda \cdot a \rangle = \langle a \cdot x, \lambda \rangle \quad (x \in X)$$

for $a \in A$ and $\lambda \in X'$; we say that X' is the *dual module* of X. In particular every closed two-sided ideal I of A is Banach A-bimodule and I' the dual space of I is a dual A-bimodule.

Successively, the duals $X^{(n)}$ are Banach A-bimodules; in particular $A^{(n)}$ is a Banach A-bimodule for each $n \in \mathbb{N}$. We take $X^{(0)} = X$.

Let A be a Banach algebra, and let X be a Banach A-bimodule. Then $\mathcal{Z}^1(A,X)$ is the space of all continuous derivations from A into X, $\mathcal{N}^1(A,X)$ is the space of all inner derivations from A into X, and the first cohomology group of A with coefficients in X is the quotient space

$$\mathcal{H}^1(A,X) = \mathcal{Z}^1(A,X)/\mathcal{N}^1(A,X)\,.$$

The Banach algebra A is *amenable* if $\mathcal{H}^1(A,X') = \{0\}$ for each Banach A-bimodule X and *weakly amenable* if $\mathcal{H}^1(A,A') = \{0\}$. For instance, the group algebra, $L^1(G)$ of a locally compact group G is always weakly amenable ([12]), and is amenable if and only if G is amenable in the classical sense ([11]). Also, a C^*-algebra is always weakly amenable ([10]) and is amenable if and only if it is nuclear ([5]).

Let A be a Banach algebra and let φ, ψ be continuous homomorphisms on A. As in [4], we consider the following module actions on A,

$$a \cdot x := \varphi(a)x, \quad x \cdot a := x\psi(a) \quad (a, x \in A).$$

The authors in [4] denote the above A-module by $A_{(\varphi,\psi)}$.

Let X be an A-module. A bounded linear mapping $d : A \to X$ is called a (φ,ψ)-derivation if

$$d(ab) = d(a)\cdot\varphi(a) + \psi(a)\cdot d(b) \quad (a, b \in A).$$

A bounded linear mapping $d : A \to X$ is called a (φ,ψ)- inner derivation if there exists $x \in X$ such that

$$d(a) = x\cdot\varphi(a) - \psi(a)\cdot x \quad (a \in A).$$

A derivation $D : A \to X$ is called approximately (φ,ψ)- inner if there exsits a net (x_α) in X such that, for all $a \in A$,

$$D(a) = lim_\alpha(x_\alpha\cdot\varphi(a) - \psi(a)\cdot x_\alpha)$$

in norm.

Derivations of this form are studied in [14,15,16].

The authors in [4] defined A to be (φ,ψ)-weakly amenable if $\mathcal{H}^1(A,(A_{(\varphi,\psi)})') = \{0\}$.

In this paper, we consider the case in which $\varphi = \psi$ and denote (φ,φ)-derivation, (φ,φ)-inner derivation by (φ)-derivation, (φ)-inner derivation respectively.

3 $(\varphi) - n$-Weak Amenability

Let A and B be Banach algebras. Suppose $\varphi : A \to B$ is a continuous homomorphism, then $B^{(n)}$ can be regarded as an A-module under the module actions

$$a \cdot m = \varphi(a) \cdot m, m \cdot a = m \cdot \varphi(a) \quad (a \in A, m \in B^{(n)}, n \in \mathbb{N}).$$

Let $\varphi : A \to A$ be a continuous homomorphism, then $A^{(n)}$ is an A-module with the module actions

$$a \cdot m = \varphi(a) \cdot m, m \cdot a = m \cdot \varphi(a) \quad (a \in A, m \in A^{(n)}, n \in \mathbb{N}).$$

A direct verification shows that the dual mappings $\varphi' : A' \to A'$ and $\varphi'' : A'' \to A''$ are A-module morphisms. This is also true for the higher dual mappings
$\varphi^{(2n-1)} : A^{(2n-1)} \to A^{(2n-1)}$ and $\varphi^{(2n)} : A^{(2n)} \to A^{(2n)}$

Proposition 3.1 *Let A and B be Banach algebras and let $\varphi : A \to A, \varphi : B \to B$ be continuous homomorphism. Let $\varphi_1 : A \to B$ and $\varphi_2 : B \to A$ be continuous homomorphisms such that $\varphi_1 \circ \varphi_2 = I_B$.*
(i) Suppose $D : B \to B^{(2n-1)}$ is a (φ)-derivation, then $\tilde{D} = (\varphi_1^{(2n-1)} \circ D \circ \varphi_1) : A \to A^{(2n-1)}$ is $(\varphi \circ \varphi_1)$-derivation.
(ii) Suppose $D : B \to B^{(2n)}$ is a (φ)-derivation, then $\bar{D} = (\varphi_2^{(2n)} \circ D \circ \varphi_1) : A \to A^{(2n)}$ is $(\varphi \circ \varphi_1)$-derivation.
(iii) Supoose $\tilde{D}$ is $(\varphi \circ \varphi_1)$-inner, then D is inner
(iv) Suppose $\bar{D}$ is $(\varphi \circ \varphi_1)$-inner, then D is (φ)-inner.
(v) Suppose A is $(\varphi \circ \varphi_1) - n$-weakly amenable for $n \in \mathbb{N}$, then B is $(\varphi) - n$-weakly amenable .

Proof (i) Let $D : B \to B^{(2n-1)}$ be a (φ)-derivation. Then, for $a, b \in A$, we have

$$\tilde{D}(ab) = (\varphi_1^{(2n-1)} \circ D \circ \varphi_1)(ab) = \varphi_1^{(2n-1)} \circ D(\varphi_1(a)\varphi_1(b))$$
$$= \varphi_1^{(2n-1)} (D(\varphi_1(a))\varphi(\varphi_1(b)) + \varphi(\varphi_1(a))D(\varphi_1(b)))$$
$$= \varphi(\varphi_1(b)) \cdot \varphi_1^{(2n-1)}(D(\varphi_1(a))) + \varphi_1^{(2n-1)}(D(\varphi_1(b))) \cdot \varphi(\varphi_1(a))$$
$$= \varphi(\varphi_1(b)) \cdot \tilde{D}(a) + \tilde{D}(b) \cdot \varphi(\varphi_1(a))$$
$$= \varphi \circ \varphi_1(b) \cdot \tilde{D}(a) + \tilde{D}(b) \cdot \varphi \circ \varphi_1(a)$$

Thus $\tilde{D}$ is $(\varphi \circ \varphi_1)$-derivation.

(ii) Let $D : B \to B^{(2n)}$ be a (φ)-derivation. Then, for $a, b \in A$, we have

$$\bar{D}(ab) = (\varphi_2^{(2n)} \circ D \circ \varphi_1)(ab) = \varphi_2^{(2n)} \circ D(\varphi_1(a)\varphi_1(b))$$

$$= \varphi_2^{(2n)} (D(\varphi_1(a))\varphi(\varphi_1(b)) + \varphi(\varphi_1(a))D(\varphi_1(b)))$$

$$= \varphi(\varphi_1(b)) \cdot \varphi_2^{(2n)}(D(\varphi_1(a))) + \varphi_2^{(2n)}(D(\varphi_1(b))) \cdot \varphi(\varphi_1(a))$$

$$= \varphi(\varphi_1(b)) \cdot \bar{D}(a) + \bar{D}(b) \cdot \varphi(\varphi_1(a))$$

$$= \varphi \circ \varphi_1(b) \cdot \bar{D}(a) + \bar{D}(b) \cdot \varphi \circ \varphi_1(a)$$

Thus $\bar{D}$ is $(\varphi \circ \varphi_1)$-derivation.

(iii) Clearly, $\varphi_2^{(2n-1)} : A^{(2n-1)} \to B^{(2n-1)}$ is a B-module morphism. Suppose $\tilde{D}$ is $(\varphi \circ \varphi_1)$-inner, then there exists $F \in A^{(2n-1)}$ with

$$\tilde{D}(a) = \varphi \circ \varphi_1(a) \cdot F - F \cdot \varphi \circ \varphi_1(a) \quad (a \in A).$$

Since $\varphi_1 \circ \varphi_2 = I_B$, we have $\varphi_1^{(2n-2)} \circ \varphi_2^{(2n-2)} = I_{B^{(2n-2)}}$, and so for every $b \in B$ and $m \in B^{(2n-2)}$, we have

$$\langle D(b), m\rangle = \langle D(\varphi_1 \circ \varphi_2(b)), \varphi_1^{(2n-2)} \circ \varphi_2^{(2n-2)}(m)\rangle$$

$$= \langle \varphi_1^{(2n-1)} \circ D \circ \varphi_1(\varphi_2(b)), \varphi_2^{(2n-2)}(m)\rangle$$

$$\langle \tilde{D}(\varphi_2(b)), \varphi_2^{(2n-2)}(m)\rangle$$

$$\langle \varphi \circ \varphi_1(\varphi_2(b)) \cdot F - F \cdot \varphi \circ \varphi_1(\varphi_2(b)), \varphi_2^{(2n-2)}(m)\rangle$$

(Since $\tilde{D}$ is $(\varphi \circ \varphi_1)$-inner)

$$\langle \varphi(b) \cdot F - F \cdot \varphi(b), \varphi_2^{(2n-2)}(m)\rangle$$

$$= \langle \varphi_2^{(2n-1)}(\varphi(b) \cdot F - F \cdot \varphi(b)), m\rangle$$

$$\langle \varphi(b) \cdot \varphi_2^{(2n-1)}(F) - \varphi_2^{(2n-1)}(F) \cdot \varphi(b), m\rangle$$

Thus, D is (φ)-inner.

(iv) The proof of (iv) is similar to that of (iii).

(v) This follows directly from (i),(ii),(iii) and (iv)

Theorem 3.2 *Let A be a Banach algebra such that $A = B \oplus I$ for some closed ideal I and closed subalgebra B. Let $\varphi : A \to A$ be a continuous homomorphism. Suppose A is $(\varphi \circ \varphi_1) - n$-weakly amenable where $\varphi_1 : A \to B$ is a natural projection of A onto B. Then B is $(\varphi) - n$-weakly amenable.*

Proof Let $\varphi_2 : B \to A$ be the natural injection into A. Clearly, φ_1 and φ_2 are continuous homomorphism with $\varphi_1 \circ \varphi_2 = I_B$. Thus, the result follows from Proposition 3.1.

We recall that a short exact sequence of Banach algebras is a triple of Banach algebras A, B and C together with a pair of continuous homomorphism $\varphi : A \to B$ and $\psi : B \to C$ such that φ is injective, its image $\varphi(A)$ equals $Kernel(\psi)$, and ψ is surjective. This short exact sequence is denoted by

$$0 \to A \to B \to C \to 0.$$

The short exact sequence is said to be split if there is a continuous homomorphism $\chi : C \to B$ with $\psi \circ \chi$ the identity map on C (see [17] for details).

Corollary 3.3 *Let A be a Banach algebra and let I be a closed ideal of A. Let $\varphi : A \to A$, and $\varphi_1 : A \to A/I$. Suppose the natural short exact sequence*

$$0 \to A \to A \to A/I \to 0$$

splits. If A is $(\varphi \circ \varphi_1) - n$-weakly amenable, then A/I is $(\varphi) - n$-weakly amenable.

Proof Since the short exact sequence split, there exists a continuous homomorphism $\varphi_2 : A/I \to A$ such that $\varphi_1 \circ \varphi_2 = I_{A/I}$. Thus the result follows from above result.

Proposition 3.4 *Let A be an algebra and let X be an A-bimodule. Define $\mathcal{A}$ to be the linear space $A \oplus X$ with the product*

$$(a, x) \cdot (b, y) = (ab, ay + xb) \quad (a, b \in A, x, y \in X).$$

(i) $\mathcal{A}$ is an algebra with respect to the specified product; $\mathcal{A}$ is commutative if and only if A is commutative and X is an A-module. The map $\Phi : \mathcal{A} \to A$ defined by $\Phi((a, x)) = a$ is an epimorphism.
(ii) Let $D : A \to X$ be a map and let $\varphi : A \to A$ be a continuous homomorphism. Define $\theta : A \to \mathcal{A}$ by $\theta(a) = (\varphi(a), D(a))(a \in A)$. Then θ is a homomorphism if and only if D is a (φ)-derivation.
(iii) Suppose $D : A \to X$ is a (φ)-derivation. Then $\tilde{D} : \mathcal{A} \to \mathcal{A}$ defined by $\tilde{D}((a, x)) = (0, D(a))$ is a $(\theta \circ \Phi)$-derivation

Proof (i) This is Theorem 1.8.14 (i) of [6].

(ii) Suppose D is a (φ)-derivation. Then, for $a, b \in A$,

$$\theta(ab) = (\varphi(ab), D(ab))$$
$$= (\varphi(a)\varphi(b), D(a)\cdot\varphi(b) + \varphi(a)\cdot D(b))$$
$$= (\varphi(a), D(a))\cdot(\varphi(b), D(b)) = \theta(a)\cdot\theta(b).$$

Conversely, suppose θ is a homomorphism. It is easy to see that $D(ab) = D(a)\cdot\varphi(b) + \varphi(a)\cdot D(b)$.

(iii) Since D is a (φ)-derivation, then θ is a homomorphism by (ii) and so by (i) Φ is a homomorphism. Let $(a, x), (b, y) \in \mathcal{A}, a, b \in \mathcal{A}, x, y \in X$. Then

$$\tilde{D}((a, x))\cdot\theta\circ\Phi((b, y)) + \theta\circ\Phi((a, x))\cdot\tilde{D}((b, y))$$
$$= (0, D(a))\cdot\theta(b) + \theta(a)\cdot(0, D(b))$$
$$(0, D(a))\cdot(\varphi(b), D(b)) + (\varphi(a), D(a))\cdot(0, D(b))$$
$$= (0, D(a)\cdot\varphi(b)) + (0, \varphi(a)\cdot D(b))$$
$$(0, D(a)\cdot\varphi(b) + \varphi(a)\cdot D(b))$$
$$= (0, D(ab)) = \tilde{D}((ab, a\cdot y + x\cdot b))$$
$$= \tilde{D}((a, x)(b, y))$$

and so $\tilde{D}$ is a $(\theta\circ\Phi)$-derivation.

Let A be a Banach algebra and let X be a Banach A-bimodule. Then the l^1-direct sum $\mathcal{A} = A \oplus X$ is a Banach algebra under the product

$$(a, x)\cdot(b, y) = (ab, ay + xb) \quad (a, b \in A, x, y \in X).$$

This is known as a module extension Banach algebra. Since X is an ideal of $\mathcal{A}$ and A is a closed subalgebra of $\mathcal{A}$, then as a consequence of Theorem 3.2, we have the next result.

Corollary 3.5 *Let A be a Banach algebra and X be a Banach A-bimodule. Let $\varphi : A \oplus X \to A \oplus X$ be a continuous homomorphism and $\varphi_1 : A \oplus X \to A$ a projection of $A \oplus X$ onto A. Suppose $A \oplus X$ is $(\varphi \circ \varphi_1) - n$-weakly amenable, then A is $(\varphi) - n$-weakly amenable.*

Let A be a Banach algebra and let X be a Banach A-bimodule. The higher duals $X^{(n)}$ are Banach A-bimodules. We recall that a Banach A-bimodule X is symmetric if $a\cdot x = x\cdot a$ for $n \in \mathbb{N}, a \in A, x \in X$. If X is symmetric, then each higher dual $X^{(n)}$ is symmetric. Let $\varphi : A \to A$ be a continuous homomorphism, since $A^{(n)}$ is a Banach A-module under the module actions

$$a\cdot m = \varphi(a)\cdot m, m\cdot a = m\cdot\varphi(a) \quad (a \in A, m \in A^{(n)})$$

By using the fact that $(A_{(\varphi,\varphi)})'$ is a symmetric Banach A-module (see Example 4.1 of [4]), we have the next result.

Proposition *Let A be a commutative weakly amenable Banach algebra and let*
$\varphi : A \to A$ be a continuous homomorphism. Then A is $(\varphi) - n$-weakly amenable.

Proof This follows from Theorem 1.5 of [3] and the above explanation.

4 $(\varphi) - n$-Weak Amenability of the Second Dual

Let A be a Banach algebra. There are two products on the second dual A'' of A, these products are denoted by $\square$ and $\diamond$ and are called the first and second Arens products on A; the original definitions of the two products were given in [1]. We recall briefly the definitions of $\square$ and $\diamond$.

First, for $\lambda \in A'$, we have

$$\langle b, a \cdot \lambda \rangle = \langle ba, \lambda \rangle, \langle b, \lambda \cdot a \rangle = \langle ab, \lambda \rangle (a, b \in A)$$

For $\lambda \in A'$ and $\psi \in A''$, define $\lambda \cdot \psi$ and $\psi \cdot \lambda$ in A' by

$$\langle a, \lambda \cdot \psi \rangle = \langle \psi, a \cdot \lambda \rangle, \langle a, \psi \cdot \lambda \rangle = \langle \psi, \lambda \cdot a \rangle (a \in A).$$

Finally, for $\psi_1, \psi_2 \in A''$, define

$$\langle \psi_1 \square \psi_2, \lambda \rangle = \langle \psi_1, \psi_2 \cdot \lambda \rangle,$$

$$\langle \psi_1 \diamond \psi_2, \lambda \rangle = \langle \psi_2, \lambda \cdot \psi_1 \rangle (\lambda \in A').$$

The Banach algebra A is said to be Arens regular if the two products $\square$ and $\diamond$ concide in A''

Suppose that $\psi_1 = lim_\alpha a_\alpha$ and $\psi_2 = lim_\beta b_\beta$ for nets (a_α) and (b_β) in A. Then

$$\psi_1 \square \psi_2 = lim_\alpha lim_\beta a_\alpha b_\beta$$

$$\psi_1 \diamond \psi_2 = lim_\beta lim_\alpha a_\alpha b_\beta,$$

where all limits are taken in the $\sigma(A'', A')$-topology on A''.

Theorem 4.1 **[9]** *Let A be a Banach algebra. Then both $(A'', \square)$ and $(A'', \diamond)$ are Banach algebras containing A as a closed subalgebra.*

Using Theorem 4.1, we have that $(A'', \square)$ and $(A'', \diamond)$ are Banach A-bimodule with respect to the product on $(A'', \square)$ and $(A'', \diamond)$ respectively.

Let A and B be Banach algebras, and let $\varphi : A \to B$ be a continuous homomorphism. Then $\varphi'' : (A'', \Box) \to (B'', \Box)$ is a continuous homomorphism. Let A be a closed subalgebra of B. Then we regard $(A'', \Box)$ as a closed subalgebra of $(B'', \Box)$. For further details on these products see [8].

We also recall that a Banach algebra A is called a dual Banach algebra if there is a closed submodule $A_\prime$ of A' such that $A_\prime' = A$.

Proposition 4.2 (See [8, Propositiom 5.2])
For a Banach algebra A the following statements are equivalent
(i) $(A'', \Box)$ *is a dual Banach algebra (with predual* A'*)*
(ii) A is Arens regular.

As a consequence of Theorem 3.2 and Theorem 4.1, we have the next result.

Proposition 4.3 *Let A be a dual Banach algebra and let $\varphi : A \to A$ be a continuous homomorphism. Suppose $(A'', \Box)$ is $(\varphi \circ \varphi_1) - n$ -weakly amenable for $\varphi_1 : A'' \to A$ a natural projection of A'' onto A, then A is $(\varphi) - n$-weakly amenable.*

Proof Let A be a dual Banach algebra with respect to the predual $A_\prime$ and let $i : A_\prime \to A'$ be the canonical embedding with adjoint φ_1 and $\varphi_2 : A \to A''$ be the canonical embedding. Clearly, $\varphi_1 \circ \varphi_2 = I_A$ and $\varphi_2 : A \to A''$ is a homomorphism. Also, $\varphi_1 : A'' \to A$ is a homomorphism since for $a \in A_\prime, \psi_1, \psi_2 \in A''$ with nets $(a_\alpha), (b_\beta)$ in A such that $\psi_1 = lim_\alpha a_\alpha, \psi_2 = lim_\beta b_\beta$, we have

$$\langle \varphi_1(\psi_1 \Box \psi_2), a\rangle = \langle \psi_1 \Box \psi_2, i(a)\rangle$$

$$= lim_\alpha lim_\beta \langle a_\alpha b_\beta, i(a)\rangle$$

$$= lim_\alpha lim_\beta \langle \varphi_1(a_\alpha)\varphi_1(b_\beta), a\rangle$$

$$= lim_\alpha \langle i(a \cdot \varphi_1(a_\alpha)), \psi_2\rangle$$

$$= lim_\alpha \langle a_\alpha, i(\varphi_1(\psi_2) \cdot a)\rangle$$

$$= \langle \psi_1, i(\varphi_1(\psi_2) \cdot a)\rangle$$

$$= \langle \varphi_1(\psi_1)\varphi_1(\psi_2), a\rangle.$$

Thus, the result follows using Theorem 3.2.

Proposition 4.4 *Let A be a Banach algebra, let $\varphi : A \to A$ be a continuous homomorphism and let X be a Banach A-bimodule. Suppose $D : A \to X$ is a continuous (φ)-derivation. Then $D'' : (A'', \Box) \to X''$ is a continuous (φ'')-derivation.*

Proof Clearly $D'' : A'' \to X''$ is a continuous linear operator. Let $\psi_1, \psi_2 \in A''$ with $\psi_1 = lim_\alpha a_\alpha$ and $\psi_2 = lim_\beta b_\beta$ in $(A'', \sigma(A'', A))$, where $(a_\alpha), (b_\beta)$ are nets in A with $\|a_\alpha\| \leq \|\psi_1\|$ and $\|b_\beta\| \leq \|\psi_2\|$. Then

$$D''(\psi_1 \Box \psi_2) = D''(lim_\alpha lim_\beta a_\alpha b_\beta)$$

$$= lim_\alpha lim_\beta D(a_\alpha b_\beta)$$

$$= lim_\alpha lim_\beta (D(a_\alpha) \cdot \varphi(b_\beta) + \varphi(a_\alpha) \cdot D(b_\beta)$$

$$= D''(\psi_1) \cdot \varphi''(\psi_2) + \varphi''(\psi_1) \cdot D''(\psi_2)$$

and so D'' is a (φ'')-derivation.

Theorem 4.5 *Let A be a Banach algebra, let $\varphi : A \to A$ be a continuous homomorphism, and let $D_\varphi : A \to (A'', \Box)$ be a continuous (φ)-derivation. Suppose A is Arens regular. Then there is a continuous (φ'')-derivation $D_{\varphi''} : (A'', \Box) \to (A'', \Box)$ such that*

$$D_{\varphi''}(\tilde{a}) = D_\varphi(a) \quad (a \in A),$$

and $\tilde{a}$ is the canonical image in A'' of $a \in A$.

Proof By Proposition 4.4, $D''_\varphi : (A'', \Box) \to A''''$ is a continuous (φ'')-derivation. By using the fact that A is Arens regular, we have that the canonical projection $P : A'''' \to A''$ is a $(A'', \Box)$-bimodule morphism. Let $\psi \in A''$ such that $a_\alpha \to \psi$ in $\sigma(A'', A')$, where (a_α) is a bounded net in A. We have $\tilde{a_\alpha} \to \tilde{\psi}$ in $\sigma(A'', A')$, where $\tilde{\psi}$ is the canonical image of ψ in A''. By taking $D_{\varphi''} = P \circ D''_\varphi$, $D_{\varphi''}$ clearly satisfy $D_{\varphi''}(\tilde{a}) = D_\varphi(a)(a \in A)$.

Corollary 4.6 *Let A be a Banach algebra which is Arens regular and let $\varphi : A \to A$ be a continuous derivation. Suppose every continuous (φ'')-derivation from $(A'', \Box)$ to $(A'', \Box)$ is (φ'')- inner. Then A is (φ)-2-weakly amenable.*

Proof Let $D : A \to A''$ be a continuous (φ)-derivation. By Theorem 4.5, there exists a continuous (φ'')-derivation $\tilde{D}$ such that $\tilde{D}(\tilde{a}) = D(a) \quad (a \in A)$. Thus, there exists $\psi_1 \in A''$ such that

$$\tilde{D}(\psi_2) = \varphi''(\psi_2)_1 - \psi_1''(\psi_2) \quad (\psi_1 \in A'').$$

In particular,

$$D(a) = \varphi(a) \cdot \psi_1 - \psi_1 \cdot \varphi(a) \quad (a \in A)$$

and so D is (φ)-inner. Thus, A is (φ)-2-weakly amenable.

Corollary 4.7 *Let A be a Banach algebra and let $\varphi : A \to A$ be a continuous homomorphism. Suppose $(A'', \square)$ is a dual Banach algebra (with predual $A\prime$) and every continuous (φ'')-derivation from A'' to A'' is (φ'')-inner. Then A is (φ)-2-weakly amenable.*

Proof This follows Proposition 4.2 and the above result.

References

[1] Arens R, *The adjoint of a bilinear operation*, Proc. Amer. Math. Soc. **2** (1951), 839–848.

[2] Arens R, *Operations induced in function classes*, Monatsh Math. **55** (1951), 1–19.

[3] Bade W.G, Curtis P.C and Dales H.G, *Amenability and weak amenability for Beurling and Lipschitz algebras*, Proc. London Math. Soc. **3** (1987), 359–377.

[4] Bodaghi A, Gordji M.E and Medghalchi A.R, *A generalization of the weak amenability of Banach algebras*, Banach J. Math. Anal. **3** (2009), 131–142.

[5] Connes A, *On the cohomology of operator algebras*, J. Funct. Anal. **28** (1978), 248–253.

[6] Dales H.G, *Banach algebras and automatic continuity*, London Mathematical Society Monographs, New Series, Volume 24, The Clarendon Press, Oxford, 2000.

[7] Dales H.G, Grahramani F, and Gronbaek N, *Derivations into iterated duals of Banach algebras*, Studia Math. **128** (1998), 19–54.

[8] Dales H.G, Rodrigues-Palacios A and Velasco M.V, *The second transpose of a derivation*, J. London Math. Soc. **64**(2) (2001), 707–721.

[9] Dales H.G and Lau A.T-M, *The second duals of Beurling algebras*, Memoirs American Math. Soc., **177** (2005), 1–191.

[10] Haagerup U, *All nuclear C*-algebras are amenable*, Invent. Math. **74** (1983), 305–319.

[11] Johnson B.E, *Cohomology in Banach algebras*, Memoirs American Math. Soc. **127** (1972).

[12] B.E. Johnson, *Weak amenability of group algebras*, Bull. London Math. Soc. **23** (1991).

[13] Mewomo O.T, *On n-weak amenability of Rees semigroup algebras*, Proc. Indian Acad. Sci. (Math. Sci.), **118**(4) (2008), 547–555.

[14] Mirzavaziri M. and Moslehian M.S, *Automatic continuity of σ-derivations in C^*-algebras*, Proc. American Math. Soc. **134**(11) (2006), 3319–3327.

[15] Mirzavaziri M. and Moslehian M.S, *σ-derivations in Banach algebras*, Bull. Iranian Math. Soc. (2006), 65–78.

[16] Moslehian M.S and Motlagh A.N, *Some notes on (σ, τ)-amenability of Banach algebras*, Stud. Univ. Babes-Bolyai Math. **53**(3) (2008), 57–68.

[17] Palmer T.M, *Banach algebra and the general theory of *-algebras*, Cambridge University Press, Vol. (1994).

[18] Yong Zhang, *Weak amenability of module extensions of Banach algebra*, Trans. Amer. Math. Soc., **354**(10) (2002), 4131–4151.

Department of Mathematics
University of Agriculture
Abeokuta, Nigeria
tosinmewomo@yahoo.com , mewomoot@unaab.edu.ng

Department of Mathematics
Obafemi Awolowo University
Ile - Ife, Nigeria
agnebg@yahoo.co.uk