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THE $L(2,1)$-LABELING ON TOTAL GRAPHS OF COMPLETE MULTIPARTITE GRAPHS

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Abstract

An $L(2,1)$-labeling of a connected graph G is defined as a function f from the vertex set $V(G)$ to the set of all nonnegative integers such that $|f(u) - f(v)| \geq 2$ if $d_G(u,v) = 1$ and $|f(u) - f(v)| \geq 1$ if $d_G(u,v) = 2$, where $d_G(u,v)$ denotes the distance between vertices u and v in G. The $L(2,1)$-labeling number of G, denoted by $\lambda(G)$, is the smallest number k such that G has an $L(2,1)$-labeling f with $\max\{f(v) : v \in V(G)\} = k$. In this paper, we consider the total graphs of the complete multipartite graphs and provide exact value for their λ-numbers.

1 Introduction

Motivated by the frequency assignment problem, Yeh [8] and Griggs and Yeh [3] proposed the notion of $L(2,1)$-labeling of a simple graph. An $L(2,1)$-labeling of a graph is a coloring of its vertices with nonnegative integers such that the labels on adjacent vertices differ by at least 2 and the labels on vertices at distance two differ by at least 1. This concept generalizes the notion of vertex coloring, because vertex coloring is the same as $L(1,0)$-labeling.

The $L(2,1)$-labeling number of G, denoted by $\lambda(G)$, is the smallest number k, such that G has a $L(2,1)$-labeling with no label greater than k.

Griggs and Yeh [3] showed that every graph with maximum degree Δ has an $L(2,1)$-labeling for which the value λ is at most $\Delta^2 + 2\Delta$. Chang and Kuo [1] provided a better upper bound $\Delta^2 + \Delta$. Griggs and Yeh [3] conjectured that the best bound is Δ^2 for any graph G with the maximum degree $\Delta \geq 2$;

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this bound is valid for graphs having diameter 2. There are many articles that are studying the problem of $L(2,1)$ - labelings ([1-7]). Most of these papers consider the values of λ on particular classes of graphs. For example, Shao, Yeh and Zhang [7] determined the λ-numbers for the total graphs of complete graphs. Determining the value of λ was proved to be NP-complete [3].

The goal of this paper is to determine the exact value of λ for total graphs of the complete multipartite graphs. It also provides a better upper bound for λ-numbers as function of Δ corresponding to this class of graphs.

For basic terminology and notation in graph theory we refer [4].

2 Total graphs of complete multipartite graphs

Let G be a graph. We denote by $\delta(G)$ its minimum degree and by $\Delta(G)$ its maximum degree.

The total graph $T(G)$ of graph G is the graph whose vertices correspond to the vertices and edges of G, and whose two vertices are joint if and only if the corresponding vertices are adjacent, edges are adjacent or vertices and edges are incident in G.

In this paper we consider the complete multipartite graphs $K_{n_1,n_2,...,n_p}$ with $n_1 \leq n_2 \leq ... \leq n_p$.

Next, we will use the following notations. If vertices x and y are adjacent in $K_{n_1,n_2,...,n_p}$, then the edge $[x,y]$ will be a vertex in the total graph $T(K_{n_1,n_2,...,n_p})$, denoted by xy.

We consider the multipartition $V(K_{n_1,n_2,...,n_p}) = V_1 \cup V_2 \cup ... \cup V_p$, where partite sets V_1, V_2, ..., V_p are disjoint and $|V_i| = n_i$ for $1 \leq i \leq p$. We also denote by ${x_k}^i$ the k-th vertex of V_i, where $1 \leq i \leq p$ and $1 \leq k \leq n_i$. The number of vertices of the complete multipartite graph $K_{n_1,n_2,...,n_p}$ is denoted by n. Thus, $n = \sum\limits_{1 \leq i \leq p} n_i$.

We have

$$|V(T(K_{n_1,n_2,...,n_p}))| = |V(K_{n_1,n_2,...,n_p})| + |E(K_{n_1,n_2,...,n_p})| = n + \sum_{1 \leq i < j \leq p} n_i n_j.$$

Lemma 2.1. *If G is the total graph $T(K_{n_1,n_2,...,n_p})$ then $\delta(G) = 2(n - n_p)$ and $\Delta(G) = 2(n - n_1)$.*

Proof. Since it is easy to see that in the total graph $T(K_{n_1,n_2,...,n_p})$ we have $d({x_k}^i) = 2(n - n_i)$ and $d({x_k}^i {x_t}^j) = 2n - n_i - n_j$, for $1 \leq i \neq j \leq p$, $1 \leq k \leq n_i$ and $1 \leq t \leq n_j$, the result follows. ∎

Let u be a vertex of the total graph $T(G)$. If u corresponds to a vertex in graph G then it is called a v-vertex. Otherwise, if u corresponds to an edge in G, then it is called an e-vertex [7].

Lemma 2.2. *The total graph $T(K_{n_1,n_2,\ldots,n_p})$ has the diameter*

$$diam(T(K_{n_1,n_2,\ldots,n_p})) = \begin{cases} 1, & if\ p = 2\ and\ n_1 = n_2 = 1 \\ 2, & otherwise. \end{cases}$$

Proof. The total graph $T(K_{1,1})$ is K_3; therefore, in this case $diam(T(K_{1,1})) = 1$. Otherwise, from the definition of the total graph $T(K_{n_1,n_2,\ldots,n_p})$, we have $d_{T(K_{n_1,n_2,\ldots,n_p})}(u_1, u_2) = 1$ if and only if u_1 and u_2 are v-vertices in different partite sets, or one of them is a v-vertex and the other is an e-vertex that has one extremity equal to the first, or u_1 and u_2 are e-vertices that have one common extremity. Otherwise, $d_{T(K_{n_1,n_2,\ldots,n_p})}(u_1, u_2) = 2$ because in all cases there exists a v-vertex or an e-vertex that is adjacent with both vertices u_1 and u_2. Moreover, for $p \geq 3$ or ($p = 2$ and $n_p \geq 2$) there exist in $K_{n_1,n_2,\ldots,n_p}$ a vertex and an edge that are not incident. Therefore, in this case $diam(T(K_{n_1,n_2,\ldots,n_p})) = 2$. ∎

3 λ-numbers for total graphs $T(K_{n_1,n_2,\ldots,n_p})$

Before proving Theorem 3.6, we need the following results. For a graph G, we denote by $\overline{G}$ its complement and by $c(\overline{G})$ the smallest number of vertex-disjoint paths in $\overline{G}$ needed to cover its vertex set.

Theorem 3.1. (Dirac). *Let G be a graph. If $\delta(G) \geq |V(G)|/2$ then there is a Hamiltonian cycle in G.*

Theorem 3.2. [2]. *Let G be a graph of order n that has diameter 2 and $\overline{G}$ its complement. If $c(\overline{G})$=1 then $\lambda(G) = n - 1$.*

Lemma 3.3. *If G is the total graph $T(K_{n_1,n_2,\ldots,n_p})$ then the minimum degree of its complement is*

$$\delta(\overline{G}) = \sum_{1 \leq i < j \leq p} n_i n_j + 2n_1 - n - 1.$$

Proof. We know that $d_{\overline{G}}(v) = |V(G)| - 1 - d_G(v)$ for all $v \in V(G)$. Next, the result follows from Lemma 2.1. ∎

Lemma 3.4. [7]

$$\lambda(T(K_n)) = \begin{cases} 4, & if\ n=2 \\ 7, & if\ n=3 \\ \binom{n}{2}, & if\ n\geq 4. \end{cases}$$

Lemma 3.5. [6]

$$\lambda(T(K_{n,m})) = \begin{cases} 4, & if\ n=m=1 \\ 2m+1, & if\ n=1\ and\ m\geq 2 \\ nm+n+m-1, & if\ m\geq n\geq 2. \end{cases}$$

Theorem 3.6. *If G is the total graph $T(K_{n_1,n_2,\ldots,n_p})$, where $p\geq 3$ and $n_p\geq 2$ then*

$$\lambda(G) = n + \sum_{1\leq i<j\leq p} n_i n_j - 1.$$

Proof. By Lemma 2.2 we have $diam(G)=2$. In order to determine $\lambda(G)$ we will use Theorem 3.2. For that, we will find $c(\overline{G})$. First we study the cases in which $\overline{G}$ satisfies condition from Dirac's Theorem 3.1. In this cases $\overline{G}$ is Hamiltonian, hence $c(\overline{G})=1$ and by Theorem 3.2 we have $\lambda(G)=|V(G)|-1 = n+\sum\limits_{1\leq i<j\leq p} n_i n_j - 1$. The other cases will be studied individually.

Let $S = 2\delta(\overline{G}) - |V(G)|$. By Lemma 3.3 we obtain

$$\begin{aligned} S &= 2\left(\sum_{1\leq i<j\leq p} n_i n_j + 2n_1 - n - 1\right) - n - \sum_{1\leq i<j\leq p} n_i n_j = \\ &= n_p(n_1+n_2+\ldots+n_{p-1}) + n_{p-1}(n_1+n_2+\ldots+n_{p-2}) + \ldots + \\ &\quad + n_2 n_1 + 4n_1 - 3n - 2. \end{aligned}$$

Dirac's condition for hamiltonicity is satisfied if and only if $S\geq 0$.

For $p\geq 4$ we will prove that the following inequality holds:

$$S \geq n_p(n_1+n_2+\ldots+n_{p-1}) - (n_1+n_2+n_3) - 3n_p + n_1 - 2. \quad (1)$$

Indeed, denote by

$$S_1 = n_p(n_1+n_2+\ldots+n_{p-1}) - 3n_p + n_1 - 2.$$

Since $1\leq n_1\leq n_2\leq \ldots \leq n_p$ we have

$$\begin{aligned} S &= S_1 + n_{p-1}(n_1+n_2+\ldots+n_{p-2}) + \ldots + n_4(n_3+n_2+n_1) + \\ &\quad + n_3(n_2+n_1) + n_2 n_1 - 3(n_2+\ldots+n_{p-1}) \geq \\ &\geq S_1 + 3(n_{p-1}+\ldots+n_4) + 2n_3 + n_3(n_2-n_1) + n_2 n_1 - \\ &\quad -3(n_2+\ldots+n_{p-1}) = \\ &= S_1 - (n_1+n_2+n_3) + n_3(n_2-n_1) + n_2 n_1 + n_1 - 2n_2. \end{aligned}$$

If $n_2 > n_1$ it follows that

$$n_3(n_2 - n_1) + n_2 n_1 + n_1 - 2n_2 \geq n_3 + n_2 - 2n_2 \geq 0.$$

If $n_2 = n_1$ then

$$n_3(n_2 - n_1) + n_2 n_1 + n_1 - 2n_2 = n_1^2 - n_1 \geq 0.$$

Hence

$$S \geq S_1 - (n_1 + n_2 + n_3)$$

and inequality (1) holds.

For $p \geq 5$, by (1) we obtain

$$\begin{aligned} S &\geq n_p(n_1 + n_2 + n_3 + n_4) - (n_1 + n_2 + n_3) - 3n_p + n_1 - 2 = \\ &= n_p n_4 - 3n_p + n_p(n_1 + n_2 + n_3) - (n_2 + n_3) - 2. \end{aligned}$$

Function $f : {\mathbb{N}_+}^3 \longrightarrow \mathbb{Z}$, defined by

$$f(n_1, n_2, n_3) = n_p(n_1 + n_2 + n_3) - (n_2 + n_3) - 2$$

is increasing in n_1, n_2, n_3, hence

$$f(n_1, n_2, n_3) \geq f(1, 1, 1) = 3n_p - 4.$$

It follows that

$$S \geq n_p n_4 - 4.$$

Then we deduce $S \geq 0$ for $n_4 \geq 2$ or $n_5 \geq 4$.

For $p = 4$, by (1) we have

$$S \geq n_4(n_1 + n_2 + n_3) - 3n_4 - (n_2 + n_3) - 2.$$

Function $g : {\mathbb{N}_+}^4 \longrightarrow \mathbb{Z}$, defined by

$$g(n_1, n_2, n_3, n_4) = n_4(n_1 + n_2 + n_3) - 3n_4 - (n_2 + n_3) - 2$$

is also increasing in n_1, n_2, n_3, n_4, hence

- for $n_1 \geq 2$, $S \geq g(2, 2, 2, 2) = 0$;
- otherwise, for $n_3 \geq 3$, $S \geq g(1, 1, 3, 3) = 0$;
- otherwise, for $n_3 = 2$ and $n_4 \geq 5$, $S \geq g(1, 1, 2, 5) = 0$;
- otherwise, for $n_2 = 2$ and $n_4 \geq 4$, $S \geq g(1, 2, 2, 4) = 2$.

If $p = 3$ then $S = (n_1 - 1)(n_2 + n_3 + 1) + (n_2 - 2)(n_3 - 2) - 5$ and it is easy to prove that $S \geq 0$ for: $n_1 \geq 2$ or $n_2 \geq 5$ or ($n_2 = 4$ and $n_3 \geq 5$) or ($n_2 = 3$ and $n_3 \geq 7$).

It remains to consider the following cases: (1) $p = 5$ and $n_1 = n_2 = n_3 = n_4 = 1$ and $n_5 \in \{2, 3\}$; (2) $p = 4$ with subcases (2.1) $n_1 = n_2 = n_3 = 1$, (2.2) $n_1 = n_2 = 1$, $n_3 = 2$ and $n_4 \in \{2, 3, 4\}$, (2.3) $n_1 = 1$ and $n_2 = n_3 = n_4 = 2$, (2.4) $n_1 = 1$, $n_2 = n_3 = 2$ and $n_4 = 3$; (3) $p = 3$ with subcases (3.1) $n_1 = n_2 = 1$, (3.2) $n_1 = 1$ and $n_2 = 2$, (3.3) $n_1 = 1$, $n_2 = 3$ and $n_3 \in \{3, 4, 5, 6\}$, (3.4) $n_1 = 1$ and $n_2 = n_3 = 4$.

Case 1. $p = 5$

We can directly verify that $\overline{T(K_{1,1,1,1,2})}$ and $\overline{T(K_{1,1,1,1,3})}$ have a Hamiltonian path. For example, $L = x_1^1, x_1^3x_1^5, x_1^2x_1^4, x_1^1x_1^5, x_2^5, x_1^5, x_1^4x_2^5, x_1^2x_1^5, x_1^3, x_1^1x_1^4, x_1^2x_1^3, x_1^1x_2^5, x_1^2, x_1^3x_1^4, x_1^1x_1^2, x_1^3x_2^5, x_1^4, x_1^1x_1^3, x_1^2x_2^5, x_1^4x_1^5$ is a Hamiltonian path in $\overline{T(K_{1,1,1,1,2})}$ and $L = x_1^1, x_1^2x_1^5, x_1^1x_2^5, x_3^5, x_1^4x_2^5, x_1^3x_1^5, x_1^4, x_1^2x_1^3, x_1^1x_1^4, x_1^2, x_1^3x_3^5, x_1^2x_2^5, x_1^1x_1^3, x_1^4x_3^5, x_1^3, x_1^2x_3^5, x_1^4x_1^5, x_1^3x_2^5, x_1^1x_1^2, x_1^3x_1^4, x_1^1x_1^5, x_2^5, x_1^2x_1^4, x_1^1x_3^5, x_1^5$ is a Hamiltonian path in $\overline{T(K_{1,1,1,1,3})}$.

Case 2. $p = 4$

(2.1) For $\overline{T(K_{1,1,1,2})}$ we can construct a Hamiltonain path, for example $L_2 = x_1^4, x_1^1x_2^4, x_1^2, x_1^3x_1^4, x_1^1x_1^2, x_1^3, x_1^1x_1^4, x_1^2x_1^3, x_1^1, x_1^2x_2^4, x_1^1x_1^3, x_2^4, x_1^2x_1^4, x_1^3x_2^4$.

For $m \geq 3$ we will prove by induction on m that $\overline{T(K_{1,1,1,m})}$ has a Hamiltonian path L_m, having the extremities x_m^4 and $x_1^2x_m^4$ and containing the subpath $x_1^ix_m^4, x_1^2, x_1^jx_m^4$, where $i, j \in \{1, 3\}$, $i \neq j$.

For $m = 3$ the graph $\overline{T(K_{1,1,1,3})}$ has such a Hamiltonian path $L_3 = x_1^2x_3^4, x_1^4, x_1^1x_2^4, x_1^3x_3^4, x_1^2, x_1^1x_3^4, x_1^3x_1^4, x_1^1x_1^2, x_1^3, x_1^1x_1^4, x_1^2x_1^3, x_1^1, x_1^2x_2^4, x_1^1x_1^3, x_2^4, x_1^2x_1^4, x_1^3x_2^4, x_3^4$.

Let $m \geq 3$ and assume that $\overline{T(K_{1,1,1,m})}$ has a Hamiltonian path denoted by L_m, having the extremities x_m^4 and $x_1^2x_m^4$ and containing the subpath $x_1^ix_m^4, x_1^2, x_1^jx_m^4$, where $i, j \in \{1, 3\}$, $i \neq j$. Since $V(\overline{T(K_{1,1,1,m+1})}) = V(\overline{T(K_{1,1,1,m})}) \cup \{x_{m+1}^4, x_1^1x_{m+1}^4, x_1^2x_{m+1}^4, x_1^3x_{m+1}^4\}$, we can obtain a Hamiltonian path L_{m+1} for $\overline{T(K_{1,1,1,m+1})}$ from L_m by connecting the vertex x_{m+1}^4 to the extremity $x_1^2x_m^4$ of L_m and the vertex $x_1^2x_{m+1}^4$ to the extremity x_m^4 of L_m, and transforming the subpath $x_1^ix_m^4, x_1^2, x_1^jx_m^4$, where $i, j \in \{1, 3\}$, $i \neq j$ of L_m into $x_1^ix_m^4, x_1^{4-i}x_{m+1}^4, x_1^2, x_1^{4-j}x_{m+1}^4, x_1^jx_m^4$. The Hamiltonian path L_{m+1} satisfies the induction hypothesis.

(2.2) We can directly verify that $\overline{T(K_{1,1,2,2})}$, $\overline{T(K_{1,1,2,3})}$ and $\overline{T(K_{1,1,2,4})}$ have a Hamiltonian path. For example, $L = x_1^1, x_1^3x_2^4, x_1^2x_1^4, x_1^1x_2^4, x_1^4, x_1^1x_1^3, x_1^2, x_2^3x_1^4, x_1^2x_2^4, x_1^1x_1^4, x_2^4, x_1^2x_2^3, x_1^3x_1^4, x_1^1x_2^3, x_1^2x_1^3, x_2^3x_2^4, x_1^1x_1^2, x_1^3, x_2^3$ is a Hamiltonian path in $\overline{T(K_{1,1,2,2})}$, $L = x_1^1, x_1^3x_2^4, x_1^4, x_1^2x_3^4, x_1^1x_1^3, x_1^2x_1^4, x_1^1x_2^4, x_1^2x_2^3, x_2^4, x_1^3x_1^4, x_1^2, x_1^3x_3^4, x_2^3x_2^4, x_1^1x_1^4, x_1^2x_1^3, x_1^1x_2^3, x_3^4, x_1^2x_2^4, x_1^1x_3^4, x_2^3x_1^4, x_1^3, x_2^3, x_1^1x_1^2, x_2^3x_3^4$ is a Hamiltonian path in $\overline{T(K_{1,1,2,3})}$ and $L = x_1^1, x_1^3x_1^4, x_2^4, x_1^2x_1^4, x_1^1x_1^3, x_1^2x_2^3, x_4^4, x_1^2x_3^4, x_2^3, x_1^3x_4^4, x_2^3x_3^4, x_1^2x_4^4, x_1^1x_3^4, x_1^2x_1^3, x_1^3x_2^4, x_2^3x_1^4, x_1^2, x_1^1x_1^4, x_1^3x_3^4, x_2^3x_2^4, x_1^1x_1^2, x_2^3x_4^4, x_1^1x_2^4, x_1^3, x_1^2x_2^4, x_1^1x_4^4, x_1^4, x_1^1x_2^3, x_3^4$ is a Hamiltonian path in $\overline{T(K_{1,1,2,4})}$.

(2.3) We can directly verify that $L = x_1^1, x_2^2x_2^3, x_1^1x_1^4, x_1^2x_2^4, x_2^3x_1^4, x_1^2, x_1^1x_2^2,$

$x_1^2x_2^3$, x_2^4, $x_1^1x_2^3$, $x_2^2x_1^3$, $x_1^1x_1^2$, $x_2^2x_2^4$, x_1^4, $x_1^3x_2^4$, x_2^2, $x_2^3x_2^4$, $x_2^2x_1^4$, $x_1^2x_1^3$, $x_1^1x_2^4$, $x_1^3x_1^4$, x_2^3, $x_1^1x_1^3$, $x_1^2x_1^4$, x_1^3 is a Hamiltonian path in $\overline{T(K_{1,2,2,2})}$.

(2.4) In this case we obtain directly $S = 1 > 0$ and the Dirac's condition for hamiltonicity is satisfied.

Case 3. $p = 3$

(3.1) For $\overline{T(K_{1,1,2})}$ we can construct a Hamiltonain path, for example $L_2 = x_1^3$, $x_1^1x_2^3$, x_1^2, $x_1^1x_1^3$, $x_1^2x_2^3$, x_1^1, $x_1^2x_1^3$, x_2^3, $x_1^1x_1^2$.

For $m \geq 3$ we will prove by induction on m that $\overline{T(K_{1,1,m})}$ has a Hamiltonian path L_m having the vertex x_m^3 as an extremity and containing the subpath $x_1^ix_m^3$, x_1^3, $x_1^jx_m^3$, where $i, j \in \{1, 2\}$, $i \neq j$.

For $m = 3$ the graph $\overline{T(K_{1,1,3})}$ has such a Hamiltonian path $L_3 = x_1^1x_3^3$, x_1^3, $x_1^2x_3^3$, $x_1^1x_2^3$, x_1^2, $x_1^1x_1^3$, $x_1^2x_2^3$, x_1^1, $x_1^2x_1^3$, x_2^3, $x_1^1x_1^2$, x_3^3.

Let $m \geq 3$ and assume that $\overline{T(K_{1,1,m})}$ has a Hamiltonian path denoted by L_m, having the vertex x_m^3 as an extremity and containing the subpath $x_1^ix_m^3$, x_1^3, $x_1^jx_m^3$, where $i, j \in \{1, 2\}$, $i \neq j$. Since

$$V(\overline{T(K_{1,1,m+1})}) = V(\overline{T(K_{1,1,m})}) \cup \{x_1^1x_{m+1}^3, x_1^2x_{m+1}^3, x_{m+1}^3\},$$

we can obtain a Hamiltonian path $L_{m+1}$for $\overline{T(K_{1,1,m+1})}$ from L_m by connecting the vertex x_{m+1}^3 to the extremity x_m^3 of L_m and transforming the subpath $x_1^ix_m^3$, x_1^3, $x_1^jx_m^3$, where $i, j \in \{1, 2\}$, $i \neq j$ of L_m into $x_1^ix_m^3$, $x_1^{3-i}x_{m+1}^3$, x_1^3, $x_1^{3-j}x_{m+1}^3$, $x_1^jx_m^3$. The Hamiltonian path L_{m+1} satisfies the induction hypothesis.

(3.2) We will prove by induction on m that $\overline{T(K_{1,2,m})}$ has a Hamiltonian path L_m containing the subpaths $x_1^1x_m^3$, x_1^3, $x_1^2x_m^3$ and $x_m^3, x_1^1x_1^3, x_2^2x_m^3$.

For $\overline{T(K_{1,2,2})}$ we can construct such a Hamiltonain path, for example L_2= $x_1^1x_2^3$, x_1^3, $x_1^2x_2^3$, x_2^2, $x_1^1x_1^2$, $x_2^2x_1^3$, x_2^3, $x_1^1x_1^3$, $x_2^2x_2^3$, x_1^1, $x_1^2x_1^3$, $x_1^1x_2^2$, x_1^2.

Let $m \geq 2$ and assume that $\overline{T(K_{1,2,m})}$ has a Hamiltonian path denoted by L_m, containing the subpaths $x_1^1x_m^3$, x_1^3, $x_1^2x_m^3$, and x_m^3, $x_1^1x_1^3$, $x_2^2x_m^3$. We have $V(\overline{T(K_{1,2,m+1})}) = V(\overline{T(K_{1,2,m})}) \cup \{x_1^1x_{m+1}^3, x_1^2x_{m+1}^3, x_2^2x_{m+1}^3, x_{m+1}^3\}$. Let L'_{m+1} be the path obtained from L_m by replacing the subpath $x_1^1x_m^3$, x_1^3, $x_1^2x_m^3$ with $x_1^1x_m^3$, $x_1^2x_{m+1}^3$, x_1^3, $x_1^1x_{m+1}^3$, $x_1^2x_m^3$ and the subpath x_m^3, $x_1^1x_1^3$, $x_2^2x_m^3$ with x_m^3, $x_2^2x_{m+1}^3$, $x_1^1x_1^3$, x_{m+1}^3, $x_2^2x_m^3$. Then the vertices of L'_{m+1} in reverse order form a path L_{m+1} which satisfies the induction hypothesis.

(3.3) For $n_3 \leq 6$, it can be verified that $\overline{T(K_{1,3,n_3})}$ has a Hamiltonian path. Moreover, it can be proved by induction on m that, for every $m \geq 3$, $\overline{T(K_{1,3,m})}$ has a Hamiltonian path L_m having the vertex x_m^3 as an extremity and containing the subpaths $x_1^1x_m^3$, x_1^3, $x_1^2x_m^3$ and $x_2^2x_m^3$, x_1^1, $x_3^2x_m^3$. Indeed, for $m = 3$, $\overline{T(K_{1,3,3})}$ has such a path $L_3 = x_1^1x_3^2$, x_1^2, x_2^2, $x_1^1x_1^2$, x_2^3, $x_1^1x_2^2$, x_3^2, $x_2^2x_1^3$, $x_3^2x_2^3$, $x_1^1x_3^3$, x_1^3, $x_1^2x_3^3$, $x_3^2x_1^3$, $x_2^2x_2^3$, $x_1^1x_1^3$, $x_1^2x_2^3$, $x_2^2x_3^3$, x_1^1, $x_3^2x_3^3$, $x_1^1x_2^3$, $x_1^2x_1^3$, x_3^3.

Let $m \geq 3$ and assume that $\overline{T(K_{1,3,m})}$ has a Hamiltonian path denoted by L_m having the vertex x_m^3 as an extremity and containing the subpaths $x_1^1x_m^3$,

x_1^3, $x_1^2x_m^3$ and $x_2^2x_m^3$, x_1^1, $x_3^2x_m^3$. We have $V(\overline{T(K_{1,3,m+1})}) = V(\overline{T(K_{1,3,m})}) \cup \{x_1^1x_{m+1}^3, x_1^2x_{m+1}^3, x_2^2x_{m+1}^3, x_3^2x_{m+1}^3, x_{m+1}^3\}$. Let L'_{m+1} be the path obtained from L_m by connecting the vertex x_{m+1}^3 to the extremity x_m^3 of L_m and replacing the subpath $x_1^1x_m^3$, x_1^3, $x_1^2x_m^3$ with $x_1^1x_m^3$, $x_1^2x_{m+1}^3$, x_1^3, $x_1^1x_{m+1}^3$, $x_1^2x_m^3$ and the subpath $x_2^2x_m^3$, x_1^1, $x_3^2x_m^3$ with $x_2^2x_m^3$, $x_3^2x_{m+1}^3$, x_1^1, $x_2^2x_{m+1}^3$, $x_3^2x_m^3$. Then the vertices of L'_{m+1} in reverse order form a path L_{m+1} which satisfies the induction hypothesis for $m+1$.

(3.4) We can see directly that $\overline{T(K_{1,4,4})}$ has a Hamiltonian path $L = x_3^2x_1^3$, x_1^1, $x_1^2x_1^3$, $x_1^1x_4^2$, x_2^3, $x_4^2x_3^3$, $x_2^2x_2^3$, $x_1^1x_3^3$, x_4^3, $x_4^2x_1^3$, x_1^2, $x_2^2x_1^3$, $x_4^2x_4^3$, $x_1^1x_1^3$, $x_1^2x_4^3$, x_2^2, $x_3^2x_4^3$, $x_1^2x_3^3$, $x_1^1x_3^2$, $x_2^2x_3^3$, $x_4^2x_2^3$, $x_1^1x_2^2$, $x_3^2x_3^3$, $x_1^1x_2^3$, x_3^2, $x_1^2x_2^3$, $x_2^2x_4^3$, x_4^2, $x_1^1x_4^3$, $x_3^2x_2^3$, x_1^3, $x_1^1x_1^2$, x_3^3.

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Corollary 3.7. *$\lambda(T(K_{n_1,n_2,\ldots,n_p})) \le \frac{p}{p-1}(\frac{\Delta^2}{8} + \frac{\Delta}{2}) - 1$ for all $p \ge 4$ or $p = 3$ and $n_3 \ge 2$.*

Proof. Let G be the total graph $T(K_{n_1,n_2,\ldots,n_p})$.

By Cauchy - Schwarz inequality for p-vectors $(n_1,\ldots,n_p)$ and $(1,\ldots,1)$ we have the inequality

$$n_1^2 + n_2^2 + \ldots + n_p^2 \ge \frac{(n_1+n_2+\ldots+n_p)^2}{p} = \frac{n^2}{p}.$$

Since, by Lemma 2.1, the total graph G has the maximum degree $\Delta = 2(n - n_1)$, and $n = n_1 + n_2 + \ldots + n_p \ge pn_1$, we obtain the inequality

$$n \le \frac{p}{2(p-1)}\Delta.$$

Using these two inequalities it follows that

$$\sum_{1\le i<j\le p} n_in_j = \frac{n^2 - (n_1^2 + \ldots + n_p^2)}{2} \le \frac{p-1}{2p}n^2 \le \frac{p}{8(p-1)}\Delta^2.$$

By Theorem 3.6 for all $p \ge 4$ or $p = 3$ and $n_3 \ge 2$ we have

$$\begin{aligned} \lambda(G) &= n + \sum_{1\le i<j\le p} n_in_j - 1 \le \frac{p}{2(p-1)}\Delta + \frac{p}{8(p-1)}\Delta^2 - 1 = \\ &= \frac{p}{p-1}\left(\frac{\Delta^2}{8} + \frac{\Delta}{2}\right) - 1. \end{aligned}$$

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