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THE COLLAPSIBILITY OF HEXAGON 2-COMPLEXES WITH THE 12-PROPERTY

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Abstract

We give some necessary and sufficient conditions for the collapsibility of a finite, 2-dimensional hexagon complex. We will show that any finite simply connected hexagon 2-complex with the 12-property is collapsible whereas any locally finite such complex has a simple combinatorial structure.

1 Introduction

We investigate in this paper combinatorial conditions which guarantee the collapsibility of a finite, 2-dimensional hexagon complex.

The combinatorial condition we consider is given by the 12-property of a hexagon complex (see [6], [7], [13], [17]). A hexagon 2-complex has the 12-property if the link of each vertex is a graph of girth at least 12. The *girth* of a graph is defined as the minimum number of edges in a circuit (see [9], [10], [11], [12]).

The collapsibility of finite simplicial complexes was studied before. W. White showed in 1967 that a finite, 2-dimensional simplicial complex is collapsible if and only if it admits a strongly convex metric (see [18]).

The collapsibility of a finite simplicial 2-complex can be also ensured by a CAT(0) metric (see [4], [1], [3], [5]). A proof for the fact that finite simplicial 2-complexes with a CAT(0) metric, retract to a point through CAT(0) spaces, is given in [14].

Key Words: hexagon 2-complex, star of a cell, link of a cell, systole of a complex, 12-property, collapsibility, van Kampen diagram.
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J. Corson and B. Trace proved further that a finite, simply connected, 2-dimensional simplicial complex with the 6-property, collapses to a point (see [7]).

K. Crowley showed, under a technical condition, that any finite simplicial complex of dimension 3 or less endowed with the standard piecewise Euclidean metric that is non-positively curved simplicially collapses to a point (see [8]).

The results obtained in this paper are based on regular tessellations of the Euclidean plane. There are only three regular tessellations of the Euclidean plane: by triangles, squares, and hexagons. J. Corson and B. Trace used in [7] the first possibility of subdividing the Euclidean plane (by triangles) when showing the collapsibility of finite, simply connected, 2-dimensional simplicial complexes with the 6-property. Using the second possibility of subdividing the Euclidean plane (by squares), it is shown in [15] that finite, simply connected, square 2-complexes with the 8-property, collapse to a point. We focus in this paper on the third possibility of subdividing the Euclidean plane (by hexagons) and obtain similar results.

The main result of the paper states that finite, simply connected, hexagon 2-complexes with the 12-property, collapse to a point. Besides, we show that any locally finite, simply connected, 2-dimensional hexagon complex with the 12-property, is an infinite ascending union of collapsible subcomplexes. The paper's results are included in the author's Ph.D. Thesis (see [16]).

Our main result on finite hexagon 2-complexes with the 12−property will reveal an important metric property of a particular class of such complexes called hexagonal 2-complexes (see [2]). What distinguishes hexagonal 2-complexes among hexagon 2-complexes is that the intersection of any two 2-cells in a hexagonal 2-complex is either the empty set, or a single common face of the two intersecting cells. In a hexagon 2-complex such intersection may be a union of faces (see [4], chapter I.7, page 114; [9], Appendix A, page 404).

2 Preliminaries

We present in this section the notions we shall work with and the results we shall refer to.

Let k be a real number. Let X_k^n denote a simply connected complete Riemannian $n-$manifold of constant curvature k.

A *convex* X_k^n*-polyhedral cell* C is the convex hull of a finite set of points in X_k^n. The *support* of a point $x \in C$, denoted $supp(x)$, is the unique face of C containing x in its interior.

Let $(C_\lambda : \lambda \in \Lambda)$ be a family of convex X_k^n-polyhedral cells and let $L = \cup_{\lambda\in\Lambda}(C_\lambda x\{\lambda\})$ denote their disjoint union. Let $\sim$ be an equivalence relation on L and let $K = L|_\sim$. Let $p : L \to K$ be the natural projection and define

$p_\lambda : C_\lambda \to K$ by $p_\lambda(x) := p(x, \lambda)$. K is called an n-dimensional X^n_k*-polyhedral complex* if:

1. for all $\lambda \in \Lambda$, the restriction of p_λ to the interior of each face of C_λ is injective;
2. for all $\lambda_1, \lambda_2 \in \Lambda$ and $x_1 \in C_{\lambda_1}$, $x_2 \in C_{\lambda_2}$, if $p_{\lambda_1}(x_1) = p_{\lambda_2}(x_2)$ then there is an isometry $h : supp(x_1) \to supp(x_2)$ such that $p_{\lambda_1}(y) = p_{\lambda_2}(h(y))$ for all $y \in supp(x_1)$.

A 2-*dimensional hexagon complex* is a 2-dimensional X^2_k-polyhedral complex whose 2-cells have six 1-dimensional faces. We note that the intersection of any two cells in a hexagon 2-complex is either the empty set, or at most six common vertices, or / and at most six common edges. So in a hexagon 2-complex such intersection may be a union of faces.

Let K be a cell complex. $|K|$ denotes the underlying space of K, and $K^{(k)}$ denotes the k-skeleton of K.

Let α be an i-cell of K. If β is a k-dimensional face of α but not of any other cell in K, then we say there is an *elementary collapse* from K to $K' = K \setminus \{\alpha, \beta\}$. We denote an elementary collapse by $K \searrow K'$. If $K = K_0 \supseteq K_1 \supseteq ... \supseteq K_n = L$ are cell complexes such that there is an elementary collapse from K_{j-1} to K_j, $1 \leq j \leq n$, then we say that K *collapses* to L.

A *closed edge* is an edge together with its endpoints. An *oriented edge* of K is an oriented 1-cell of K, $e = [v_0, v_1]$. We denote by $i(e) = v_0$, the *initial* vertex of e, by $t(e) = v_1$, the *terminus* of e, and by $e^{-1} = [v_1, v_0]$, the *inverse* of e. A finite sequence $\alpha = e_1 e_2 ... e_n$ of oriented closed edges in K such that $t(e_i) = i(e_{i+1})$ for all $1 \leq i \leq n-1$, is called an *edge-path* in K. If $t(e_n) = i(e_0)$, then we call α a *closed edge-path* or *cycle*. We denote by $|\alpha|$ the number of 1-cells contained in α and we call $|\alpha|$ the *length* of α.

Let σ be a cell of K. The *star* of σ in K, denoted $St(\sigma, K)$, is the union of all cells that contain σ. The *link* of σ in K, denoted $Lk(\sigma, K)$, consists of all cells in the star of σ in K which are disjoint from σ and which, together with σ, span a cell of K.

A subcomplex L in K is called *full* (in K) if any cell of K spanned by a set of vertices in L, is a cell of L. A *full cycle* in K is a cycle that is full as subcomplex of K. The *systole* of K is given by

$$sys(K) = \min\{|\alpha| : \alpha \text{ is a full cycle in } K\}.$$

A cell 2-complex has the k*-property* if the link of each of its vertices is a graph of systole at least k, $k \in \{6, 8, 12\}$.

Let v be a vertex of K. The *degree* of v, denoted by degv, is the number of oriented edges having v as initial vertex.

A *combinatorial map* $f : K_1 \to K_2$ between the cell complexes K_1 and K_2 is a continuous map such that each open cell of K_1 is mapped homeomorphically onto an open cell of K_2. A *combinatorial* 2*-complex* is a 2-dimensional cell complex whose 2-cells are attached along immersions of S^1 into the 1-skeleton of the complex. A *combinatorial disk* is a combinatorial 2-complex homeomorphic to a disk.

Van Kampen diagrams constitute a significant part of small cancellation theory (see [17]). They are an adequate tool for studying the collapsibility of finite cell complexes of dimension 2. We shall study a cell complex K by associating to each closed edge-path α in K a diagram in the Euclidean plane, called a van Kampen diagram, which contains all the essential information about α.

Let $\alpha = e_0e_1...e_n$ be a closed edge-path in K. A *van Kampen diagram* for α is a pair (D, ϕ). D is a finite, simply connected combinatorial disk embedded in the plane, bounded by the closed edge-path $\beta = f_0f_1...f_n$. $\phi : D \to K$ is a combinatorial map assigning to each oriented edge f_i of β in D an oriented edge $\phi(f_i) = e_i$ of α in K such that $\phi(f_i^{-1}) = \phi(f_i)^{-1}$ for all $0 \leq i \leq n$.

A *region* is a 2-cell of K. Let $\alpha = e_0e_1...e_n$ be a closed edge-path in K and let (D, ϕ) be a van Kampen diagram for α. The *area* of the diagram is given by the number of regions of D.

We present further some important results dealing with van Kampen diagrams.

Theorem 2.1. *Let K be a* 2*-dimensional cell complex and let α be a closed edge-path in K. α is null-homotopic if and only if there exists a van Kampen diagram for α.*

For the proof see [17], chapter V, page $237 - 239$.

Let α be a closed edge-path in K and let (D, ϕ) be a van Kampen diagram for α. Let D_1, D_2 be regions (not necessarily distinct) of D with an edge $e \subseteq \partial D_1 \bigcap \partial D_2$. Let $e\delta_1$ and $\delta_2 e^{-1}$ be boundary cycles of D_1 and D_2, respectively. Let $\phi(\delta_1) = f_1$ and $\phi(\delta_2) = f_2$. The diagram (D, ϕ) is called *reduced* if one never has $f_2 = f_1^{-1}$.

Theorem 2.2. *Let K be a* 2*-dimensional cell complex and let α be a closed edge-path in K. Let (D, ϕ) be a van Kampen diagram for α. If (D, ϕ) is a van Kampen diagram of minimal area, then the diagram is reduced.*

For the proof we refer to [17], chapter V, page 241.

Theorem 2.3. *Let K be a* 2*-dimensional cell complex with the k-property, $k \in \{6, 8, 12\}$, and let α be a closed edge-path in K. If (D, ϕ) is a reduced van Kampen diagram for α, then D also has the k-property.*

For the proof we refer to [17], chapter V, page 242.

3 Collapsing a 2-dimensional hexagon complex with the 12-property

This section provides a combinatorial characterization of collapsible 2-complexes. Namely, we prove that finite, simply connected, 2-dimensional hexagon complexes with the 12-property, collapse to a point. Besides, we show that any locally finite, simply connected, 2-dimensional hexagon complex with the 12-property is an infinite ascending union of collapsible subcomplexes.

The following lemma gives an important property of hexagon disk whose interior vertices have degree at least 3.

Lemma 3.1. *Let D be a hexagon disk whose interior vertices have degree at least 3. Then*

$$\sum_{v\in\partial D}(\tfrac{5}{2}-\mathrm{degv})\geq 3,$$

summing over the boundary vertices of D.

Proof. We denote the set of interior vertices of D by $int(D)$. We denote by V, V_{int}, V_{ext}, E, E_{ext} and F, the number of vertices, interior vertices, exterior vertices, edges, exterior edges and 2-cells of D, respectively. The following relations hold in any hexagon disk: $1 = V - E + F$ (Euler's characteristic), $2E - E_{ext} = 6F$, $V_{ext} = E_{ext}$, $\sum_v \mathrm{degv} = 2\mathrm{E}$. Using these relations, we obtain

$$\begin{aligned}6 &= 6(V - E + F) = \\ &= 6V - 4E - V_{ext} = \\ &= (6V_{int} - 2\sum_{v\in int(D)}\mathrm{degv}) + (5\mathrm{V_{ext}} - 2\sum_{\mathrm{v}\in\partial\mathrm{D}}\mathrm{degv}).\end{aligned}$$

So

$$3 = \sum_{v\in int(D)}(3-\mathrm{degv}) + \sum_{\mathrm{v}\in\partial\mathrm{D}}(\tfrac{5}{2}-\mathrm{degv}).$$

Because D is a hexagon disk whose interior vertices have degree at least 3, we have: $\sum_{v\in int(D)}(3-\mathrm{degv})\leq 0$. The above relation further implies:

$$\sum_{v\in\partial D}(\tfrac{5}{2}-\mathrm{degv})\geq 3.$$

□

Let K be a hexagon complex. We introduce the following elementary operations on edge-paths in K.

1. *Free reduction*: Let α be an edge-path containing a subpath of the form ee^{-1}. Let β be the edge-path obtained by deleting this subpath. We call the passage from α to β a *free reduction*.

2. *Short-cut*: Let $v_0, v_1, v_2, v_3, v_4, v_5$ be the vertices of a 2-cell in K. The passage from one edge-path to another, obtained

a. either by replacing an occurrence of an edge-path $[v_0, v_1]$ by the edge-path $[v_0, v_5][v_5, v_4][v_4, v_3][v_3, v_2][v_2, v_1]$,

b. or by replacing an occurrence of an edge-path $[v_0, v_1]$ $[v_1, v_2]$ by the edge-path $[v_0, v_5][v_5, v_4][v_4, v_3][v_3, v_2]$,

c. or by replacing an occurrence of an edge-path $[v_0, v_1]$ $[v_1, v_2][v_2, v_3][v_3, v_4]$ by the edge-path $[v_0, v_5][v_5, v_4]$,

d. or by replacing an occurrence of an edge-path $[v_0, v_1]$ $[v_1, v_2][v_2, v_3][v_3, v_4]$ $[v_4, v_5]$ by the edge-path $[v_0, v_5]$,

is called a *short-cut*.

3. *Elementary exchange*: Let $v_0, v_1, v_2, v_3, v_4, v_5$ be the vertices of a 2-cell in K. The passage from one edge-path to another, obtained by replacing an occurrence of an edge-path $[v_0, v_1][v_1, v_2][v_2, v_3]$ by the edge-path $[v_0, v_5][v_5, v_4][v_4, v_3]$, is called an *elementary exchange*.

One can notice that none of the elementary operations alter the endpoints of an edge-path. Free reductions and short-cuts change the length of an edge-path, whereas elementary exchanges do not. This will be important. If we can pass from one edge-path α to another edge-path β via a finite sequence of elementary exchanges, then we call the edge-paths α and β *exchangeable* and we write $\alpha \equiv \beta$. An edge-path α is *strongly reduced* if for any edge-path β, exchangeable with α, β does not admit a free reduction or short-cut.

The following theorem proves that in a hexagon 2-complex with the 12-property, one can essentially pass from one edge-path to any path-homotopic edge-path, via a finite sequence of elementary operations.

Theorem 3.2. *Let K be a 2-dimensional hexagon complex with the* 12*-property. Let β be a strongly reduced edge-path in K. If α is any edge-path that is path-homotopic to β, then there exists a finite sequence of elementary operations passing from α to β.*

Proof. We apply elementary operations on α until it becomes strongly reduced. We must show that there exists a finite sequence of elementary exchanges from α to β.

Since α and β are path-homotopic, the edge-path $\alpha\beta^{-1}$ is null-homotopic. So there exists a van Kampen diagram (D, ϕ) for $\alpha\beta^{-1}$. D is a hexagon disk. We choose a van Kampen diagram for $\alpha\beta^{-1}$ of minimal area. Since the diagram (D, ϕ) is reduced and the edge-paths α and β in K are strongly reduced, D has no boundary vertex with degree smaller than 2.

Because K has the 12-property and (D, ϕ) is a reduced van Kampen diagram for a closed edge-path in K, D also has the 12-property. Let v_0, v_1, v_2, v_3 and v_4 be boundary vertices of D. Since $(\frac{5}{2} - \mathrm{degv_i}) \leq \frac{1}{2}$, $0 \leq i \leq 4$, Lemma 3.1 implies that

$$3 \leq (\tfrac{5}{2} - \mathrm{degv_0}) + (\tfrac{5}{2} - \mathrm{degv_1}) + (\tfrac{5}{2} - \mathrm{degv_2}) + (\tfrac{5}{2} - \mathrm{degv_3}) +$$
$$+ (\tfrac{5}{2} - \mathrm{degv_4}) + \sum_{\mathrm{v} \in \partial \mathrm{D}, \mathrm{v} \notin \{\mathrm{v_0}, \mathrm{v_1}, \mathrm{v_2}, \mathrm{v_3}, \mathrm{v_4}\}} (\tfrac{5}{2} - \mathrm{degv}) \leq$$
$$\leq \tfrac{5}{2} + \sum_{v \in \partial D, v \notin \{v_0, v_1, v_2, v_3, v_4\}} (\tfrac{5}{2} - \mathrm{degv}).$$

So D has an exterior vertex $v \notin \{v_0, v_1, v_2, v_3, v_4\}$ such that $\mathrm{degv} \leq 2$. Because $\mathrm{degv} = 2$, the open star of v in D contains one 2-cell.

Because α and β are strongly reduced, we can not perform short-cuts or free reductions on them. By deleting the open star of v in D, we obtain a disk $D' = D \setminus Stv$. So D' is constructed by performing an elementary exchange on, say α. We reach hence another edge-path γ. So $\gamma \equiv \alpha$.

Since elementary exchanges do not change the length of edge-paths, the diagram (D', ϕ) is a van Kampen diagram for $\alpha\beta^{-1}$ as well. But the disk D' has one 2-cell less than D, a contradiction because D has minimal area. Therefore, by performing an elementary exchange either on α or on β, we reach either β or α. So $\alpha \equiv \beta$.

□

Let K be a 2-dimensional hexagon complex. Let e, e' be two directed edges in K such that $i(e) = i(e') = v$. We denote by $\rho(e, e')$ the length of a shortest edge-path in $\mathrm{Lk}(v, K)$ joining $t(e)$ and $t(e')$. We define $\rho(e, e')$ to be infinite, if there does not exist an edge-path joining $t(e)$ and $t(e')$ in K. We call an edge-path $e_1 ... e_n$ in K a *locally geodesic* if $\rho(e_i^{-1}, e_{i+1}) \geq 6$ for all $1 \leq i < n$. The term 'locally' does not have its traditional meaning. Instead, it suggests that such an edge-path can not be deleted by any elementary operations on K. So there exists an edge-path between any two points in K joined by a locally geodesic, no matter the elementary operations we perform on the complex. By its definition, a locally geodesic edge-path does not admit any elementary operations. So, if K is a 2-dimensional hexagon complex with the 12-property, any closed locally geodesic edge-path in K is, by Theorem 3.2, essential.

Applying Theorem 3.2, the collapsibility of any finite, simply connected, hexagon 2-complex with the 12-property, follows.

Corollary 3.3. *Let K be a finite simply connected 2-dimensional hexagon complex that has the 12-property. Then K is collapsible.*

Proof. Suppose that there exists a connected subcomplex of K, L, that has more than one vertex and that does not admit any elementary collapses. If L is 1-dimensional, its fundamental group is nontrivial. If L is 2-dimensional, each 0- and 1-cell of L is contained in at least two cells of L. Because a locally geodesic edge-path does not permit any elementary collapses, by choosing edges in succession, we can construct in L a locally geodesic edge-path α. L being finite, α is eventually closed. Hence, because L inherits from K the 12-property, Theorem 3.2 implies that α is not null-homotopic in L. The fundamental group of L is therefore nontrivial. But $|K|$ is simply connected; the fundamental group of any connected subcomplex of K is therefore trivial. This implies a contradiction. So any connected subcomplex of K admits an elementary collapse. K being finite, there exists a finite sequence of elementary collapses, starting with K, which eventually terminates in a one-point subcomplex. So K collapses to a point.

□

We show further that locally finite simply connected hexagon complexes of dimension 2 with the 12-property, possess a simple combinatorial structure.

Corollary 3.4. *Let K be a locally finite simply connected 2-dimensional hexagon complex with the 12-property. Then K is a monotone union of a sequence of collapsible subcomplexes.*

Proof. Let v_0 be a fixed vertex in K. For each integer n, let L_n be the full subcomplex of K generated by the vertices that can be joined to v_o by an edge-path of length at most n. Thus, for each n, L_n is finite and connected. Since $K = \bigcup_{n=1}^{\infty} L_n$, the corollary follows due to the above result, once we have shown that, for each n, $|L_n|$ is simply connected.

Let $\alpha = e_1...e_n$ be a closed edge-path in L_n with endpoints at v_0. We denote by $v_i = t(e_i) = i(e_{i+1})$ and by γ_i an edge-path in K from v_o to v_i of minimal length. We consider the edge-path $\gamma = e_1\gamma_1^{-1}\gamma_1 e_2\gamma_2^{-1}\gamma_2...e_{n-1}\gamma_{n-1}^{-1}\gamma_{n-1}e_n$ in L_n that freely reduces to α.

Because $|K|$ is simply connected, the edge-paths $\gamma_i e_{i+1}$ and γ_{i+1} are path-homotopic. Because K has the 12-property, Theorem 3.2 implies that there exists, for each i, a finite sequence of elementary operations passing from $\gamma_i e_{i+1}$ to γ_{i+1}. Each edge-path in this sequence lies in L_n. The edge-path γ is therefore null-homotopic in L_n and so is α. $|L_n|$ is hence simply connected.

□

Since finite, 2-dimensional hexagon complex with the 12-property are collapsible, the weaker condition of contractibility does also characterize these spaces.

Corollary 3.5. *Let K be a finite simply connected 2-dimensional hexagon complex with the 12-property. Then K is contractible.*

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