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A HAHN-BANACH TYPE GENERALIZATION OF THE HYERS–ULAM THEOREM

Tamás Glavosits and Árpád Száz

Abstract

Having in mind a generalization of the classical Hahn-Banach extension theorem, we give a simple generalization of the classical Hyers–Ulam stability theorem.

In 1941, giving a partial answer to a general question of S.M. Ulam, Hyers [13] proved a Banach space particular case of the subsequent stability theorem. (For some relevant generalizations, see [24], [33], [3], [20], [1] and [30].)

Hyers's theorem was the starting point of an extensive theory of the stability of functional equations and inequalities. (For a rapid overview on the subject, the reader may be referred to the pioneering book of Hyers, Isac and Rassias [14].)

Theorem 0.1. *If f is an ε–approximately additive function of a commutative semigroup U to a Banach space X, for some $\varepsilon \geq 0$, in the sense that*

$$\|f(u+v) - f(u) - f(v)\| \leq \varepsilon$$

for all $u, v \in U$, then there exists an additive function g of U to X which is ε–near to f in the sense that

$$\|f(u) - g(u)\| \leq \varepsilon$$

for all $u \in U$.

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Remark 0.2. At an international conference on functional equations, M. Laczkovich informed (see Ger [10, p. 4]) that the $U = \mathbb{N}$, $X = \mathbb{R}$ and $\varepsilon = 1$ particular case of the above generalized Hyers's theorem was already proved by Pólya and Szegő [22, Aufgabe 99, p.17] in 1925.

Moreover, he also noted that the real-valued particular case of Hyers's theorem can be easily derived from the result of the above mentioned authors. Thus, Hyers's theorem is essentially equivalent to that of Pólya and Szegő by the observations of Székelyhidi [34] and Gajda [6].

The following generalization of the classical Hahn–Banach extension theorem is a particular case [4, Corollary 1.3] of Fuchssteiner. (For some more readable treatments, see [5, 1.3.2.Theorem] and [31, Theorem 3.3].)

Theorem 0.3. *If p is a subadditive function of a commutative semigroup U to $\mathbb{R}$ and φ is an additive function of a subsemigroup V of U to $\mathbb{R}$ such that:*

1. $\varphi(v) \leq p(v)$ *for all* $v \in V$*;*
2. $\varphi(u+v) \leq p(u) + \varphi(v)$ *for all* $u \in U$ *and* $v \in V$ *with* $u + v \in V$*;*

then φ can be extended to an additive function ψ of U to $\mathbb{R}$ such that $\psi(u) \leq p(u)$ for all $u \in U$.

Remark 0.4. To see the necessity of condition (2), note that if ψ is as above, then

$$\varphi(u+v) = \psi(u+v) = \psi(u) + \psi(v) \leq p(u) + \varphi(v)$$

for all $u \in U$ and $v \in V$ with $u + v \in V$.

Now, to have a close analogue of Theorem 0.3, we shall prove a partial generalization of Theorem 0.1. For this, in addition to Theorem 0.1, we shall also need the following

Lemma 0.5. *If f is a function of a semigroup U to a normed space X and φ is a function of a subsemigroup V of U to X such that*

1. φ *is 2–homogeneous in the sense that* $\varphi(2v) = 2\varphi(v)$ *for all* $v \in V$*;*
2. φ *is* ε*–near to* f*, for some* $\varepsilon \geq 0$*, in the sense that* $\|f(v) - \varphi(v)\| \leq \varepsilon$ *for all* $v \in V$*;*

then

$$\varphi(v) = \lim_{n\to\infty} \frac{1}{2^n} f(2^n v)$$

for all $v \in V$.

Proof. From (1), by induction, we can easily infer that $2^n v \in V$ and

$$\varphi(2^n v) = 2^n \varphi(v)$$

for all $v \in V$ and $n \in \mathbb{N}$. Now, by using (2), we can also see that

$$\left\| \frac{1}{2^n} f(2^n v) - \varphi(v) \right\| = \frac{1}{2^n} \left\| f(2^n v) - \varphi(2^n v) \right\| \leq \frac{1}{2^n} \varepsilon$$

for all $v \in V$ and $n \in \mathbb{N}$. Hence, since $\lim\limits_{n \to \infty} 2^{-n} = 0$, we can already infer that

$$\lim_{n \to \infty} \left\| \frac{1}{2^n} f(2^n v) - \varphi(v) \right\| = 0$$

for all $v \in V$. Therefore, the required assertion is also true. □

Theorem 0.6. *If f is an ε–approximately additive function of a commutative semigroup U to a Banach space X, for some $\varepsilon \geq 0$, and φ is a 2–homogeneous function of a subsemigroup V of U to X which is δ–near to f, for some $\delta \geq 0$, then φ can be extended to an additive function ψ of U to X which is ε–near to f.*

Proof. Now, by Theorem 0.1, we can state that there exists an additive function ψ of U to X which is ε–near to f. Moreover, because of the additivity of ψ, we can also state that ψ is 2–homogeneous. Furthermore, by using Lemma 0.5, we can see that

$$\varphi(v) = \lim_{n \to \infty} \frac{1}{2^n} f(2^n v)$$

for all $v \in V$ and

$$\psi(u) = \lim_{n \to \infty} \frac{1}{2^n} f(2^n u)$$

for all $u \in U$. Therefore, $\psi(v) = \varphi(v)$ also holds for all $v \in V$. □

Remark 0.7. To see that the above theorem is more general than that of Hyers, note that if in particular U has a zero element 0, then

$$\|f(0)\| = \|f(0+0) - f(0) - f(0)\| \leq \varepsilon.$$

Thus, $\varphi = \{(0,0)\}$ is an additive function of the subgroup $\{0\}$ of U to X such that φ is ε–near to f. Therefore, by the Theorem 0.6, there exists an additive function ψ of U to X which is ε–near to f.

Moreover, we can note that if p and φ are as in Theorem 0.3, then by defining a relation F of U to $\mathbb{R}$ such that

$$F(u) =]-\infty, p(u)]$$

for all $u \in U$, we can see that $\varphi(v) \in F(v)$ for all $v \in V$.

While, if f and φ are as in Theorem 0.6, then by defining a relation F of U to X such that

$$F(u) = f(u) + B_\delta(0), \qquad \text{with} \qquad B_\delta(0) = \{x \in X : \ \|x\| \le \delta\},$$

for all $u \in U$, we can again see that $\varphi(v) \in F(v)$ for all $v \in V$.

Therefore, the essence of Theorems 0.3 and 0.6 is nothing else but the observation that an additive partial selection function φ of a certain relation F of U to $\mathbb{R}$ and X, respectively, can be extended to an additive total selection function of ψ of F.

The corresponding fact in connection with the Hahn–Banach extension theorem was already recognized by Rodríguez-Salinas and Bou [25]. (For some further developments, see [15], [9], [27], [28] and [11].)

Moreover, Smajdor [26] and Gajda and Ger [7] observed that the essence of the Hyers–Ulam stability theorem is the existence of an additive selection function of a certain relation. (For some further developments, see [23], [2] and [29].)

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Department of Mathematics,
Dumlupınar University, Kütahya Turkey
ekinci@dumlupinar.edu.tr

Department of Mathematics,
Dumlupınar University, Kütahya Turkey
ckeskin@dumlupinar.edu.tr