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AN INTEGRAL GEOMETRY PROBLEM ALONG GEODESICS AND A COMPUTATIONAL APPROACH

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Abstract

In this paper, we prove the existence, uniqueness and stability of the solution of an integral geometry problem (IGP) for a family of curves of given curvature. The functions in the statement of the curvature depend on two variables, which is occured especially in the case of IGP along geodesics. To prove the solvability of the problem, we reduce the IGP to an overdetermined inverse problem for the transport equation. We also develop a new symbolic algorithm to compute the approximate solution of the problem and present two computational experiments to show the accuracy of the algorithm. The results show that the proposed approach provides highly accurate solutions and it is robust against data noises.

1 Introduction

Since the famous paper by I. Radon in 1917, it has been agreed that integral geometry problems (IGPs) consist in determining some function or a more general quantity (chomology class, tensor field, etc.), which is defined on a manifold, given its integrals over submanifolds of a prescribed class, [18]. It can be formulated as follows:

Let $\lambda(x)$ be a sufficiently smooth function which is defined in n-dimensional space and assume that $\{\Gamma(r)\}$ is a family of smooth manifolds in this space

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which depend on the paraemeter $r = (r_1, r_2, ..., r_k)$. Suppose that the integrals

$$\int_{\Gamma(r)} \lambda(x)\, d\sigma = J(r)$$

are known, where $d\sigma$ is the measure element on $\Gamma(r)$ and $x = (x_1, x_2, ..., x_n)$. Here, it is required to determine the function $\lambda(x)$, provided that $J(r)$ is given.

In this work, we consider an integral geometry problem (IGP) in the case of one-dimensional manifolds $\Gamma(r)$, or more precisely, in the case where $\Gamma(r)$ are curves and $d\sigma$ is the arc length element of a curve. We invesigate the solvability conditions and approximate solution of the IGP for a family of curves of given curvature. The case when the curvature depends on φ in a special manner as $K(x, \varphi) = f_2(x)\cos\varphi - f_1(x)\sin\varphi$ was previously discussed in [4]. Here, we consider more general functions $K(x, \varphi)$. The functions in the statement of the curvature depend on two variables, which is especially occured in the case of IGP along geodesics. Therefore, IGP along geodesics are considered as a special case in the second section of the paper. Also the way of specifying dependence of λ upon φ (or determining $\widehat{L}$, see section 3) and the spaces where the problem is investigated are new. In the last section, a new symbolic algorithm based on the Galerkin method is developed to compute approximate solution of the problem and some computational experiments are presented which show that the proposed algorithm gives efficient and reliable results. The proposed approximation method is important, because there has been no numerical study for such IGPs and the related inverse problems.

The main method proposed here for investigating the solvability of IGP is to reduce it to the equivalent Dirichlet type problem for a third order differential equation. Such a reduction is demonstrated for Problem 1 below. Here, it is assumed that a family of regular curves $\{\Gamma\}$ passing from each point $x \in D$ and in any direction $\nu = (\cos\varphi, \sin\varphi)$ is given by curvature $K(x, \varphi) = F_1(x, \varphi)\cos\varphi + F_2(x, \varphi)\sin\varphi$, and there exists a curve passing from every $x \in D$ in the arbitrary direction ν, with endpoints on the boundary of D. Moreover, the curves are specified using the angle variables $\varphi = (\varphi_1, \varphi_2)$, and these angles are defined as the solution to the Cauchy problem:

$$\frac{\partial\tilde{\varphi}}{\partial s} = F(x, \tilde{\varphi}), \quad \tilde{\varphi}(0) = \varphi,$$

where $\tilde{\varphi} = (\varphi_1, \varphi_2)$ and $F = (F_1, F_2)$.

2 Statement of the Problem

Problem 1. *Determine a function $\lambda(x)$ in a bounded domain D from the integrals of λ along the curves of a given family of curves $\{\Gamma\}$.*

It is assumed that there exists a unique sufficiently smooth curve in $\{\Gamma\}$ with endpoints on the boundary of D, passing through the point x in direction v. Suppose lengths of these curves in D are bounded above by the same constant. Take the curve $l_+(x,\varphi)$ with the endpoint $x \in D$, direction $\nu = (\cos\varphi, \sin\varphi)$ and with the other endpoint on ∂D and also with the curvature $K(x,\varphi)$ at the point x. The curve $l_+(x,\varphi)$ is a part of a curve with the same property that belongs to $\{\Gamma\}$. We introduce an auxiliary function

$$u(x,\varphi) = \int_{l_+(x,\varphi)} \lambda ds, \tag{1}$$

where $\lambda(x) \in C\left(\mathbb{R}^2\right)$ vanishes outside D and ds is the arc length element along $l_+(x,\varphi)$. If we differentiate (1) at the point x in the direction ν, i.e., differentiating with respect to the parameter s we obtain the following transport-like equation

$$Lu \equiv u_{x_1}\cos\varphi + u_{x_2}\sin\varphi + u_\varphi \frac{\partial\varphi}{\partial s} = \lambda(x), \tag{2}$$

in $\Omega = \{(x,\varphi) : x \in D \subset R^2,\ \varphi \in (0,2\pi),\ \partial D \in C^3\}$, where $\frac{\partial\varphi}{\partial s} = K(x,\varphi)$ and $F_1, F_2 \in C^3\left(\bar{D}\right)$. From the nature of IGP, we have

$$u|_{\Gamma_1} = u_0(x,\varphi),\ u(x,\varphi) = u(x,\varphi+2\pi). \tag{3}$$

where $\Gamma_1 = \partial D \times (0,2\pi)$.

Problem 2. *Given the function $K(x,\varphi)$, determine a pair of functions (u,λ) defined in Ω from the transport-like equation (2) and conditions (3).*

Given the function K, we construct a set of curves $\{\Gamma\}$ such that K is the curvature of the curve that passes through $x \in D$ in the direction $(\cos\varphi, \sin\varphi)$. It is always possible to construct such a set of curves for a sufficiently smooth function $K(x,\varphi)$ with certain convexity properties. Integrating both sides of equality (2) along the curve $l_+(x,\varphi)$ and observing (3), we arrive at Problem 1. Thus we have proved that Problem 1 is equivalent to the Problem 2 in the corresponding spaces. Reduction of an IGP to a Dirichlet problem was first carried out by Lavrent'ev and Anikonov in [13].

IGPs and inverse problems for transport equation are important both from theoretical and practical points of view. For example, IGP provide mathematical background for computerized tomography (CT), [17]. In CT, the object under investigation is exposed to radiation at different angles, and the radiation parameters are measured at the points of observation. The results are digitized and processed by computers which calculate spatial distribution of quantitative physical parameters of the object. The obtained results are then visualized by means of special devices. The basic equation in the mathematical model of CT can be written in general form by the first equation in the introduction, where $\Gamma(r)$ is the ray connecting the source point of the object radiation with the observation point, $\lambda(x)$ characterizes the object under study. CT has important applications in many fields, some of them are geophysics, astronomy, seismology, diagnostic radiology, etc.

Transport equations arise in radiative transfer, spread of neutrons, plasma theory, sound propagation, and in other fields of physics. Historically, the earliest work in transport theory was performed in connection with astrophysical problems. They are related to radiative transfer. Analysis of temperature distribution and radiative fields in the photospheres of stars is a classical problem. Transport problems are actively used in many other areas of contemporary physics such as radiative transfer in gas dynamics and the theory of highly intensive shock waves. Radiative transport is of great importance in plasma theory and processes in laser and quantum generators. Transport problems are also employed in investigating spreading of sound waves, electrical charges in gases and in a number of other phenomena, [1].

3 Main Definitions and Notations

In this section, some necessary definitions and notations are presented which will be used throughout the paper. We use some standard function spaces below such as $C^k(\Omega)$, $L_2(\Omega)$ and $H^k(\Omega)$ which are described in detail, for example, in [15, 16].

Definition 1. *By $C^3_\pi(\Omega)$ we denote the space of all real-valued functions $u(x,\varphi) \in C^3(\Omega)$ which are 2π-periodic with respect to the argument φ in the domain Ω, i.e., the values of the function u and its derivatives up to third order at $\varphi = 0$ are equal to those at $\varphi = 2\pi$. We define the following scalar product in $C^3_\pi(\Omega)$:*

$$(u,z)_{1,c} = \int_\Omega [uz + (u_{x_1} + F_1 u_\varphi)(z_{x_1} + F_1 z_\varphi) + (u_{x_2} + F_2 u_\varphi)(z_{x_2} + F_2 z_\varphi)]\, d\Omega,$$

$d\Omega = dx_1 dx_2 d\varphi$, and introduce the norms

$$\|u\|_{1,c} = [(u,u)_{1,c}]^{1/2},\ \|u\|_1 = [(u,u)_{1,c} + \int_\Omega u_\varphi^2 d\Omega]^{1/2}.$$

The completions of the set $C^3_\pi(\Omega)$ with respect to the norms $\|\cdot\|_{1,c}$ and $\|\cdot\|_{H^m(\Omega)}$

($m = 1, 2, 3$) are denoted by $H^{\pi}_{1,c}(\Omega)$ and $H^{\pi}_{m}(\Omega)$, respectively,[3].

Definition 2. *The set of functions $\psi(x,\varphi) \in C^3_{\pi}(\Omega)$ such that $\psi = 0$ on Γ_1 is denoted by $C^3_{\pi 0}(\Omega)$. The spaces $\mathring{H}^{\pi}_{1,c}(\Omega)$ and $\mathring{H}^{\pi}_{m}(\Omega)$ are the completions of the set $C^3_{\pi 0}$ with respect to the norm $\|\cdot\|_{1,c}$ and $\|\cdot\|_{H^m(\Omega)}$ ($m = 1, 2, 3$), [3].*

Definition 3. *Let us introduce the following designations:*

$$\begin{aligned} Au &= \widehat{L}Lu = \frac{\partial^2}{\partial l \partial \varphi}(Lu), \\ \frac{\partial}{\partial l} &= \sin\varphi \left(\frac{\partial}{\partial x_1} + F_1 \frac{\partial}{\partial \varphi} + \frac{\partial F_1}{\partial \varphi} + F_2 \right) - \cos\varphi \left(\frac{\partial}{\partial x_2} + F_2 \frac{\partial}{\partial \varphi} + \frac{\partial F_2}{\partial \varphi} - F_1 \right). \end{aligned}$$

The conjugate of the operator $\frac{\partial}{\partial l}$ in the sense of Lagrange can be obtained as follows:

$$\left(\frac{\partial}{\partial l} \right)^* = -\sin\varphi \left(\frac{\partial}{\partial x_1} + F_1 \frac{\partial}{\partial \varphi} \right) + \cos\varphi \left(\frac{\partial}{\partial x_2} + F_2 \frac{\partial}{\partial \varphi} \right).$$

By $\Gamma''(A)$ we denote the set of functions $u(x,\varphi) \in L_2(\Omega)$ with the property that for any $u \in \Gamma''(A)$ there exists a function $y \in L_2(\Omega)$ such that $\forall \eta \in C^{\infty}_0(\Omega)$

$$(u, A^*\eta)_{L_2(\Omega)} = (y, \eta)_{L_2(\Omega)}$$

and $y = Au$. Here $(u,v)_{L_2(\Omega)}$ is a scalar product of functions u and v in $L_2(\Omega)$, A^* is the differential expression conjugate to A in the sense of Lagrange, and $C^{\infty}_0(\Omega)$ is the set of all functions defined in Ω which have continuous partial derivatives of order up to all $k < \infty$, whose supports are compact subsets of Ω. So the equality $y = Au$ is satisfied in the sense of generalized functions.

Definition 4. *The subset $\Gamma(A) \subset \Gamma''(A)$ is such that for any $u \in \Gamma(A)$ there is a sequence $\{u_k\} \subset C^3_{\pi 0}$ with the following properties:*

i) $u_k \to u$ weakly in $L_2(\Omega)$

ii) $(Au_k, u_k)_{L_2(\Omega)} \to (Au, u)_{L_2(\Omega)}$ as $k \to \infty$.

Let $\Gamma'(A)$ be the closure of $C^3_{\pi 0}$ with respect to the norm $\|u\|_{\Gamma(A)} = \|u\| + \|Au\|$, where $\|.\|$ is the norm in $L_2(\Omega)$. Then the inclusions

$$\Gamma'(A) \subset \Gamma(A) \subset \Gamma''(A) \subset L_2(\Omega),\ \mathring{H}^{\pi}_3(\Omega) \subset \Gamma''(A) \cap \mathring{H}^{\pi}_{1,c}(\Omega) \subset \Gamma(A) \subset L_2(\Omega)$$

hold.

4 Existence, Uniqueness and Stability of the Solution

At present, there are a great number of publications devoted to the uniqueness of solutions to IGP, while the problem of existence has been given much less attention. Since the underlying operator of the related IGP is compact and its inverse operator is unbounded, the issue of existence of solution of the problem is basically unsolvable, as it is the case of all inverse/ill-posed problems. In other words, the main difficulty in studying the solvability of such problems is overdeterminancy. To overcome this difficulty, a new method of investigating the solvability of overdetermined inverse problems was firstly proposed by Amirov (1986) for transport equation.

The way of proving the solvability of Problem 2 can be outlined as follows: the class of the unknown functions λ is extended so that the IGP becomes well posed for the new class. And this extension is not arbitrary: it should contain the functions depending only on x (as in classical problems of integral geometry), [3]. In other words, we immerse equation (2) into a system of two equations (4) and (6) in which a new unknown function $\widetilde{\lambda}$ is involved and $\widetilde{\lambda} = \widetilde{\lambda}(x, \varphi)$. Here, φ-dependence of the function $\widetilde{\lambda}(x, \varphi)$ is via a nontrivial manner, because this function is assumed to satisfy the new equation (6).

Hence, Problem 2 is replaced by the following determined problem:

Problem 3. *Determine the functions $\widetilde{u}\,(x, \varphi)$ and $\widetilde{\lambda}(x, \varphi)$ defined in the domain Ω that satisfy the equations*

$$
\begin{aligned}
L\widetilde{u} &= \widetilde{\lambda}(x, \varphi), &(4)\\
\widetilde{u}|_{\Gamma_1} &= \widetilde{u}_0, \quad \widetilde{u}(x, 0) = \widetilde{u}(x, 2\pi), &(5)\\
\widehat{L}\widetilde{\lambda} &= 0 &(6)
\end{aligned}
$$

provided that the function K is known. Equation (6) is satisfied in the generalized functions sense, i.e., $(\widetilde{\lambda}, \left(\widehat{L}\right)^* \eta)_{L_2(\Omega)} = 0$ for any $\eta \in C_0^\infty(\Omega)$, where $\left(\widehat{L}\right)^*$ is the conjugate operator to $\widehat{L}$ in the Lagrange sense, if $\widetilde{\lambda}(x, \varphi)$ does not depend on φ, then $\widetilde{\lambda}$ satisfies condition (6).

It is important to mention here that, if $u_0 \in C^3(\Gamma_1)$, $u(x, \varphi) \in \Gamma(A) \cap H_1^\pi(\Omega)$, $\lambda\,(x) \in C^3(\bar{D})$, and (u, λ) is the solution of Problem 2 then from the equality $\widehat{L}\widetilde{\lambda} = 0$ (because of $\lambda = \lambda(x)$) it follows that (u, λ) is also a solution to Problem 3.

Suppose that, a priori we know a function u_0^e to be the exact data of Problem 1 related to a function λ depending only on x. Then, utilizing u_0^e, we can construct a solution $\breve{\lambda}$ to Problem 1. By uniqueness of a solution, $\breve{\lambda}$ coincides with $\lambda(x)$. If we know the approximate data u_0^a with $\|u_0^e - u_0^a\|_{H^3(\partial\Omega)} \le \varepsilon$, we can construct an approximate solution $\lambda^a(x,\varphi)$ such that $\|\lambda - \lambda^a\|_{L_2(\Omega)} \le C\varepsilon$. Recall that, if λ depends only on x and u_0^a does not satisfy the "solvability conditions", the solution λ^a depending only x does not exist. Here the data are specified on $\partial\Omega$ and $C > 0$ is not dependent on u_0^e and u_0^a. In other words, we construct a regularizing procedure for Problem 1.

Since $\widetilde{u}_0 \in C^3(\Gamma_1)$ and $\partial D \in C^3$ then by Theorem 2, p. 130 in [16], there is a function $\Psi \in C^3(\Omega)$ such that $\Psi|_{\Gamma_1} = \widetilde{u}_0$. And with the aid of substitution $\bar{u} = \widetilde{u} - \Psi$, Problem 3 can be reduced to the following one with homogenous data on Γ_1.

Problem 4. *Determine a pair of functions $(\bar{u}, \widetilde{\lambda})$ defined in Ω and satisfying the equations*

$$L\bar{u} = \widetilde{\lambda} + G, \tag{8}$$

$$\bar{u}|_{\Gamma_1} = 0, \quad \bar{u}(x,0) = \bar{u}(x,2\pi), \tag{9}$$

$$\widehat{L}\widetilde{\lambda} = 0, \tag{10}$$

provided that the functions K and *G are known,* where $G = -L\Psi$.

The following theorem states the existence, uniqueness and stability of the solution of Problem 4. The uniqueness of the solution of Problem 3 follows from Theorem 1, since the corresponding homogeneous versions of both problems are the same. Hence, if $\left(\widetilde{u}, \widetilde{\lambda}\right)$ is a solution to Problem 3, then because of uniqueness of solution to Problem 3, the function $\widetilde{u} = \bar{u} + w$ does not depend on choice of Ψ (also on G) and it depends only on $\widetilde{u}_0$. For the notational simplicity, we will denote $\bar{u}$ by u and $\widetilde{\lambda}$ by λ.

Theorem 1. Assume that $F_1(x,\varphi)$, $F_2(x,\varphi) \in C^2(\bar{D} \times (0,2\pi))$ and the inequality $F_{1x_2} - F_{2x_1} + F_{1\varphi}F_2 - F_1F_{2\varphi} > 0$ holds for all $x \in \bar{D}$ then Problem 4 has a unique solution (u,λ), such that $u \in \Gamma(A) \cap \mathring{H}_1^\pi(\Omega)$, $\lambda \in L_2(\Omega)$. Also, the inequality

$$\|u\|_{\mathring{H}_1^\pi(\Omega)} + \|\lambda\|_{L_2(\Omega)} \le C(\|G\|_{L_2(\Omega)} + \|G_\varphi\|_{L_2(\Omega)}) \tag{11}$$

holds, where $G \in H_2^\pi(\Omega)$, $C > 0$ depends on F_1, F_2 and the Lebesgue measure of D, and $\bar{D}$ is the closure of D.

Remark 1. *If $F_1 = F_1(x)$, $F_2 = F_2(x)$ then for the validity of Theorem 1 it is enough condition $F_{1x_2} - F_{2x_1} > 0$.*

Proof. Firstly we will prove uniqueness of a solution to Problem 4. Suppose that (u,λ) is a solution to the homogeneous version of Problem 4 ($G = 0$)

such that $u \in \Gamma(A) \cap \overset{\circ}{H}{}_1^{\pi}(\Omega)$ and $\lambda \in L_2(\Omega)$. Equation (8) and condition (10) imply $Au = 0$. Since $u \in \Gamma(A)$, there exists a sequence $\{u_k\} \subset C^3_{\pi 0}$ such that $u_k \to u$ weakly in $L_2(\Omega)$ and $(Au_k, u_k)_{L_2(\Omega)} \to 0$ as $k \to \infty$. It can be easily verified that

$$\begin{aligned}(Au_k)\, u_k &= \left(\widehat{L} L u_k\right) u_k = \left(\frac{\partial}{\partial l}\left(\frac{\partial}{\partial \varphi} L u_k\right)\right) u_k = \frac{\partial}{\partial \varphi} L u_k \left(\frac{\partial}{\partial l}\right)^* u_k \\ &+ \frac{\partial}{\partial x_1}\left(u_k \left(\frac{\partial}{\partial \varphi} L u_k\right) \sin\varphi\right) - \frac{\partial}{\partial x_2}\left(u_k \left(\frac{\partial}{\partial \varphi} L u_k\right) \cos\varphi\right) \\ &+ \frac{\partial}{\partial \varphi}\left(u_k \left(\frac{\partial}{\partial \varphi} L u_k\right)(F_1 \sin\varphi - F_2 \cos\varphi)\right), \qquad (12)\end{aligned}$$

and

$$\begin{aligned}&2\frac{\partial}{\partial \varphi} L u_k \left(\frac{\partial}{\partial l}\right)^* u_k \\ =\ & 2\left[u_{kx_1\varphi}\cos\varphi - u_{kx_1}\sin\varphi + u_{kx_2\varphi}\sin\varphi + u_{kx_2}\cos\varphi\right. \\ &\left. +(F_1\cos\varphi + F_2\sin\varphi)u_{k\varphi\varphi} + (\cos\varphi\,(F_{1\varphi} + F_2) + \sin\varphi\,(F_{2\varphi} - F_1))u_{k\varphi}\right] \\ &\times\left[-(\sin\varphi)\,(u_{kx_1} + F_1 u_{k\varphi}) + \cos\varphi\,(u_{kx_2} + F_2 u_{k\varphi})\right]\end{aligned}$$

$$\begin{aligned} =\ & (u_{kx_1} + F_1 u_{k\varphi})^2 + (u_{kx_2} + F_2 u_{k\varphi})^2 - \frac{\partial}{\partial x_2}[u_{k\varphi}(u_{kx_1} + F_1 u_{k\varphi})] \\ &+ (F_{1x_2} - F_{2x_1} + F_{1\varphi}F_2 - F_1 F_{2\varphi})u_{k\varphi}^2 + \frac{\partial}{\partial x_1}[u_{k\varphi}(u_{kx_2} + F_2 u_{k\varphi})] \\ &+ \frac{\partial}{\partial \varphi}[(u_{kx_2} + F_2 u_{k\varphi})^2 \sin\varphi\cos\varphi + u_{k\varphi}(F_1 u_{kx_2} - F_2 u_{kx_1}) \\ &- (u_{kx_1} + F_1 u_{k\varphi})^2 \sin\varphi\cos\varphi + (u_{kx_1} + F_1 u_{k\varphi})(u_{kx_2} + F_2 u_{k\varphi})\cos 2\varphi].\end{aligned}$$

If we integrate (12) over the domain Ω, since $u_k \in C^3_{\pi 0}$ the divergent terms will disappear, so we get

$$\begin{aligned}2\,(Au_k, u_k)_{L_2(\Omega)} = \int_\Omega &[(u_{kx_1} + F_1 u_{k\varphi})^2 + (u_{kx_2} + F_2 u_{k\varphi})^2 + \\ &+ (F_{1x_2} - F_{2x_1} + F_{1\varphi}F_2 - F_1 F_{2\varphi})u_{k\varphi}^2]d\Omega\ . \qquad (13)\end{aligned}$$

It can be seen that the quadratic form

$$\begin{aligned}J(\nabla u_k) &= (u_{kx_1} + F_1 u_{k\varphi})^2 + (u_{kx_2} + F_2 u_{k\varphi})^2 + (F_{1x_2} - F_{2x_1} + F_{1\varphi}F_2 - \\ &- F_1 F_{2\varphi})u_{k\varphi}^2 = u_{kx_1}^2 + F_1^2 u_{k\varphi}^2 + 2F_1 u_{kx_1} u_{k\varphi} + u_{kx_2}^2 + F_2^2 u_{k\varphi}^2 \\ &\quad + 2F_2 u_{kx_2} u_{k\varphi} + (F_{1x_2} - F_{2x_1} + F_{1\varphi}F_2 - F_1 F_{2\varphi})u_{k\varphi}^2\end{aligned}$$

is positive definite in $(u_{kx_1} + F_1 u_{k\varphi})$, $(u_{kx_2} + F_2 u_{k\varphi})$, $u_{k\varphi}$ under the condition that $F_{1x_2} - F_{2x_1} + F_{1\varphi}F_2 - F_1 F_{2\varphi} > 0$ for all $(x, \varphi) \in \Omega$. Taking into account the estimates

$$\begin{aligned} 2F_1 u_{kx_1} u_{k\varphi} &\geq -\varepsilon u_{kx_1}^2 - \varepsilon^{-1} F_1^2 u_{k\varphi}^2,\ 0 < \varepsilon < 1 \\ 2F_2 u_{kx_2} u_{k\varphi} &\geq -\varepsilon u_{kx_2}^2 - \varepsilon^{-1} F_2^2 u_{k\varphi}^2 \end{aligned}$$

and the conditions of the theorem, we obtain

$$\begin{aligned} J(\nabla u_k) &\geq (1-\varepsilon)\left(u_{kx_1}^2 + u_{kx_2}^2\right) + \left(1-\varepsilon^{-1}\right) K u_{k\varphi}^2 + \eta_0 u_{k\varphi}^2 \\ &\geq (1-\varepsilon)\left|\nabla_x u_k\right|^2 + \left(\eta_0 + K\left(1-\varepsilon^{-1}\right)\right) u_{k\varphi}^2, \end{aligned}$$

where $\eta_0, K \in \mathbb{R}$ such that $F_{1x_2} - F_{2x_1} + F_{1\varphi}F_2 - F_1 F_{2\varphi} \geq \eta_0 > 0$ and $F_1^2 + F_2^2 \leq K$. For sufficiently close value of ε to 1 we have $\eta_0 + K\left(1-\varepsilon^{-1}\right) > \dfrac{\eta_0}{2}$, hence

$$J(\nabla u_k) \geq (1-\varepsilon)\left|\nabla_x u_k\right|^2 + \frac{\eta_0}{2} u_{k\varphi}^2 \geq \gamma_0 \left(\left|\nabla_x u_k\right|^2 + u_{k\varphi}^2\right), \qquad (14)$$

where $\gamma_0 = \min\left\{(1-\varepsilon), \dfrac{\eta_0}{2}\right\}$. Since the domain D is bounded and $u_k = 0$ on Γ_1, it can be easily obtained that $\|u_k\|_{L_2(\Omega)}^2 \leq C_0 \int_\Omega \left|\nabla_x u_k\right|^2 d\Omega$, so we have $\|u_k\|_{L_2(\Omega)}^2 \leq C \int_\Omega J(\nabla u_k) d\Omega$, where $C = C_0 \gamma_0^{-1}$ and $C_0 > 0$ is independent of k and depends on Lebesgue measure of D. Consequently, by virtue (13) and the definition of $\Gamma(A)$ we have

$$\|u\|_{L_2(\Omega)}^2 \leq \varliminf_{k\to\infty} \|u_k\|_{L_2(\Omega)}^2 \leq C \lim_{k\to\infty} \int_\Omega J(\nabla u_k) d\Omega = 2C \lim_{k\to\infty} (Au_k, u_k)_{L_2(\Omega)} = 0. \qquad (15)$$

From (15), it folllows that $\|u\|_{L_2(\Omega)}^2 = 0$, i.e., $u = 0$ and from (8), $\lambda = 0$. Hence, the uniqueness of the solution of the problem is proven.

We now prove the existence of a solution (u, λ) of the problem in the set: $(\Gamma(A) \cap \mathring{H}_1^\pi(\Omega)) \times L_2(\Omega)$.

Consider the following auxiliary problem:

Find u defined in Ω that satisfies

$$Au = \mathcal{F}, \qquad (16)$$

$$u|_{\Gamma_1} = 0, \quad u(x,0) = u(x,2\pi), \qquad (17)$$

where $\mathcal{F} = \widehat{L}G$.

Select a set $\{e_1, e_2, e_3, ...\} \subset C_{\pi 0}^3$ which is complete and orthonormal in $L_2(\Omega)$. We may assume here that the linear span of this set is everywhere dense in $\overset{\circ}{H}{}^{\pi}_{1,c}(\Omega)$. In fact, the space $\overset{\circ}{H}{}^{\pi}_{1,c}(\Omega) \cap \overset{\circ}{H}_1(\Omega)$ being seperable, there exists a countable set $\{\varphi_i\}_{i=1}^{\infty} \subset C^3_{\pi 0}$ which is everywhere dense in this space. If necessary, this set up can be extended to a set which is everywhere dense in $L_2(\Omega)$. Orthonormalizing the latter in $L_2(\Omega)$, we obtain $\{e_1, e_2, e_3, ...\}$.We denote the orthogonal projector of $L_2(\Omega)$ onto $\mathcal{M}_n$ by $\mathcal{P}_n$, where $\mathcal{M}_n$ is the linear span of $\{e_1, e_2, ..., e_n\}$.

An approximate solution to problem (16)-(17) is sought in the form

$$u_N = \sum_{i=1}^{N} \alpha_{N_i} e_i(x, \varphi); \qquad \alpha_N = (\alpha_{N_1}, \alpha_{N_2}, ..., \alpha_{N_N}) \in \mathbb{R}^N.$$

The unknown coefficients α_{N_i} are determined from the following system of linear algebraic equations:

$$\int_{\Omega} \widehat{L}(Lu_N - G)e_j d\Omega = 0, \quad j = 1, 2, ..., N, \quad d\Omega = dx_1 dx_2 d\varphi. \tag{18}$$

We now prove that under the assumptions of the theorem, system (18) has a unique solution for any $G \in H_2^{\pi}(\Omega)$. For this purpose, we consider the homogeneous version of system (18) ($G = 0$). Let's substitute $\bar{\alpha}_N$ for α_N, multiply the jth equation by $2\bar{\alpha}_{N_j}$ and sum with respect to j from 1 to N, then we obtain

$$2\int_{\Omega} \widehat{L}L\bar{u}_N \bar{u}_N d\Omega = 0, \tag{19}$$

where $\bar{u}_N = \sum\limits_{i=1}^{N} \bar{\alpha}_{N_i} e_i$. Then equality (13) yields

$$\int_{\Omega} [(\bar{u}_{Nx_1} + F_1\bar{u}_{N\varphi})^2 + (\bar{u}_{Nx_2} + F_2\bar{u}_{N\varphi})^2 + (F_{1x_2} - F_{2x_1} + F_{1\varphi}F_2 - F_1F_{2\varphi})\bar{u}^2_{N\varphi}] d\Omega = 0. \tag{20}$$

Using the fact that, $J(\nabla \bar{u}_N)$ is positive definite and $\bar{u}_N = 0$ on Γ_1, from(20) we have $\bar{u}_N = 0$ in Ω. Since the system $\{e_i\}$, $(i = 1, 2, ...)$ is linearly independent, we get $\bar{\alpha}_{N_i} = 0$, $i = 1, 2, ..., N$. Thus, the homogeneous version of system (18) has only trivial solution and therefore the original inhomogeneous system (18) has a unique solution α_N for any $G \in H_2^{\pi}(\Omega)$.

Now we estimate the solution u_N of system (18) in terms of the right hand side G. If we multiply the jth equation of (18) by $2\alpha_{N_j}$ and sum from 1 to N

with respect to j, then we obtain

$$2\int_{\Omega} u_N \widehat{L} L u_N d\Omega = 2\int_{\Omega} u_N \widehat{L} G d\Omega. \tag{21}$$

Observing that $u_N \in C^3_{\pi 0}$ and transferring operator $\frac{\partial}{\partial l}$ in $\widehat{L}$ to the function u_N, the right hand side of (21) can be estimated as follows:

$$2\left|\int_{\Omega} u_N \widehat{L} G d\Omega\right| \leq \alpha_0 \int_{\Omega} G_{\varphi}^2 d\Omega + \alpha_0^{-1} \int_{\Omega} \left(\left(\frac{\partial}{\partial l}\right)^* u_N\right)^2 d\Omega.$$

Since the left hand side of (21) equals $\int_{\Omega} J(\nabla u_N) d\Omega$, from (21) for sufficiently large $\alpha_0 > 0$, we get

$$\int_{\Omega} J(\nabla u_N) d\Omega \leq \alpha_0 \int_{\Omega} G_{\varphi}^2 d\Omega + \alpha_0^{-1} \int_{\Omega} \left(\left(\frac{\partial}{\partial l}\right)^* u_N\right)^2 d\Omega.$$

Hence, using the inequality (14), we obtain

$$\|u_N\|_{\mathring{H}_1^{\pi}(\Omega)} \leq C \|G_{\varphi}\|_{L_2(\Omega)}, \tag{22}$$

where the constant C doesn't depend on N. Thus, the set of functions $\{u_N\}$ is bounded in $\mathring{H}_1^{\pi}(\Omega)$. Since $\mathring{H}_1^{\pi}(\Omega)$ is a Hilbert space, the set $\{u_N\}$ is weakly compact in it. Therefore, there exists a subsequence (we again denote it by $\{u_N\}$) such that $u_N \to u$ weakly in $\mathring{H}_1^{\pi}(\Omega)$ as $N \to \infty$, so it follows that

$$\|u\|_{\mathring{H}_1^{\pi}(\Omega)} \leq C \|G_{\varphi}\|_{L_2(\Omega)}.$$

Since $u \in \mathring{H}_1^{\pi}(\Omega)$, by the definition of $\mathring{H}_1^{\pi}(\Omega)$, we have $u|_{\Gamma_1} = 0$. From estimate (22), it can be easily proved that there exists a subsequence of $\{u_N\}$, which is again denoted by $\{u_N\}$, such that u_{Nx_1}, u_{Nx_2} and $u_{N\varphi}$ converge weakly in $L_2(\Omega)$ to u_{x_1}, u_{x_2} and u_{φ} respectively. Transferring the operator $\widehat{L}$ to e_j in (18) and taking into account the conditions $u_N, w_j \in C^3_{\pi 0}$ and $G \in H_2^{\pi}(\Omega)$, we have

$$\int_{\Omega} (Lu_N - G)(\widehat{L})^* e_j d\Omega = 0, \; N \geq j.$$

Since the linear span of $\{e_j\}$ is eveywhere dense in the space $\mathring{H}_{1,c}^{\pi}(\Omega)$, passing to the limit as $N \to \infty$ we get

$$\int\limits_{\Omega} (Lu - G)(\widehat{L})^* \zeta d\Omega = 0, \tag{23}$$

for every $\zeta \in \mathring{H}^{\pi}_{1,c}(\Omega)$. If we set $\lambda = Lu - G$, from (23) we see that λ satisfies condition (6) for any $\zeta \in C_0^{\infty}(\Omega) \subset \mathring{H}^{\pi}_{1,c}(\Omega)$, and the following estimate is valid:

$$\|\lambda\|_{L_2(\Omega)} \leq C \|u\|_{\mathring{H}_1^{\pi}(\Omega)} + \|G\|_{L_2(\Omega)} .$$

Thus, by using the inequality $\|u\|_{\mathring{H}_1^{\pi}(\Omega)} \leq C \|G_{\varphi}\|_{L_2(\Omega)}$, we obtain (11). In the expressions above , C stands for different constants that depend only on the given functions and Lebesgue measure of the domain D. Consequently, we have found a solution (u, λ) to problem 4, where $u \in \mathring{H}_1^{\pi}(\Omega)$ and $\lambda \in L_2(\Omega)$.

Now it will be proven that $u \in \Gamma(A)$. Since $u \in L_2(\Omega)$ and $G \in H_2^{\pi}(\Omega)$, from (23) it follows that $\mathcal{F} = Au \in L_2(\Omega)$ in the generalized sense, i.e., $u \in \Gamma''(A)$. Indeed, for any $\zeta \in C_0^{\infty}(\Omega)$ we have

$$(u, A^*\zeta)_{L_2(\Omega)} = (u, L^*(\widehat{L})^*\zeta)_{L_2(\Omega)} = (Lu, (\widehat{L})^*\zeta)_{L_2(\Omega)} = (G, (\widehat{L})^*\zeta)_{L_2(\Omega)} =$$

$(\mathcal{F}, \zeta)_{L_2(\Omega)}$, where $\mathcal{F} = \widehat{L}G \in L_2(\Omega)$.

To complete the proof, it remains to show the convergence $(Au_N, u_N)_{L_2(\Omega)} \to (Au, u)_{L_2(\Omega)}$ as $N \to \infty$. From (18), it follows that $\mathcal{P}_N Au_N = \mathcal{P}_N \mathcal{F}$. Since the system $\{e_1, e_2, ..., e_N, ...\}$ is orthogonal and complete in $L_2(\Omega)$, $\mathcal{P}_N \mathcal{F}$ converges strongly to $\mathcal{F}$ in $L_2(\Omega)$ as $N \to \infty$, i.e., we get $\mathcal{P}_N Au_N \to \mathcal{F} = Au$ strongly in $L_2(\Omega)$ as $N \to \infty$. Then, $(\mathcal{P}_N Au_N, u_N)_{L_2(\Omega)} \to (Au, u)_{L_2(\Omega)}$ as $N \to \infty$ because $\{u_N\}$ weakly converges to u in $L_2(\Omega)$ as $N \to \infty$. By the definitions of $\mathcal{P}_N$ and u_N (since $\mathcal{P}_N$ is self adjoint in $L_2(\Omega)$, see p 481 in [12]) we obtain

$$(\mathcal{P}_N Au_N, u_N)_{L_2(\Omega)} = (Au_N, \mathcal{P}_N^* u_N)_{L_2(\Omega)} = (Au_N, \mathcal{P}_N u_N)_{L_2(\Omega)} = (Au_N, u_N)_{L_2(\Omega)} .$$

Hence $(Au_N, u_N)_{L_2(\Omega)} \to (Au, u)_{L_2(\Omega)}$ as $N \to \infty$ which completes the proof of Theorem 1. ∎

Now we consider a special case of Problem 1.

5 Solvability of an IGP along Geodesics

In this section we investigate the solvability of an integral geometry problem along geodesics and the related inverse problem for a special kinetic equation. We take the curve passing from every $x \in D$ in the arbitrary direction $\nu =$

$(\cos\varphi, \sin\varphi)$ given by curvature $K(x,\varphi) = F_1(x,\varphi)\cos\varphi + F_2(x,\varphi)\sin\varphi$, with end points on the boundary of D. Moreover, we consider the Cauchy problem for the system

$$\ddot{z}^i = -\sum_{i,j,k=1}^{2} \Gamma^i_{jk} \dot{z}^j \dot{z}^k, \tag{24}$$

with the data

$$z^i(0) = z^i_0, \; \dot{z}^i(0) = \dot{z}^i_0 = \xi^i_0, \; i = 1, 2, \tag{25}$$

where $\dot{z}^i = \dfrac{dz^i}{dt}$.

System (24) coincides with the equation for the geodesics under the assumption that the parameter of a curve is chosen to be proportional to the natural one (see, [9]). It is possible to choose such parameter if the tangent vector of the curve is nonzero. Thus, we shall assume that $|\dot{z}| \neq 0$ in (24).

Problem 1′. *Given the integrals of λ along the geodesics of a given family of curves $\{\Gamma\}$, determine $\lambda(x)$ in the domain D.*

We take a geodesic $\gamma(x,\xi)$ with endpoint $x \in D$ and "direction" ξ at x. The geodesic $\gamma(x,\xi)$ is the projection of the solution of the Cauchy problem (24)-(25) with the data $z^i_0 = x^i$, $\xi^i_0 = \xi^i$, $x = (x_1, x_2)$, $\xi = (\xi_1, \xi_2)$, onto the domain D. By assumption, this geodesic intersects the boundary of D. We introduce a function $u(x,\xi)$ as follows and write the integral of λ over the part of $\gamma(x,\xi)$ that lies in D (denoted by $\gamma_D(x,\xi)$):

$$u = \int_{\gamma_D(x,\xi)} \lambda dS, \tag{26}$$

where λ is the same function as in Problem 1′ and dS is the arc length element of the geodesic γ_D. Differentiating (26) at the point x in the "direction" ξ, i.e., differentiating (26) with respect to the parameter t and taking into account that $\gamma(x,\xi)$ is the solution of the Cauchy problem (24)-(25) in D with the data x^i, ξ^i,

$$Lu \equiv \sum_{i=1}^{2} \xi^i \frac{\partial u}{\partial x^i} - \sum_{i,j,k=1}^{2} \Gamma^i_{jk} \xi^j \xi^k \frac{\partial u}{\partial \xi^i} = \lambda(x), \tag{27}$$

is obtained, where Γ^i_{jk} is the symmetric connection (Christoffel symbols), [9].

Utilizing the change of variables $\xi_1 = r\cos\varphi$, $\xi_2 = r\sin\varphi$ yields to

$$\begin{aligned} u_{\xi_1} &= -\frac{\sin\varphi}{r}\tilde{u}_\varphi + \cos\varphi\tilde{u}_r, \\ u_{\xi_2} &= \frac{\cos\varphi}{r}\tilde{u}_\varphi + \sin\varphi\tilde{u}_r, \end{aligned}$$

where $u(x,\xi) \equiv \tilde{u}(x,r,\varphi)$. Hence, equation (27) takes the following form:

$$\begin{aligned} L\tilde{u} \equiv\; & r\cos\varphi\frac{\partial\tilde{u}}{\partial x_1} + r\sin\varphi\frac{\partial\tilde{u}}{\partial x_2} + \\ & + \left[\left(\Gamma^1_{12}\sin^2\varphi + \left(\Gamma^1_{11} - \Gamma^2_{12}\right)\cos\varphi\sin\varphi - \Gamma^2_{11}\cos^2\varphi\right)\cos\varphi + \right. \\ & \left. + \left(\Gamma^1_{22}\sin^2\varphi + \left(\Gamma^1_{21} - \Gamma^2_{22}\right)\cos\varphi\sin\varphi - \Gamma^2_{21}\cos^2\varphi\right)\sin\varphi\right] r\tilde{u}_\varphi \qquad (28) \\ & - \left[\left(\Gamma^1_{11}\cos^2\varphi + \left(\Gamma^1_{12} + \Gamma^2_{11}\right)\cos\varphi\sin\varphi + \Gamma^2_{12}\sin^2\varphi\right) r^2\cos\varphi \right. \\ & \left. + \left(\Gamma^1_{21}\cos^2\varphi + \left(\Gamma^1_{22} + \Gamma^2_{21}\right)\cos\varphi\sin\varphi + \Gamma^2_{22}\sin^2\varphi\right) r^2\sin\varphi\right]\tilde{u}_r \\ =\; & \tilde{\lambda}(x,r). \end{aligned}$$

Here, if we take $r = 1$, $\tilde{u}_r = 0$ and for the notational simplicity, denote $\tilde{u}$, $\tilde{\lambda}$ by u and λ, respectively, from (28), we obtain

$$Lu \equiv u_{x_1}\cos\varphi + u_{x_2}\sin\varphi + \left(F_1(x,\varphi)\cos\varphi + F_2(x,\varphi)\sin\varphi\right)u_\varphi = \lambda(x), \quad (29)$$

where

$$\begin{aligned} F_1(x,\varphi) &= \Gamma^1_{12}\sin^2\varphi + \left(\Gamma^1_{11} - \Gamma^2_{12}\right)\cos\varphi\sin\varphi - \Gamma^2_{11}\cos^2\varphi, \\ F_2(x,\varphi) &= \Gamma^1_{22}\sin^2\varphi + \left(\Gamma^1_{21} - \Gamma^2_{22}\right)\cos\varphi\sin\varphi - \Gamma^2_{21}\cos^2\varphi. \end{aligned}$$

Problem 2′. *Given the function $K(x,\varphi)$, find a pair of functions (u,λ) from equation (29), provided that the equations*

$$u|_{\Gamma_1} = u_0, \quad u(x,0) = u(x,2\pi), \qquad (30)$$

in $\Omega = \{(x,\varphi) : x \in D \subset R^2,\ \varphi \in (0,2\pi),\ \partial D \in C^3\}$.

Here,

$$\widehat{L}u = \frac{\partial^2}{\partial l\partial\varphi}u = \frac{\partial}{\partial l}u_\varphi,$$

$$\begin{aligned}\frac{\partial}{\partial l} \quad = \quad & \sin\varphi\left(\frac{\partial}{\partial x_1} + \left(\Gamma_{12}^1\sin^2\varphi + \left(\Gamma_{11}^1 - \Gamma_{12}^2\right)\cos\varphi\sin\varphi - \Gamma_{11}^2\cos^2\varphi\right)\frac{\partial}{\partial\varphi}\right. \\ & + \left(\Gamma_{11}^1 - \Gamma_{12}^2\right)\cos 2\varphi + \left(\Gamma_{12}^1 + \Gamma_{11}^2\right)\sin 2\varphi \\ & \left. + \Gamma_{22}^1\sin^2\varphi + \left(\Gamma_{21}^1 - \Gamma_{22}^2\right)\cos\varphi\sin\varphi - \Gamma_{21}^2\cos^2\varphi\right) \\ & - \cos\varphi\left(\frac{\partial}{\partial x_2} + \left(\Gamma_{22}^1\sin^2\varphi + \left(\Gamma_{21}^1 - \Gamma_{22}^2\right)\cos\varphi\sin\varphi - \Gamma_{21}^2\cos^2\varphi\right)\frac{\partial}{\partial\varphi}\right. \\ & + \left(\Gamma_{21}^1 - \Gamma_{22}^2\right)\cos 2\varphi + \left(\Gamma_{22}^1 + \Gamma_{21}^2\right)\sin 2\varphi \\ & \left. - \Gamma_{12}^1\sin^2\varphi + \left(\Gamma_{12}^2 - \Gamma_{11}^1\right)\cos\varphi\sin\varphi + \Gamma_{11}^2\cos^2\varphi\right),\end{aligned}$$

and the conjugate of the operator $\frac{\partial}{\partial l}$ in the sense of Lagrange is

$$\begin{aligned}\left(\frac{\partial}{\partial l}\right)^* = & -\sin\varphi\left(\frac{\partial}{\partial x_1} + \left(\Gamma_{12}^1\sin^2\varphi + \left(\Gamma_{11}^1 - \Gamma_{12}^2\right)\cos\varphi\sin\varphi - \Gamma_{11}^2\cos^2\varphi\right)\frac{\partial}{\partial\varphi}\right) \\ & + \cos\varphi\left(\frac{\partial}{\partial x_2} + \left(\Gamma_{22}^1\sin^2\varphi + \left(\Gamma_{21}^1 - \Gamma_{22}^2\right)\cos\varphi\sin\varphi - \Gamma_{21}^2\cos^2\varphi\right)\frac{\partial}{\partial\varphi}\right).\end{aligned}$$

Problem 3′. *Given the integrals of $\lambda(x,\varphi)$ along the geodesics with endpoint $x \in \partial D$ and direction $\xi = (\cos\varphi, \sin\varphi)$ at x, determine the functions $u(x,\varphi)$ and $\lambda(x,\varphi)$ defined in the domain Ω that satisfy the equations*

$$Lu = \lambda(x,\varphi), \tag{31}$$

$$u|_{\Gamma_1} = u_0, \quad u(x,0) = u(x,2\pi), \tag{32}$$

$$\widehat{L}\lambda = 0, \tag{33}$$

provided that the function K is known.

Problem 4′. *Find a pair of functions (u,λ) defined in Ω and satisfying the equation*

$$Lu = \lambda + G,$$

provided that the functions K and G are known, u satisfies condition (32), and for λ condition (33) holds.

Now, we establish the solvability theorem for IGP along geodesics. For this aim, we first estimate the term $F_{1x_2} - F_{2x_1} + F_{1\varphi}F_2 - F_1F_{2\varphi}$ as follows:

$$F_{1x_2} - F_{2x_1} + F_{1\varphi}F_2 - F_1F_{2\varphi} = a\sin^2\varphi + b\sin\varphi\cos\varphi + c\cos^2\varphi,$$

where

$$a=\left(\left(\Gamma^1_{12}\right)_{x_2}-\left(\Gamma^1_{22}\right)_{x_1}-\Gamma^1_{11}\Gamma^1_{22}-\Gamma^1_{12}\Gamma^2_{22}+\left(\Gamma^1_{12}\right)^2+\Gamma^2_{12}\Gamma^1_{22}\right),$$

$$b=\left(\left(\Gamma^1_{11}-\Gamma^2_{12}\right)_{x_2}+\left(\Gamma^2_{22}-\Gamma^1_{21}\right)_{x_1}-2\Gamma^1_{12}\Gamma^2_{21}+2\Gamma^2_{11}\Gamma^1_{22}\right),$$

$$c=\left(\left(\Gamma^2_{21}\right)_{x_1}-\left(\Gamma^2_{11}\right)_{x_2}-\Gamma^1_{11}\Gamma^2_{21}-\Gamma^2_{11}\Gamma^2_{22}+\left(\Gamma^2_{12}\right)^2+\Gamma^2_{11}\Gamma^1_{21}\right).$$

From the condition $F_{1x_2}-F_{2x_1}+F_{1\varphi}F_2-F_1F_{2\varphi}>0$, we have the inequalities

$$a>0 \text{ and } 4ac-b^2>0. \tag{34}$$

In the case when $F_1=F_2$, this condition takes the following form

$$\left(\Gamma^1_{12}\right)_{x_2}-\left(\Gamma^1_{12}\right)_{x_1}>0,$$

$$4\left(\left(\Gamma^1_{12}\right)_{x_2}-\left(\Gamma^1_{12}\right)_{x_1}\right)\left(\left(\Gamma^2_{11}\right)_{x_1}-\left(\Gamma^2_{11}\right)_{x_2}\right)-\left(\left(\Gamma^1_{11}-\Gamma^2_{12}\right)_{x_2}+\left(\Gamma^2_{12}-\Gamma^1_{11}\right)_{x_1}\right)^2>0.$$

Hence, for $F_1=F_2$, we give the new version of Theorem 1:

Theorem 1′. *Let the inequalities*

$$\left(\Gamma^1_{12}\right)_{x_2}-\left(\Gamma^1_{12}\right)_{x_1}>0,$$

$$4\left(\left(\Gamma^1_{12}\right)_{x_2}-\left(\Gamma^1_{12}\right)_{x_1}\right)\left(\left(\Gamma^2_{11}\right)_{x_1}-\left(\Gamma^2_{11}\right)_{x_2}\right)-\left(\left(\Gamma^1_{11}-\Gamma^2_{12}\right)_{x_2}+\left(\Gamma^2_{12}-\Gamma^1_{11}\right)_{x_1}\right)^2>0,$$

hold for all $x\in\bar{D}$ and $G\in H_2^\pi(\Omega)$ then Problem 4′ has a unique solution (u,λ), that satisfies the conditions $u\in\Gamma(A)\cap \overset{\circ}{H}{}_1^\pi(\Omega)$, $\lambda\in L_2(\Omega)$, and the inequality

$$\|u\|_{\overset{\circ}{H}{}_1^\pi(\Omega)}+\|\lambda\|_{L_2(\Omega)}\le C(\|G\|_{L_2(\Omega)}+\|G_\varphi\|_{L_2(\Omega)})$$

holds, where $C>0$ depends on F_1,F_2 and the Lebesgue measure of D and $\bar{D}$ is the closure of D.

Proof. The proof can be carried out in a similar way to that of Theorem 1. ∎

6 A Symbolic Algorithm for the Approximate solution of the Problem

In this section, we construct a symbolic algorithm for computing an approximate solution pair $\left(\bar{u}_N, \widetilde{\lambda}_N\right)$ of Problem 4. The approximate solution to the problem is sought in the following form:

$$\bar{u}_N = \sum_{i,j,k=0}^{N} \left(\alpha_{i,j,k} v_{i,j,k} + \beta_{i,j,k} w_{i,j,k}\right) \mu, \tag{35}$$

in the domain $\Omega = D \times (0, 2\pi)$, where, for example, $D = (-1,1) \times (-1,1)$ and

$$\mu = \mu(x_1, x_2) = \begin{cases} \left(1 - x_1^2\right)\left(1 - x_2^2\right), & |x| \le 1 \\ 0, & |x| > 1 \end{cases}, \tag{36}$$

are choosen. The unknown coefficients $\alpha_{i,j,k}$ and $\beta_{i,j,k}$, $i,j,k = 0, ..., N$ in (35), are determined from the following system of linear algebraic equations:

$$\sum_{i,j,k=0}^{N} \left(A\left(\alpha_{i,j,k} v_{i,j,k} + \beta_{i,j,k} w_{i,j,k}\right)\mu, v_{i',j',k'}\mu\right)_{L_2(\Omega)} = \left(\mathcal{F}, v_{i',j',k'}\mu\right)_{L_2(\Omega)}, \tag{37}$$

$$\sum_{i,j,k=0}^{N} \left(A\left(\alpha_{i,j,k} v_{i,j,k} + \beta_{i,j,k} w_{i,j,k}\right)\eta, w_{i',j',k'}\mu\right)_{L_2(\Omega)} = \left(\mathcal{F}, w_{i',j',k'}\mu\right)_{L_2(\Omega)}, \tag{38}$$

where $i', j', k' = 0, ..., N$. In (35), $\{v_{i,j,k}\}_{i,j,k=0}^{\infty}$ and $\{w_{i,j,k}\}_{i,j,k=0}^{\infty}$ are complete systems in $L_2(\Omega)$ where $v_{i,j,k} = x_1^i x_2^j \sin(k\varphi)$ and $w_{i,j,k} = x_1^i x_2^j \cos(k\varphi)$.

Algorithm 1.

INPUT : Order of calculation N, given functions in the curvature: $F_1(x_1, x_2, \varphi)$, $F_2(x_1, x_2, \varphi)$ and the known function in the right hand side of equation (8): G.

OUTPUT : Approximate solution U_N and λ_N

Step 1 *Construct the left hand side of system* (37) for each (i', j', k')

$LeftSys1(i', j', k') := 0,$

for $i = 0, ..., N$,for $j = 0, ..., N$,for $k = 0, ..., N$

$LeftSys1(i', j', k') :=$
$LeftSys1(i', j', k') + \left(A\left(\alpha_{i,j,k} v_{i,j,k} + \beta_{i,j,k} w_{i,j,k}\right)\mu,\ v_{i',j',k'}\mu\right)_{L_2}$

Step 2 *Construct the left hand side of system* (38) for each (i', j', k')

$LeftSys2(i', j', k') := 0,$

for $i = 0, ..., N$, for $j = 0, ..., N$, for $k = 0, ..., N$

$LeftSys2(i',j',k') :=$
$LeftSys2(i',j',k') + \left(A\left(\alpha_{i,j,k}v_{i,j,k} + \beta_{i,j,k}w_{i,j,k}\right)\mu,\, w_{i',j',k'}\mu\right)_{L_2}$
Step 3 *Construct the linear algebraic equations* (37)-(38)
$System := \{\}$, $\mathcal{F} := \widehat{L}G$
for $i = 0,...,N$, for $j = 0,...,N$, for $k = 0,...,N$
$System := System \cup \Big\{LeftSys1\,(i',j',k') = (\mathcal{F},\, v_{i',j',k'}\mu)_{L_2(\Omega)}\,,$
$LeftSys2\,(i',j',k') = (\mathcal{F},\, w_{i',j',k'}\mu)_{L_2(\Omega)}\Big\}$
Step 4 Solve the systems and find the coefficients $\{\alpha_{i,j,k}\}, \{\beta_{i,j,k}\}$
Step 5 Compute (U_N, λ_N)
for $i = 0,...,N$, for $j = 0,...,N$, for $k = 0,...,N$
$U_N = U_N + \left(\alpha_{i,j,k}v_{i,j,k} + \beta_{i,j,k}w_{i,j,k}\right)\mu$
$\lambda_N = L\,(U_N) - G$
Step 6 Output (U_N, λ_N) end.

7 Computational Experiments

Proposed solution algorithm has been implemented in the computer algebra system Maple and tested for several inverse problems. Two examples are presented below. In the computational experiments, we use noisy data G_σ, which is obtained by adding a random perturbation to the exact data G according to the formula $G_\sigma = G\left(1 + \frac{\alpha\sigma}{100}\right)$, where α is a random number in the interval [-1,1] and σ is the noise level in percents.

Example 1. *In the domain* $\Omega = \{\,(x_1,x_2,\varphi)|\, x_1 \in (0,1)\,, x_2 \in (0,1)\,, \varphi \in (0,2\pi)\}$, *according to the given functions,*

$$F\,(x_1,x_2,\varphi) = \frac{1}{2}\left((x_1+x_2)\,(1+2x_1x_2) - \left(x_1^2+4x_1x_2+x_2^2\right)\right)\sin 2\varphi$$
$$+\frac{1}{2}\,(x_1-x_2)\,(x_1-1+x_2\,(1-2x_1))\cos 2\varphi + x_1\,(x_1-1)\left(\left(x_2^2+1\right)(x_2-1)\right.$$
$$\left.+x_2\right)\sin\varphi + \frac{1}{2}x_1x_2\,(x_1+x_2)\,(x_1-1)\,(1-x_2)\,(\sin\varphi-\cos\varphi)\,(1+\cos 2\varphi)$$
$$+x_2\,(x_2-1)\left(2x_1+x_1x_2-x_1^2x_2-1\right)\cos\varphi,$$

$F_1\,(x_1,x_2,\varphi) = x_1\cos\varphi$ *and* $F_2\,(x_1,x_2,\varphi) = -x_2\sin\varphi$, *computed solution pair of the problem is*

$$U_1 = \left(x_1 - x_1^2\right)\left(x_2 - x_2^2\right)(1+\cos\varphi+\sin\varphi),$$
$$\lambda_1 = \frac{1}{2}\left(\left(x_1^2 - x_1\right)(2x_2-1) + \left(x_2^2 - x_2\right)(2x_1-1)\right),$$

which is also the exact solution of the problem. On Figure 1 below, a comparison between exact solution and the approximate solution of the inverse

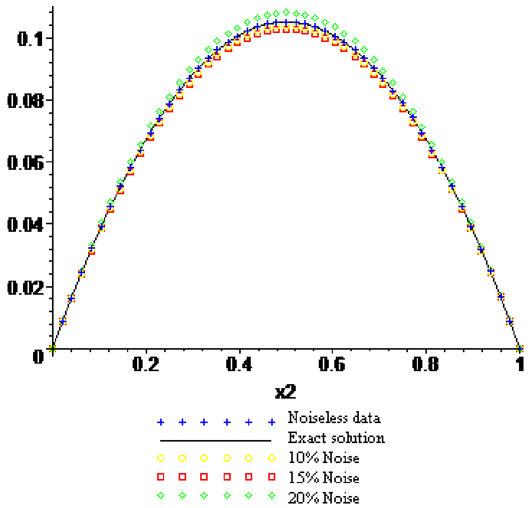

Figure 1: Computed approximate solutions for different noise levels.

problem for different noise levels ($\sigma = 0\%, 10\%, 15\%, 20\%$) *is presented by one dimensional cross sections* ($x_1 = 0.3$, $\varphi = \dfrac{\pi}{2}$).

Example 2. *Let the domain* $\Omega = (-1,1) \times (-1,1) \times (0, 2\pi)$ *and the functions* $F(x_1, x_2, \varphi) = 2\cos\varphi \left(e^{-x_1} x_2 (1 - x_1^2) \sin^2\varphi + x_1 \left(1 - x_2^2\right) \left(x_2^2 - e^{x_2}\left(x_2^2 - 1\right)\right)\right)$
$+ \cos^2\varphi e^{-x_1} \left(1 - x_2^2\right) \left(\left(1 - x_1^2\right) \left(2\cos^2\varphi \left((x_2 + 2)^2 + x_1^2 + 1\right)\right.\right.$
$\left.-3\left(x_2 + 2\right)^2 - x_1^2 - 1\right) + \sin\varphi \left(-x_1^2 + 2x_1 + 1\right)\Big) + \sin\varphi (1 - x_1^2)(2x_2 \left(2x_2^2 - 1\right)$
$-e^{x_2} x_1 \left(x_2^2 + 2x_2 - 1\right)),$
$F_1(x_1, x_2, \varphi) = -(x_1^2 + 1)\cos\varphi$ *and* $F_2(x_1, x_2, \varphi) = (x_2 + 2)^2 \sin\varphi$ *are given. According to these known data, computed approximate solutions* (U_1, U_3) *of the problem at* $N = 1$ *and* $N = 3$ *are shown in Figure 2: (a),(b) respectively.*

Here, the exact solution is

$$u(x_1, x_2, \varphi) = \left(1 - x_1^2\right) \left(1 - x_2^2\right) \left(x_1^2 e^{x_2} - \cos\varphi \sin\varphi e^{-x_1} - x_2^2\right),$$

$$\lambda(x_1, x_2, \varphi) = e^{-x_1} \left(x_1^2 - 1\right) \left(x_2^2 - 1\right) (x_2 + 2)^2.$$

As it can be seen from Figure 2, approximate solution at $N = 3$ *is very closed to the exact solution. Similar accuracy is obtained for* λ.

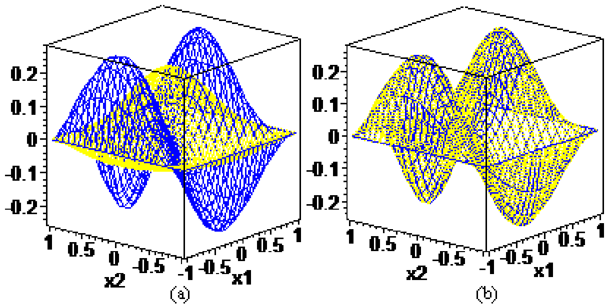

Figure 2: A comparison between the approximate (yellow graph) and exact solution
$u(x,p)$ (blue graph) of the problem at $\psi = \pi$: (a) $N = 1$, (b) $N = 3$.

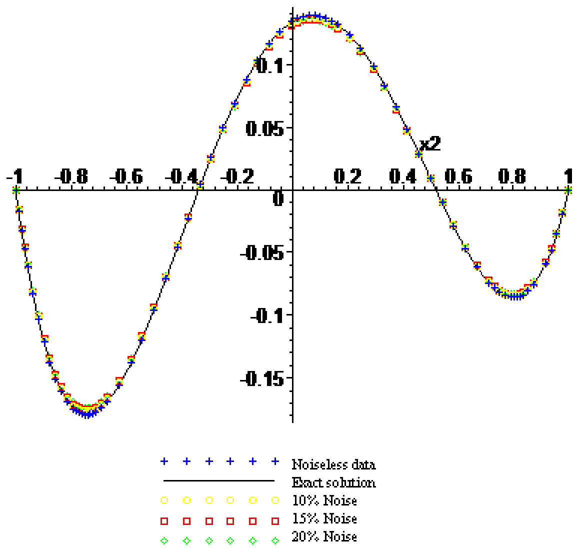

Figure 3: Computed solutions for different noise levels and the exact solution of the inverse problem.

Figure 3 above displays the one dimensional cross sections ($x_1 = 0.4$, $\varphi = \pi$) of computed approximate solutions at $N = 3$ for different noise levels ($\sigma = 0\%$, 10%, 15%, 20%) superimposed with the exact solution $u(x_1, x_2, \varphi)$ of the inverse problem.

Consequently, the computational experiments show that proposed method provides highly accurate and reliable results. Strong random noise in the input data is acceptable without bad deterioration of the solution.

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