An. Şt. Univ. Ovidius Constanţa Vol. 18(2), 2010, 23–40

INDUCED REPRESENTATIONS OF GROUPOID CROSSED PRODUCTS

Massoud Amini

Abstract

We use Green-Riefel machinary to induce representations from a closed subgroupoid crossed product to the groupoid crossed product with a lower semicontinuous bundle of C^*-algebras .

1 Introduction

Marc Rieffel has provided us a powerful machine for inducing representations from a closed subalgebra to a C^*-algebra [Ri] which generalizes the well-known Mackey machine [M]. Jean Renault has generalized the Mackey machine to closed subgroupoids of a locally compact groupoid [R, section 2.2]. Phillip Green has sucessfully used the Rieffel machinary to induce representations from the crossed product of a closed subgroup to the group crossed product with a C^*-algebra [G]. We use Green-Riefel machinary to induce representations from a closed subgroupoid crossed product to the groupoid crossed product with a lower semicontinuous bundle of C^*-algebras . Our approach heavily relies on calculations in [R].

2 Groupoid crossed product

Recall that a *groupoid* is a small category whose arrows are invertible. If G is a groupoid, G^0 is the set of objects and G^2 is the set of composable pairs, and s, r are the source and range maps from G onto G^0. In particular

$$G^2 = \{(x, y) \in G \times G : r(y) = s(x)\}.$$

Key Words: groupoid, $C_0(X)$-algebra, crossed product, induced representation
2010 Mathematics Subject Classification: 46L05, 22D30
Received: November, 2009
Accepted: January, 2010

Also we write

$$G^u = \{x \in G : r(x) = u\}, \quad G_u = \{x \in G : s(x) = u\} \quad (u \in G^0).$$

G is a topological groupoid if the product map from G^2 (with induced product topology) to G and the inversion map from G onto G are continuous. If moreover the topology of G is locally compact (each point in G has a relatively compact Hausdorff neighborhood), the unit space G^0 is Hausdorff, and the source and range maps are open, we call it a locally compact groupoid. Note that G is not necessarily Hausdorff.

Groupoids act on bundles of C^*-algebras in the following sense defined by P-Y. Le Gall [L]. We first need to explain the concept of $C_0(X)$-algebras due to G.G. Kasparov. For a locally compact space X, a $C_0(X)$-algebra is a C^*-algebra A with a morphism ρ from $C_0(X)$ into the center $Z(M(A))$ of the multiplier algebra of A such that $\rho(C_0(X))A = A$. It is more convenient to omit ρ and consider A as an $C_0(X)$-bimodule with $f.a = a.f = \rho(f)a$. Given open subset Ω of X, the closed ideal $A_\Omega = C_0(\Omega).A$ is a $C_0(\Omega)$-algebra. Next for each closed subset F of X we consider the quotient $A/A_{X\setminus F}$. We write A_u for $A_{\{x\}}$, $x \in X$. Now we could identify A with the C^*-algebra bundle $\{A_x\}$. In general this is not a continuous bundle, but one can show that it is always upper semi-continuous. If $p : Y \to X$ is a continuous map between locally compact spaces, and A is a $C_0(X)$-algebra, then we can naturally construct a $C_0(Y)$-algebra $p^*(A)$ by considering the $C_0(Y \times X)$-algebra $B = C_0(Y) \otimes A$ and putting $p^*A = B_{G_p}$, where $G_p \subseteq Y \times X$ is the graph of p [L]. A morphism $\phi : A \to B$ of $C_0(X)$-algebras is a homomorphism of C^*-algebras which is $C_0(X)$-linear. Alternatively we can say that we have a C^*-algebra homomorphism $\phi_x : A_x \to B_x$, at each fiber at $x \in X$.

Now we can define the action of a locally compact Hausdorff groupoid G with unit space $X = G^0$ on a $C_0(X)$-algebra A is an isomorphism $\alpha : s^*A \to r^*A$ of $C_0(G)$-algebras (or equivalently a bundle of C^*-algebra isomorphisms $\alpha_x : A_{s(x)} \to A_{r(x)}$) such that $\alpha_{xy} = \alpha_x \circ \alpha_y$, for each $(x, y) \in G^2$. When G is not Hausdorff, we have to modify this definition as follows: We assume that for each open Hausdorff subset U of G, there is an isomorphism $\alpha_U : s|_U^* A \to r|_U^* A$ of $C_0(U)$-algebras such that for any pair $U \subseteq V$ of Hausdorff open subsets of G, $\alpha_U = \alpha_V|_U$. Now for each $x \in G$ and each open Hausdorff neighborhood U of x, the restriction of α_U to $A_{s(x)}$ is independent of U and is denoted by α_x. Now we assume that $\alpha_{xy} = \alpha_x \circ \alpha_y$, for each $(x, y) \in G^2$.

Next we can define the crossed product of A by G as follows. Let $B = \cup_{u\in X} A_u$. Consider the space of compactly supported continuous sections $C_c(G, A$. More precisely, this is the space of all continuous functions $f : G \to B$ with compact support, such that $f(x) \in A_{s(x)}(x \in G)\}$. This could naturally be identified with $C_c(G).s^*A$. When G is not Hausdorff we need to modify by putting $C_c(G,A)$ to be the linear span (in $\prod_{x\in A_{s(x)}}$) of the union of all sets $C_c(U).s|_U^*A$, where U runs over all open Hausdorff subsets of G. Now we define the convolution and involution for $f,g\in C_c(G,A)$ as follows

$$f*g(x)=\int \alpha_{y^{-1}}(f(xy^{-1})g(y)d\lambda_{s(x)}(y) \quad (x\in G)$$

and

$$f^*(x)=\alpha_{x^{-1}}(f(x^{-1})^*) \quad (x\in G).$$

It is easy to see that these are well defined and $C_c(G,A)$ is an $*$-algebra under these operations. We define the norm of $f\in C_c(G,A)$ by

$$\|f\|_1=sup_{u\in X}\{max\{\int \|f(x)\|d\lambda_u(x),\int \|f(x)\|d\lambda^u(x)\}\}.$$

Again when G is not Hausdorff we have to modify this as follows. We consider a covering $\{U_i\}_{i\in I}$ of G consisting of open Hausdorff subsets of G and take the disjoint union Ω of U_i's, namely $\Omega=\{(x,i)\in G\times I : x\in U_i\}$, and note that there is a continuous map $s_\Omega:\Omega\to X$ defined by $(x,i)\mapsto s(x)$. Then for each $g\in C_c(\Omega,s_\Omega^*A)$ we put

$$\|g\|_1=sup_{u\in X}\{max\{\sum_{i\in I}\int \|g(x,i)\|d\lambda_u(x),\sum_{i\in I}\int \|g(x,i)\|d\lambda^u(x)\}\},$$

then for each g as above one can easily see that the function defined on G by $\phi(g)(x)=\sum_i g(x,i)$ is in $C_c(G,A)$ and the map $\phi: C_c(\Omega,s_\Omega^*A)\to C_c(G,A)$ is surjective. Finally for each $f\in C_c(G,A)$ we define

$$\|f\|_1=\inf\{\|g\|_1 : g\in C_c(\Omega,s_\Omega^*A),\phi(g)=f\}.$$

Now $C_c(G,A)$ with this norm and above operations is a normed $*$-algebra, and the *full crossed product* $A\rtimes_\alpha G$ of A by G is the completion of $C_c(G,A)$ with respect to the above norm.

The construction of the reduced crossed product is based on the regular representation of the groupoid dynamical system. For each $u\in X$, consider the Hilbert A_u-module $L^2(G_u,\lambda_u)\otimes A_u$ which is the completion of the space $C_c(G_u,A_u)$ with respect to the A_u-valued inner product $\langle g,h\rangle=g^**h(u)$. Next define

$$L_u(f)(g)=f*g \quad (f\in C_c(G,A), g\in C_c(G_u,A_u)),$$

this extends to a bounded operator on the Hilbert C^*-module $L^2(G_u,\lambda_u)\otimes A_u$, and thereby yields a $*$-representation of $A\rtimes_\alpha G$. Now the *reduced crossed product* $A\rtimes_{\alpha,r}G$ of A by G is the quotient of the full crossed product $A\rtimes_\alpha G$ by the family $\{L_u\}$ of the *regular representations* of the groupoid dynamical system $\{A, \alpha, G\}$. The details of this construction and two alternative formulations could be found in [KS].

Now we discuss the representation theory of the crossed product $A \rtimes_\alpha G$. Our main objective is to show that there is a one-to-one correspondence between the representations of the C^*-algebra $A \rtimes_\alpha G$, and the so called covariant representations of the system $\{A, \alpha, G\}$. This has been proved in a somewhat more general setting in [R2], but we give an alternative proof which is adapted to the language of $C_0(X)$-algebras.

3 Induced representations

In this section we use the Rieffel machine to induce representations of $B = A \rtimes_\alpha H$ up to representations of $K = A \rtimes_\alpha G$.

Let G be a locally compact groupoid with unit space $X = G^0$ and Haar system $\{\lambda^u\}_{u\in X}$ and H be a closed subgroupoid of G containing X and admitting a Haar system $\{\lambda_H^u\}_{u\in X}$. Consider the relation on G defined by $x \sim y$ if and only if $s(x) = s(y)$ and $xy^{-1} \in H$. This is an equivalence relation and the quotient space $Y = H\backslash G$ is Hausdorff and locally compact, the quotient map $q : G \to Y$ is open, and the source map induces a surjective , continuous and open map $s : Y \to X$ [R,2.2.1]. Next consider the relation on G^2 defined by $(x, y) \sim (x', y')$ if and only if $y = y'$ and $xx'^{-1} \in H$, then the quotient space $Z = H\backslash G^2$ is a locally compact groupoid with unit space $Z^0 = Y$ and Haar system $\{\delta_{\dot{x}} \times \lambda^{s(\dot{x})}\}_{\dot{x}\in Y}$ [R, 2.2.3]. Indeed

$$H\backslash G^2 = \{(\dot{x}, y) \in Y \times G : s(x) = r(y)\}$$

and $s(\dot{x}, y) = (\dot{x}, s(x))$ and $r(\dot{x}, y) = ((xy)^{\cdot}, r(y))$ are identified with $\dot{x}, (xy)^{\cdot} \in Y$, respectively.

Now assume that A is a $C_0(X)$-algebra and there is an action α of G on A.

Proposition 3.1. *(i) $H^0 = X$ and H acts on A by restriction of α.*

(ii) Let $s : Y \to X$ be as above, then s^*A is a $C_0(Y)$-algebra and $H\backslash G^2$ acts on s^*A by the diagonal action α^2

$$\alpha^2_{(\dot{x},y)}(a) = \alpha_{y^{-1}}(a) \quad (x, y \in G, a \in A_{r(y)}).$$

Proof (i) is trivial and (ii) follows from Example (d) after Proposition 3.1 in [L] and the fact that $A_{s(s(\dot{x},y))} = A_{s(\dot{x})} = A_{r(y)}$ and $A_{s(r(\dot{x},y))} = A_{s((xy)^{\cdot})} = A_{s(y)}$. □

Let $K = A \rtimes_\alpha G$, $B = A \rtimes_\alpha H$, and $E = s^*A \rtimes_{\alpha^2} H\backslash G^2$ be the corresponding crossed products and $K_0 = C_c(G, A)$, $B_0 = C_c(H, A)$ and $E_0 =$

$C_c(H\backslash G^2, s^*A)$ be the corresponding dense pre-C^*-algebras . Let B_0 and E_0 act on K_0 from both sides via

$$\phi.f(x) = \int f(h^{-1}x)\alpha_{x^{-1}h}(\phi(h))d\lambda_H^{r(x)}(h)$$
$$f.\phi(x) = \int \phi(h^{-1})\alpha_h(f(xh))d\lambda_H^{s(x)}(h)$$
$$\psi.f(x) = \int f(y^{-1})\alpha_{x^{-1}}(\psi(\dot{x}^{-1}, xy))d\lambda^{s(x)}(y)$$
$$f.\psi(x) = \int \alpha_{x^{-1}y}(\psi(\dot{y}, y^{-1}x))\alpha_{x^{-1}y}(f(y))d\lambda^{r(x)}(y)$$

where $\phi \in B_0, \psi \in E_0$, and $f \in K_0$. One can easily check that these functions belong to K_0.

Lemma 3.2. *For each $\phi, \psi \in B_0$, and $f, g \in K_0$ we have*
(i) $(\phi * \psi).f = \phi.(\psi.f)$
(ii) $f.(\phi * \psi) = (f.\phi).\psi$
(iii) $\phi.(f.\psi) = (\phi.f).\psi$
(iv) $f * (\phi.g) = (f.\phi) * g$
(v) $(\phi.f)^* = f^*.\phi^*$
(vi) $\|\phi.f\|_I \leq \|\phi\|_I.\|f\|_I$.
The same relations hold if $\phi\psi \in E_0$. Moreover if $\phi \in B_0$ and $\psi \in E_0$ then
(vii) $\phi.(f.\psi) = (\phi.f).\psi$.

Proof The proofs are straightforward and follows exactly like the proof of [R, 2.2.4]. For instance (ii) for B_0 could be checked as follows

$$\begin{aligned} f.(\phi * \psi)(x) &= \int (\phi * \psi)(y^{-1})\alpha_y(f(xy))d\lambda_H^{s(x)}(y) \\ &= \int\int \alpha_h(\phi(y^{-1}h))\psi(h^{-1})\alpha_y(f(xy))d\lambda_H^{s(x)}(h)d\lambda_H^{s(x)}(y), \end{aligned}$$

on the other hand

$$\begin{aligned} (f.\phi).\psi(x) &= \int \psi(h^{-1})(f.\phi)(xh))d\lambda_H^{s(x)}(h) \\ &= \int\int \psi((h^{-1})\alpha_h(\phi(y^{-1}))\alpha_{hy}(f(xhy))d\lambda_H^{s(xh)}(y)d\lambda_H^{s(x)}(h) \\ &= \int\int \psi((h^{-1})\alpha_h(\phi(y^{-1}h))\alpha_y(f(xy))d\lambda_H^{s(x)}(h)d\lambda_H^{s(x)}(y), \end{aligned}$$

Similarly (i) for E_0 works as follows

$$(\phi * \psi).f(x) = \int f(y^{-1})\alpha_{x^{-1}}((\phi*\psi)(\dot{x}^{-1}, xy)d\lambda^{s(x)}(y)$$
$$= \int\int f(y^{-1})\alpha_{x^{-1}}(\phi(\dot{x}^{-1}, xyz)\alpha_{yz}(\psi((yz)^{\dot{}}, z^{-1}))d\lambda^{s(y)}(z)d\lambda^{s(x)}(y)$$

on the other hand

$$\phi.(\psi.f)(x) = \int(\psi.f)(z^{-1})\alpha_{x^{-1}}(\phi(\dot{x}^{-1}, xz)d\lambda^{s(x)}(z)$$
$$= \int\int f(y^{-1})\alpha_z(\psi(\dot{z}, z^{-1}y))\alpha_{x^{-1}}(\phi(\dot{x}^{-1}, xz)d\lambda^{s(z^{-1})}(y)d\lambda^{s(x)}(z)$$
$$= \int\int f(y^{-1})\alpha_{yz}(\psi(\dot{(yz)}^{\dot{}}, z^{-1}))\alpha_{x^{-1}}(\phi(\dot{x}^{-1}, xyz)d\lambda^{s(y)}(z)d\lambda^{s(x)}(y).$$

Also to check (v) for E_0 note that for $f \in K_0$ and $\phi \in E_0$ we have $f^*(x) = \alpha_{x^{-1}}(f(x^{-1})^*)$ and $\phi^*(\dot{x}, y) = \alpha^2_{(\dot{x},y)^{-1}}(\phi(\dot{x}, y)^{-1})^*) = \alpha_y(\phi((xy)^{\dot{}}, y^{-1})^*)$. Hence

$$(\phi.f)^*(x) = \alpha_{x^{-1}}((\phi.f)(x^{-1})^*)$$
$$= \int \alpha_{x^{-1}}(f(y^{-1})^*)\alpha_{x^{-1}}(\alpha_x(\phi(\dot{x}, x^{-1}y)^*))d\lambda^{s(x^{-1})}(y)$$
$$= \int \alpha_{x^{-1}}(f(y^{-1})^*)\alpha_{x^{-1}x}(\phi(\dot{x}, x^{-1}y)^*)d\lambda^{r(x)}(y),$$

on the other hand

$$(f^*.\phi^*)(x) = \int \alpha_{x^{-1}y}(f^*(y)\alpha_{x^{-1}y}(\phi^*(\dot{y}, y^{-1}x)))d\lambda^{r(x)}(y)$$
$$= \int \alpha_{x^{-1}y}(f^*(y)\alpha_{x^{-1}y}(\phi^*(\dot{y}, y^{-1}x)))d\lambda^{r(x)}(y)$$
$$= \int \alpha_{x^{-1}y}(\alpha_{y^{-1}}(f(y^{-1})^*))\alpha_{x^{-1}y}(\alpha_{y^{-1}x}\phi(\dot{x}, x^{-1}y)^*))d\lambda^{r(x)}(y)$$
$$= \int \alpha_{x^{-1}}(f(y^{-1})^*)\alpha_{x^{-1}x}(\phi(\dot{x}, x^{-1}y)^*)d\lambda^{r(x)}(y).$$

Finally let's check (vii). All the other relations are checked similarly.

$$\phi.(f.\psi)(x) = \int(f.\psi)(h^{-1}x)\alpha_{x^{-1}h}(\phi(h))d\lambda_H^{r(x)}(h)$$
$$= \int\int \alpha_{x^{-1}hy}(\psi(\dot{y}, y^{-1}h^{-1}x))\alpha_{x^{-1}hy}(f(y))\alpha_{x^{-1}h}(\phi(h))d\lambda^{s(h)}(y)d\lambda_H^{r(x)}(h),$$

also

$$(\phi.f).\psi(x) = \int \alpha_{x^{-1}y}(\psi(\dot{y}, y^{-1}x))\alpha_{x^{-1}y}(\phi.f)(y))d\lambda r(x)(y)$$

$$= \int\int \alpha_{x^{-1}y}(\psi(\dot{y}, y^{-1}x))\alpha_{x^{-1}y}(f(h^{-1}y))\alpha_{x^{-1}y}\alpha_{y^{-1}h}(\phi(h))d\lambda_H^{r(y)}(h)d\lambda^{r(x)}(y)$$

$$= \int\int \alpha_{x^{-1}hy}(\psi((hy)^{\dot{}}, y^{-1}h^{-1}x))\alpha_{x^{-1}hy}(f(y))\alpha_{x^{-1}h}(\phi(h))d\lambda^{s(h)}(y)d\lambda_H^{r(x)}(h),$$

but for $r(y) = s(h)$ we have $hyy^{-1} = hs(h) = h \in H$, so $(hy)^{\dot{}} = \dot{y}$ and so both sides are equal. □

Lemma 3.3. *For each bounded representation L of K_0 there is a unique bounded representation L_H of B_0 such that $L(\phi.f) = L_H(\phi)L(f)$ and $L(f.\phi) = L(f)L_H(\phi)$, for each $\phi \in B_0$ and $f \in K_0$.*

Proposition 3.4. *(i) K_0 is a B_0-bimodule and a E_0-bimodule such that their actions on opposite sides commute.*

(ii) B_0 acts as a $*$-algebra of double centralizers on the algebra K_0. This action extends to the C^*-algebra K and gives a $*$-homomorphism of B into the multiplier algebra $M(K)$.

Proof (i) and the first part of (ii) are already proved. Also by above lemma we have a norm-decreasing faithful $*$-homomorphism of B_0 into $M(K)$, which extends to a $*$-homomorphism of B into $M(K)$. □

Next we define an E_0-valued and a B_0-valued inner product on K_0 by

$$\langle f, g\rangle_{B_0}(h) = \int \alpha_{h^{-1}}(f(y^{-1})^*)g(y^{-1}h)d\lambda^{r(h)}(y)$$

$$\langle f, g\rangle_{E_0}(\dot{x}, x^{-1}y) = \int \alpha_{y^{-1}h}(f(x^{-1}h))\alpha_{x^{-1}h}(g(y^{-1}h))d\lambda_H^{r(x)}(h),$$

The fact that these are functions in B_0 and E_0 could easily be checked. Moreover we have

Lemma 3.5. *For each $f, g, k \in K_0$, $\phi \in B_0$, and $\psi \in E_0$*

(i) $\langle f, g.\phi\rangle_{B_0} = \langle f, g\rangle_{B_0} * \phi$ and $\langle \psi.f, g\rangle_{B_0} = \langle f, \psi^*.g\rangle_{B_0}$

(ii) $\langle \psi.f, g\rangle_{E_0} = \psi * \langle f, g\rangle_{E_0}$ and $\langle f, g.\phi\rangle_{E_0} = \langle f.\phi^*, g\rangle_{E_0}$

(iii) $f.\langle g, k\rangle_{B_0} = \langle f, g\rangle_{E_0}.k$.

Proof This is proved like in [R]. For instance let us check (iii).

$$\begin{aligned} f.\langle g,k\rangle_{B_0}(x) &= \int \langle g,k\rangle_{B_0}(h^{-1})\alpha_h(f(xh))d\lambda_H^{s(x)}(h) \\ &= \int\int \alpha_h(g(y^{-1})^*)k(y^{-1}h^{-1})\alpha_h(f(xh))d\lambda^{s(h)}(y)d\lambda_H^{s(x)}(h), \end{aligned}$$

whereas

$$\begin{aligned} \langle f.g\rangle_{E_0}.k(x) &= \int k(y^{-1})\alpha_{x^{-1}}(\langle f,g\rangle_{E_0}(\dot{x}^{-1},xy))d\lambda^{s(x)}(y) \\ &= \int\int k(y^{-1})\alpha_{x^{-1}}(\alpha_{xh}(f(xh))\alpha_{xh}(g(y^{-1}h)^*)d\lambda_H^{r(x^{-1})}(h)d\lambda^{s(x)}(y) \\ &= \int\int k(y^{-1})\alpha_h(f(xh))\alpha_h(g(y^{-1}h)^*)d\lambda_H^{s(x)}(h)d\lambda^{s(x)}(y) \\ &= \int\int k(y^{-1}h^{-1})\alpha_h(f(xh))\alpha_h(g(y^{-1})^*)d\lambda^{s(h)}(y)d\lambda_H^{s(x)}(h). \quad \square \end{aligned}$$

We need the following lemma which is taken from [R,2.2.2].

Lemma 3.6. *There is a Bruhat approximate cross-section for G over Y, that is a continuous function $b : G \to \mathbb{C}$ whose support has compact intersection with the saturation HD of any compact subset D of G and is such that*

$$\int b(h^{-1}x)d\lambda_H^{r(x)}(h) = 1 \quad (x \in G).$$

Also one can truncate b so that $b \in C_c(G)$ but then we only have

$$\int b(h^{-1}x)d\lambda_H^{r(x)}(h) = 1 \quad (x \in D).$$

Consider the inner products $\langle , \rangle_{B_0}$ and $\langle , \rangle_{E_0}$ defined in previous section. Following [R,2.2.5] we have

Lemma 3.7. *The linear span of the range of $\langle , \rangle_{B_0}$ contains a left bounded approximate identity for B_0 with the inductive limit topology. The same statement holds for E_0.*

Proof (i) Let $\{a_j^u\}_{j\in J}$ be an approximate identity of A_u, such that $a_j^u \geq 0$, $\|a_j^u\| \leq 1$, for each $j \in J, u \in X$. We may assume that there is a neighborhood N of $X = G^0$ in G such that

$$\|\alpha_x(a_j^{s(x)}) - a_j^{r(x)}\| < \varepsilon \quad (x \in N).$$

Let C be a compact subset of $Y = H\backslash G$ and $\varepsilon\rangle 0$. Choose a compact set $K \subseteq G$ such that $q(K) = C$, where $q : G \to H\backslash G$ is the quotient map. There is a locally finite cover of G consisting of open relatively compact sets V_i such that $V_i^{-1}V_i \subseteq N$, for each i. Let $\{b_i\}$ be the partition of unity subordinate to it. Let b be a trancated Bruhat approximate cross section so that $b \in C_c(G)$ and

$$\int b(h^{-1}x)d\lambda_H^{r(x)}(h) = 1 \quad (x \in K).$$

Put $h_i = b_i b$. Then for each i, $h_i \in C_c(G)$ and $supp(h_i) \subseteq V_i$. Also there is finitely many V_i's, say $V_1, \ldots, V_n$ such that

$$\sum_{i=1}^{n} \int h_i(h^{-1}x)d\lambda_H^{r(x)}(h) = 1 \quad (x \in K).$$

For each i, there is a function $k_i \in C_c(G^0)$ such that $k_i(u) = (\int h_i(y)d\lambda^u(y))^{-1}$ $(u \in s^{-1}(K))$. Then for each $x \in C, h \in H^{r(x)}$ we have

$$\int k_i(s(h))h_i(h^{-1}xy)d\lambda^{s(x)}(y) = \int k_i(s(h))h_i(y)d\lambda^{s(h)}(y) = 1.$$

Hence

$$\sum_{i=1}^{n} \int \int k_i(s(h))h_i(h^{-1}x)h_i(h^{-1}xy)d\lambda^{s(x)}(y)d\lambda_H^{r(x}(h)$$

$$= \sum_{i=1}^{n} \int h_i(h^{-1}x)d\lambda_H^{r(x)}(h) = 1.$$

Let $j \in J$ and put $f_i(x) = k_i(r(x))^{1/2}h_i(x)(a_j^{r(x)})^{1/2}$, $x \in G, i = 1, \ldots, n$. Then clearly $f_i \in K_0$ and

$$\begin{aligned}
\langle \tilde{f}_i, \tilde{f}_i\rangle_{E_0}(\dot{x}, y) &= \int \alpha_{x^{-1}h}(\tilde{f}_i(x^{-1}h)\alpha_{x^{-1}h}(\tilde{f}_i((xy)^{-1}h)^*)d\lambda_H^{r(x)}(h) \\
&= \int \alpha_{x^{-1}h}(k_i(s(h))h_i(h^{-1}x)h_i(h^{-1}xy)a_j^{s(h)})d\lambda_H^{r(x)}(h) \\
&= \int k_i(s(h))h_i(h^{-1}x)h_i(h^{-1}xy)\alpha_{x^{-1}h}(a_j^{s(h)})d\lambda_H^{r(x)}(h),
\end{aligned}$$

which is equal to 0 unless $y \in N$.

Now for $\gamma = (C, N, j, \varepsilon)$ put $f_\gamma = \sum_{i=1}^{n} \langle \tilde{f}_i, \tilde{f}_i \rangle_{E_0}$, then

$$\int \|f_\gamma(\dot{x}, y)\| d\lambda^{s(x)}(y) \leq \sum_{i=1}^{n} \int \|\langle \tilde{f}_i, \tilde{f}_i \rangle_{E_0}(\dot{x}, y)\| d\lambda^{s(x)}(y)$$
$$\leq \sum_{i=1}^{n} \int\int k_i(s(h))h_i(h^{-1}x)h_i(h^{-1}xy)\|\alpha_{x^{-1}h}(a_j^{s(h)})\| d\lambda_H^{r(x)}(h) d\lambda^{s(x)}(y)$$
$$\leq \sum_{i=1}^{n} \int\int k_i(s(h))h_i(h^{-1}x)h_i(h^{-1}xy) d\lambda_H^{r(x)}(h) d\lambda^{s(x)}(y) = 1,$$

and

$$\| \int f_\gamma(\dot{x}, y) d\lambda^{s(x)}(y) - a_j^{s(x)} \| =$$
$$\| \sum_{i=1}^{n} \int\int k_i(s(h))h_i(h^{-1}x)h_i(h^{-1}xy)\alpha_{x^{-1}h}(a_j^{s(h)}) - a_j^{s(x)}) d\lambda_H^{r(x)}(h) d\lambda^{s(x)}(y) \|$$
$$\leq \sum_{i=1}^{n} \int\int k_i(s(h))h_i(h^{-1}x)h_i(h^{-1}xy)\|\alpha_{x^{-1}h}(a_j^{s(x^{-1}h)})$$
$$- a_j^{r(x^{-1}h)}\| d\lambda_H^{r(x)}(h) d\lambda^{s(x)}(y)$$
$$\leq \varepsilon \sum_{i=1}^{n} \int\int k_i(s(h))h_i(h^{-1}x)h_i(h^{-1}xy) d\lambda_H^{r(x)}(h) d\lambda^{s(x)}(y) = \varepsilon.$$

Now direct γ's by $\gamma \leq \gamma'$ iff $C' \subseteq C$, $K' \supseteq K$, $j' \geq j$, and $\varepsilon' \leq \varepsilon$, then given $f \in E_0$ and $\varepsilon \rangle 0$, put $K = p(supp(f))$, where $p : H\backslash G^2 \to H\backslash G$ is the map $(\dot{x}, y) \mapsto \dot{x}$. By a compactness argument we may choose $j \in J$ and $C \subseteq G$ such that

$$\|a_j^{r(y)} f(\dot{x}_0, y) - f(\dot{x}_0, y)\| < \varepsilon$$

and

$$\|\alpha_{yz}(f((x_0yz)^{\dot{}}, z^{-1}) - f(\dot{x}_0, y)\| < \varepsilon$$

for each $x_0, y \in G, z \in C$ with $r(z) = s(y)$. Take $\gamma_0 = (C, N, j, \varepsilon)$ for $K, C, j,$

and ε as above, then for each $\gamma \geq \gamma_0$ we have

$$\begin{aligned}
\|f_\gamma * f(\dot{x}_0, y) - f(\dot{x}_0, y)\| &= \|\int f_\gamma(\dot{x}_0, yz)\alpha_{yz}(f((x_0yz)\dot{} z^{-1})d\lambda^{s(y)}(z) - f(\dot{x}_0, y)\| \\
&\leq \int \|f_\gamma(\dot{x}_0, yz)\|\|\alpha_{yz}(f((x_0yz)\dot{} z^{-1})d - f(\dot{x}_0, y)\|d\lambda^{s(y)}(z) \\
&+ \|\int f_\gamma(\dot{x}_0, yz)d\lambda^{s(y)}(z).f(\dot{x}_0, y) - f(\dot{x}_0, y)\| \\
&\leq \varepsilon \int \|f_\gamma(\dot{x}_0, yz)\|d\lambda^{s(y)}(z) \\
&+ \|\int f_\gamma(\dot{x}_0, yz)d\lambda^{s(y)}(z) - a_j^{r(y)}\|.\|f(\dot{x}_0, y)\| \\
&+ \|a_j^{r(y)} f(\dot{x}_0, y) - f(\dot{x}_0, y)\| \\
&\leq 2\varepsilon + \varepsilon\|f(\dot{x}_0, y)\|.
\end{aligned}$$

Hence $f_\gamma * f \to f$ in the inductive limit topology.

Next we show that $\{f_\gamma\}$ is a bounded approximate identity for the left action of E_0 on K_0. Given $f \in K_0$ and $\varepsilon\rangle 0$, let $C = q(supp(f))$. Then as above choose j and N so that

$$\|a_j^{s(y)} f(y) - f(y)\| < \varepsilon, \quad \|f(z^{-1}) - f(y)\| < \varepsilon,$$

for each $y \in G$ and $z \in N$ with $r(z) = s(y)$. Taking $\gamma_0 = (C, K, j, \varepsilon)$, for each $\gamma \geq \gamma_0$ we have

$$\begin{aligned}
\|f_\gamma.f(y) - f(y)\| &= \|\int f(z^{-1})\alpha_{y^{-1}}(f_\gamma(\dot{y}^{-1}, yz))d\lambda^{s(y)}(z) - f(y)\| \\
&\leq \int \|f_\gamma(\dot{y}^{-1}, yz)\|\|f((z^{-1}) - f(y)\|d\lambda^{s(y)}(z) \\
&+ \|\int \alpha_{y^{-1}}(f_\gamma(\dot{y}^{-1}, yz)d\lambda^{s(y)}(z).f(y) - f(y)\| \\
&\leq \varepsilon \int \|f_\gamma(\dot{y}^{-1}, yz)\|d\lambda^{s(y)}(z) \\
&+ \|\int \alpha_{y^{-1}}(f_\gamma(\dot{y}^{-1}, yz)d\lambda^{s(y)}(z) - a_j^{s(y)}\|.\|f(y)\| + \|a_j^{s(y)} f(y) - f(y)\| \\
&\leq \varepsilon + \|\int f_\gamma(\dot{y}^{-1}, yz)d\lambda^{s(y)}(z) - \alpha_{y^{-1}}(a_j^{s(y)})\|.\|f(y)\| + \varepsilon \\
&\leq 2\varepsilon + \varepsilon\|f(y)\|.
\end{aligned}$$

(ii) Choose $\{a_j^u\}$ as above. Let $\varepsilon\rangle 0$ and K be a compact subset of X such that

$$\|\alpha_{h^{-1}}((a_j^{r(h)})^{1/2}) - (a_j^{s(h)})^{1/2}\| < \varepsilon \quad (h \in s^{-1}(K) \cap H).$$

Let N be a r-relatively compact neighborhood of $X = G^0$ in G [R]. Then there is an r-relatively compact neighborhood U of G^0 in G and a non negative real valued continuous function g on G such that $UU^{-1} \subseteq N$, $supp(g) \subseteq U$, and $supp(g) \cap HL$ is compact, for each compact subset L of G, and

$$\int g(h^{-1}x)d\lambda_H^{r(x)}(h) = 1 \quad (x \in r^{-1}(K) \cap U).$$

Choose $k \in C_c(G^0)$ such that $k(u) = (\int h(y)d\lambda^u(y))^{-1} \quad (u \in s^{-1}(K))$. Then given $j \in J$, put $f(x) = k(r(x))g(x)(a_j^{r(x)})^{1/2} \quad (x \in G)$. For $\gamma = (K, N, j, \varepsilon)$ then put $g_\gamma = \langle \tilde{f}, \tilde{f} \rangle_{B_0}$, then

$$\begin{aligned} g_\gamma(h) &= \int \alpha_{h^{-1}}(\tilde{f}(y^{-1})^*)\tilde{f}(y^{-1}h)d\lambda^{r(h)}(y) \\ &= \int \alpha_{h^{-1}}(\tilde{f}(y^{-1}h^{-1})^*)\tilde{f}(y^{-1})d\lambda^{r(h)}(y) \\ &= \int k(r(y))g(hy)g(y)\alpha_{h^{-1}}((a_j^{r(h)})^{1/2})(a_j^{s(h)})^{1/2}d\lambda^{r(h)}(y), \end{aligned}$$

which is 0 unless $y \in N$. Also clearly

$$\int\int k(r(y))g(hy)g(y)d\lambda^{s(h)}(y)d(\lambda_H)_u(h) = 1,$$

and so for each $u \in K$ we have

$$\begin{aligned} &\| \int g_\gamma(h)d(\lambda_H)_u(h) - a_j^u \| \\ &= \| \int\int k(r(y))g(hy)g(y)\alpha_{h^{-1}}((a_j^{r(h)})^{1/2})(a_j^{s(h)})^{1/2}d\lambda^{r(h)}(y)d(\lambda_H)_u(h) - a_j^u \| \\ &= \| \int\int k(r(y))g(hy)g(y)(\alpha_{h^{-1}}((a_j^{r(h)})^{1/2})(a_j^{s(h)})^{1/2} \\ &- a_j^{s(h)})d\lambda^{r(h)}(y)d(\lambda_H)_u(h)\| \\ &\leq \int\int k(r(y))g(hy)g(y)\|(a_j^{s(h)})^{1/2}\|\|\alpha_{h^{-1}}(a_j^{r(h)})^{1/2}) \\ &- (a_j^{s(h)})^{1/2}\|d\lambda^{r(h)}(y)d(\lambda_H)_u(h)\| \\ &\leq \varepsilon \int\int k(r(y))g(hy)g(y)d\lambda^{s(h)}(y)d(\lambda_H)_u(h) = \varepsilon. \end{aligned}$$

Hence given $g_0 \in B_0$, let $K = s(supp(g_0))$ and by compactness choose j and N so that

$$\|a_j^{s(h)}g(h) - g(h)\| < \varepsilon, \quad \|\alpha_y(g(hy)) - g(h)\| < \varepsilon,$$

for each $h \in H, y \in N \cap r^{-1} \circ s(K))$ with $r(y) = s(h)$. Put $\gamma_0 = (K, N, j, \varepsilon)$, for K, N, j, and ε as above, then for each $\gamma \geq \gamma_0$ we have

$$\begin{aligned}
\|g_0 * g_\gamma(h) - g_0(h)\| &= \| \int \alpha_y(g_0(hy))g_\gamma(y^{-1})d\lambda_H^{s(h)}(y) - g_0(h)\| \\
&\leq \int \|g_\gamma(y^{-1})\|\|\alpha_y(g_0(hy)) - g_0(h)\|d\lambda_H^{s(h)}(y) \\
&+ \| \int g_\gamma(y^{-1})d\lambda_H^{s(h)}.g_0(h) - g_0(h)\| \\
&\leq \varepsilon \int \|g_\gamma(y^{-1})\|d\lambda_H^{s(h)}(y) \\
&+ \| \int g_\gamma(y^{-1})d\lambda_H^{s(h)} - a_j^{s(h)}\|.\|g_0(h)\| + \|a_j^{s(h)}g_0(h) - g_0(h)\| \\
&\leq 2\varepsilon + \varepsilon\|g_0(h)\|.
\end{aligned}$$

Hence $g_0 * g_\gamma \to g_0$ in the inductive limit topology.

Next we show that $\{g_\gamma\}$ is a bounded approximate identity for the left action of B_0 on K_0. Given $g_0 \in K_0$ and $\varepsilon\rangle 0$, let $K = s(supp(g_0))$. Choose an r-relatively compact neighborhood U of G^0 in G such that $UU^{-1} \subseteq N$ and $r(U) = G^0$. (Here we need the fact that H is standard). Then as above choose j and N so that

$$\|a_j^{s(x)}g_0(x) - g_0(x)\| < \varepsilon, \quad \|\alpha_y(g_0(xy) - g_0(x)\| < \varepsilon,$$

for each $x \in G$ and $y \in N \cap r^{-1} \circ s(K)$ with $r(y) = s(x)$. Taking $\gamma_0 = (K, N, j, \varepsilon)$, for each $\gamma \geq \gamma_0$ we have

$$\begin{aligned}
\|g_0.g_\gamma(x) - g_0(x)\| &= \| \int \alpha_y(g_0(xy))g_\gamma(y^{-1})d\lambda_H^{s(x)}(y) - g_0(x)\| \\
&\leq \int \|g_\gamma(y^{-1})\|\|\alpha_y(g_0(xy)) - g_0(x)\|d\lambda_H^{s(x)}(y) \\
&+ \| \int g_\gamma(y^{-1})d\lambda_H^{s(x)}.g_0(x) - g_0(x)\| \\
&\leq \varepsilon \int \|g_\gamma(y^{-1})\|d\lambda_H^{s(x)}(y) \\
&+ \| \int g_\gamma(y^{-1})d\lambda_H^{s(x)} - a_j^{s(x)}\|.\|g_0(x)\| + \|a_j^{s(x)}g_0(x) - g_0(x)\| \\
&\leq 2\varepsilon + \varepsilon\|g_0(x)\|. \quad \square
\end{aligned}$$

Corollary 3.8. *The linear span of the range of $\langle , \rangle_{B_0}$ is dense in B_0 and B. Same is true for E.* $\square$

Lemma 3.9. *The inner products $\langle , \rangle_{B_0}$ and $\langle , \rangle_{E_0}$ are positive.*

Proof Consider any $f \in K_0$, then by the notation of the proof of the above lemma, $f_\gamma . f = \sum_{i=1}^{n} \langle \tilde{f}_i, \tilde{f}_i \rangle_{E_0} . f$ tends to f in the inductive limit topology. Hence by Lemma 3.5(iii)

$$\begin{aligned}\langle f, f_\gamma . f \rangle_{E_0} &= \langle f, \sum_{i=1}^{n} \langle \tilde{f}_i, \tilde{f}_i \rangle_{E_0} . f \rangle_{B_0} \\ &= \sum_{i=1}^{n} \langle f, \tilde{f}_i . \langle \tilde{f}_i . f \rangle_{B_0} \rangle_{B_0} = \sum_{i=1}^{n} \langle f, \tilde{f}_i \rangle_{B_0} . \langle f, \tilde{f}_i \rangle_{B_0}^{*} \geq 0.\end{aligned}$$

But clearly $\langle f, f_\gamma . f \rangle_{E_0} \to \langle f, f \rangle_{E_0}$ in the inductive limit topology, and so in the C^*-topology, so $\langle f, f \rangle_{E_0} \geq 0$. The proof for B_0 is similar. □

Lemma 3.10. *For each $\phi \in B_0$, $\psi \in E_0$ and $f \in K_0$*
(i) $\langle f.\phi, f.\phi \rangle_{E_0} \leq \|\phi\|^2 \langle f, f \rangle_{E_0}$
(ii) $\langle \psi.f, \psi.f \rangle_{B_0} \leq \|\psi\|^2 \langle f, f \rangle_{B_0}$,
where the norms on the right hand side are the C^*-norms of B and E, respectively.

Definition 3.11. *The closed subgroupoid H is called standard if there is a locally finite cover of G consisting of open sets $\{V_i\}$ such that $q(V_i) = G^0 = X$, where $q : G \to H \backslash G$ is the quotient map.*

Proposition 3.12. *The subgroupoid G^0 of G is standard.*

Lemma 3.13. *IF H is standard then there is a bounded approximate identity in the range of $\langle , \rangle_{B_0}$ for the right action of B_0 on K_0.*

Theorem 3.14. *If H is standard, then K_0 is an E_0-B_0 imprimitivity bimodule in the sense of Rieffel.*

Proof It is clear that the B_0-valued and E_0-valued inner products are positive. The rest of conditions needed in [Ri] are already proved. □

Corollary 3.15. *If H is standard, then $A \rtimes_\alpha H$ and $s^* A \rtimes_{\alpha^2} H \backslash G^2$ are strongly Morita equivalent.*

Corollary 3.16. *If H is standard, each representation of $A \rtimes_\alpha H$ can be induced up to a representation of $s^* A \rtimes_{\alpha^2} H \backslash G^2$.*

Proof This follows from above theorem and Rieffel's tensor product construction [R, 6.15]. □

Now if we note that $A \rtimes_\alpha G$ acts on $s^*A \rtimes_{\alpha^2} H\backslash G^2$ as double centralizers, then we obtain a representation of $A \rtimes_\alpha G$ from the above induced representation . An alternative way of getting such a representation is using generalized conditional expectations in the sense of Rieffel. The following definition is due to Jean Renualt [R, 1.3.27].

Definition 3.17. *We say that G has sufficiently many non-singular Borel G-sets if for every measure μ on G^0 with induced measure ν on G, every Borel set in G of positive ν-measure contains a non-singular Borel G-set of positive $\mu \circ r$-measure.*

Examples are the transformation groups, r-discrete groupoids, and transitive principal groupoids [R, 1.3.28]. Now consider the restriction map $P : K_0 \to B_0$, then following [R, 2.2.9] we have

Lemma 3.18. *For each representation $\{\mu, \mathfrak{H}, L\}$ of H let Δ_H be the modular function of μ relative to the Haar system $\{\lambda_H^u\}_{u\in X}$ and put*

$$\pi(f,\zeta)(x) = \int f(x^{-1}k)L(k)\zeta \circ s(k)\Delta_H^{-\frac{1}{2}}(k)d\lambda_H^{r(x)}(k),$$

let b be a Bruhat cross-section for G over $Y = H\backslash G$ and $\nu = \int \lambda^u d\mu(u)$, then for each $\zeta, \eta \in L^2(\mathfrak{H}, \mu)$ and $f, g \in K_0$ we have

$$\langle L \circ P(g^* * f)\zeta, \eta\rangle = \int b(x)\langle \pi(f,\zeta), \pi(g,\eta)\rangle d\nu(x).$$

Theorem 3.19. *If G is second countable and H, G both have sufficiently many non-singular Borel G-sets, then the restriction map $P : C_c(G, A) \to C_c(H, A)$ is a generalized conditional expectation.*

Corollary 3.20. *If G is second countable and H, G both have sufficiently many non-singular Borel G-sets, then each representation of $A \rtimes_\alpha H$ can be induced up to a representation of $A \rtimes_\alpha G$ and these C^*-algebras are strongly Morita equivalent.*

Corollary 3.21. *If G is second countable and has sufficiently many non-singular Borel G-sets with respect to two Haar systems, then the corresponding crossed products of G and A are strongly Morita equivalent.*

4 Applications

In this final section we give some applications of the induction procedure described in previous section. Following [G] to each C^*-algebra D we associate the space $\mathfrak{I}(D)$ of all closed two sided ideals of D with the topology coming from the subbase consisting of the sets $Q_I = \{J \in \mathfrak{I}(D) : J \cap I^c \neq \emptyset\}$, where $I \in \mathfrak{I}(D)$ and I^c is the complement of I. The restriction of this topology to $Prim(D)$ is the Jacobson hull-kernel topology. Then any E-B-imprimitivity bimodule induces a canonical bijection of ideal spaces $\mathfrak{I}(B)\mathfrak{I}(E)$ which is also a homeomorphism [G].

Coming back to the situation of the previous section, let H be a closed subgroupoid of the locally compact groupoid G acting by α on a C^*-bundle A. For a representation $L = \pi \times \sigma$ of the crossed product $A \rtimes_\alpha G$, let $Res^G_H L$ be the representation of $A \rtimes_\alpha H$ given by the covariant representation $(\pi, \sigma|_H)$. As before we set $B = A \rtimes_\alpha H$, $E = s^*A \rtimes_{\alpha^2} H\backslash G^2$, and $K = A \rtimes_\alpha G$, and let $P : B \to M(K)$ be the canonical homomorphism obtained in the previous section. Consider the corresponding induced maps

$$Res^G_H = P^* : \mathfrak{I}(A \rtimes_\alpha G) \to \mathfrak{I}(A \rtimes_\alpha H),$$

and

$$Ext^G_H = P_* : \mathfrak{I}(A \rtimes_\alpha H) \to \mathfrak{I}(A \rtimes_\alpha G).$$

Lemma 4.1. *For any representation L of $A \rtimes_\alpha G$, $Res^G_H(kerL) = ker(Res^G_H L)$.*

Proof This follows from the fact that L is non degenerate. □

Recall from the previous section that we have a canonical homomorphism $Q : K \to M(E)$.

Proposition 4.2. *If $H\backslash G^2$ is amenable, then $Q : K \to M(E)$ is faithful and $Ind^G_H(0) = (0)$.*

Proof Let $L = \pi \times \sigma$ be a faithful representation of $A \rtimes_\alpha G$ in $\mathfrak{H}$. Let $L^{'}$ be the representation of $E = s^*A \rtimes_{\alpha^2} H\backslash G^2$ in $\mathfrak{H} \otimes L^2(H\backslash G, L^2(H\backslash G^2, \lambda^2))$ given by the covariant representation $(\sigma \otimes \Lambda, \pi \otimes M)$, where λ is the Λ is the left regular representation of $H\backslash G^2$ in $L^2(H\backslash G^2, \lambda^2))$ and M is the multiplication representation of $C_0(H\backslash G, L^2(H\backslash G^2, \lambda^2))$ also in $L^2(H\backslash G^2, \lambda^2))$. We claim that $L^{''} = Res(ExtL^{'})$ is faithful. Let $(\pi^{''}, \sigma^{''})$ be the corresponding covariant representation . Take $D = L(A \rtimes_\alpha G) \otimes \Lambda(C^*(G))$, then $M(D) \subseteq \mathfrak{B}(\mathfrak{H} \otimes L^2(H\backslash G^2, \lambda^2))$, $\sigma^{''} = \sigma \otimes \Lambda$, and $\pi^{''} = \pi \otimes 1$. Therefore $\sigma^{''}(G) \cup \pi^{''}(A) \subseteq L(M(A \rtimes_\alpha G) \otimes \Lambda(M(C^*(G))) \subseteq M(D)$. Hence $L(A \rtimes_\alpha G) \subseteq M(D)$, and so $L : A \rtimes_\alpha G \to M(D)$ is a homomorphism. Let Λ_0 be the direct sum of Λ with the trivial representation on a one dimensional space $\mathfrak{H}_1$. By our hypothesis that $H\backslash G^2$ is amenable, Λ_0 factors through $\Lambda(C^*(G))$, and so can be regarded as a representation of $\Lambda(C^*(G))$. Let 1 be the identity representation of $L(A \rtimes_\alpha G)$ and extend $1 \otimes \Lambda_0$ to $M(D)$, still denoted with the same notation, then put $L_0 = L^{''} \circ 1 \otimes \Lambda_0$. This is a representation of $A \rtimes_\alpha G$ which clearly contains a sub representation on $\mathfrak{H} \otimes \mathfrak{H}_1$ equivalent to L. As L is faithful by assumption, so is $L^{''}$, as claimed and the first statement is proved. The second statement now follows easily. □

Corollary 4.3. *If $H \backslash G^2$ is amenable and $A \rtimes_\alpha H$ is nuclear, then $A \rtimes_\alpha G$ is also nuclear. In particular, for $H = G^0$, the amenability of G^2 and nuclearity of $A \rtimes_\alpha G^0$ imply the nuclearity of $A \rtimes_\alpha G$.*

Proof Let C be an arbitrary C^*-algebra , we show that the maximal and minimal tensor products of $A \rtimes_\alpha G$ by C are equal. Now G acts on the bundle $A \otimes_{max} C$ via the inner tensor product of the action on A with the trivial action on C. A covariant representation L of this system is a triple (π_A, π_C, σ), where (π_A, σ) is a covariant representation of (A, α, G), and π_C is a representation of C whose image commutes with $\Lambda(G)$ and $\pi_A(A)$ (and hence with $\pi_A \times \sigma((A \otimes_{max} C) \rtimes G)$). As $L(A \otimes_{max} C) \rtimes G)$ is generated by $\Lambda(C^*(G)).\pi_A(A)\pi_C(C)$ and so by $\pi_A \times \sigma((A \otimes_{max} C) \rtimes G)\pi_C(C)$, it follows easily that $(A \otimes_{max} C) \rtimes G$ is naturally isomorphic to $(A \rtimes G) \otimes_{max} C$. Similarly $(A \otimes_{max} C) \rtimes H$ is isomorphic to $(A \rtimes H) \otimes_{max} C$. Choose faithful representations L_1 of $A \rtimes H$ and π_1 of C, then our assumption that $A \rtimes H$ is nuclear implies that $L_2 = L_1 \otimes \pi_1$ is a faithful representation of $(A \rtimes H) \otimes_{max} C$, which could be viewed as a faithful representation of $(A \otimes_{max} C) \rtimes H$. Put $L = Ind_H^G L_2$. Let $K_0^{'}$ be the imprimitivity bimodule of the $(A \otimes_{max} C, \alpha \times tr, H) - (A \otimes_{max} C, \alpha \times tr, G)$ induction process. Then $K^{'}$ contains a dense subspace of the form $K_0 \otimes C$, where K_0 is the imprimitivity bimodule of the (A, α, H)-(A, α, G) induction process. Hence L decomposes as $Ind_H^G L_1 \otimes \pi_1$. But by above proposition, L and $Ind_H^G \pi_1$ are both faithful, hence $(A \rtimes G) \otimes_{max} C$ and $(A \rtimes G) \otimes_{min} C$ coincide. □

Remark 4.4. *There is an alternative proof showing the injectivity of the enveloping von Neumann algebra.*

5 Acknowledgment

The author was visiting the University of Saskatchewan during the preparation of the first draft of this work. He would like to thank University of Saskatchewan and for the hospitality and support. This paper is dedicated to the memory of Professor Mahmood Khoshkam who invited the author to U of S and showed him the path to groupoid crossed products.

References

[G] Phillip Green, The local structure of twisted covariance algebras, Acta Math. 140 (1978), 191-250.

[KS] Mahmood Khoshkam, Georges Skandalis, Crossed products of C^*-algebras by groupoids and inverse semigroups, J. Operator Theory 51 (2004), no. 2, 255-279.

[L] Pierre-Yves Le Gall, Th*é*orie de Kasparov *é*quivariante et groupodes, K-Theory 16 (1999), no. 4, 361-390.

[M] George W. Mackey, Ergodic theory and virtual groups, Math. Ann. 166 (1966), 187-207.

[R] Jean Renault, A groupoid approach to C^*-algebras, Lecture Notes in Mathematics 793, Springer Verlag, Berlin, 1980.

[R2] Jean Renault, Repr*é*sentation des produits crois*é*s d'alg*é*bres de groupo*ï*des, J. Operator Theory 18 (1987), no. 1, 67-97.

[Ri] Marc Rieffel, Induced representations of C^*-algebras , Advances in Math. 13 (1974), 176-257.

Tarbiat Modares University,
Department of Mathematics,
Faculty of Mathematical Sciences,
P.O.Box 14115-134, Tehran, Iran
e-mail: mamini@modares.ac.ir