An. Şt. Univ. Ovidius Constanţa Vol. 18(2), 2010, 315–336

SCREEN TRANSVERSAL LIGHTLIKE SUBMANIFOLDS OF INDEFINITE SASAKIAN MANIFOLDS

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Abstract

We introduce screen transversal lightlike submanifolds of indefinite almost contact manifolds and show that such submanifolds contain lightlike real curves. We give examples, investigate the geometry of distributions and obtain necessary and sufficient conditions for the induced connection on these submanifolds to be metric connection. We also check the existence of screen transversal lightlike submanifolds in indefinite Sasakian manifolds.

1. Introduction

A submanifold M of a semi-Riemannian manifold $\bar{M}$ is called lightlike (degenerate) submanifold if the induced metric on M is degenerate. Lightlike submanifolds of a semi-Riemannian manifold have been studied by Duggal-Bejancu and Kupeli in [3] and [11], respectively. Kupeli's approach is intrinsic while Duggal-Bejancu's approach is extrinsic. Lightlike submanifolds have their applications in mathematical physics. Indeed, lightlike submanifolds appear in general relativity as some smooth parts of event horizons of the Kruskal and Kerr black holes [9].

Key Words: Indefinite Sasakian manifold, Lightlike submanifold, Screen transversal lightlike submanifold.
2010 Mathematics Subject Classification: 53C15, 53C40, 53C50.
Received: February, 2010
Accepted: September, 2010

Lightlike submanifolds of indefinite Sasakian manifolds are defined according to the behaviour of the almost contact structure of indefinite Sasakian manifolds and such submanifolds were studied by Duggal-Şahin in [7]. They defined and studied, invariant, screen real, contact CR-lightlike and screen CR-lightlike submanifolds of indefinite Sasakian manifolds. Later on, Duggal and Şahin studied contact generalized CR-lightlike submanifolds of indefinite Sasakian manifolds [8].

It is known that a real curve of a Sasakian manifold is an example of contact anti-invariant submanifold. One can see that invariant, screen real, CR-lightlike, screen CR-lightlike and transversal lightlike submanifolds of indefinite Sasakian manifolds do not include real lightlike curves (also see [3], [5], [6], [12], [14] for lightlike submanifolds of indefinite Kaehlerian manifolds). Lightlike real curves have many applications in mathematical physics. For instance, a null(lightlike) geodesic curve represents the path of a photon in general relativity.

All these submanifolds of indefinite Sasakian manifolds we mentioned above have invariant radical distribution on their tangent bundles i.e $\phi(RadTM) \subset TM$, where ϕ is the almost contact structure of indefinite Sasakian manifold, $RadTM$ is the radical distribution and TM is the tangent bundle. The above property is also valid for lightlike hypersurfaces [10] which are examples of contact CR-lightlike submanifolds of indefinite Sasakian manifolds. Therefore, In [16], we studied transversal lightlike submanifolds of indefinite Sasakian manifolds. More precisely, a lightlike submanifold of an indefinite Sasakian manifold is called transversal if $\phi(RadTM) = ltr(TM)$. In this paper we introduce screen transversal lightlike submanifolds of indefinite Sasakian manifolds as a generalization of lightlike real curves and study the geometry of such submanifolds.

The paper is organized as follows: In section 2, we give basic information needed for this paper. In section 3, we introduce screen transversal lightlike submanifolds, then we define its subclasses(radical screen transversal, screen transversal anti-invariant lightlike submanifolds) and show that isotropic screen transversal lightlike submanifolds contain real lightlike curves. In section 4, we obtain a characterization of screen transversal anti-invariant lightlike submanifolds. We give an example of such submanifolds, investigate the geometry of distributions and obtain necessary and sufficient condition for induced connection to be a metric connection. In section 5, we study radical screen transversal lightlike submanifolds and find the integrability of distributions. We give example, investigate the existence(or non-existence) of transversal lightlike submanifolds in an indefinite Sasakian space form. We obtain integrability conditions of distributions and give geometric conditions for the induced connection to be metric connection. In section 6, we give a characterization for an isotropic screen transversal lightlike submanifold.

2. Preliminaries

A submanifold M^m immersed in a semi-Riemannian manifold $(\bar{M}^{m+k}, \bar{g})$ is called a lightlike submanifold if it admits a degenerate metric g induced from $\bar{g}$ whose radical distribution $Rad(TM)$ is of rank r, where $1 \leq r \leq m$. $Rad(TM) = TM \cap TM^{\perp}$, where

$$TM^{\perp} = \cup_{x \in M}\{u \in T_x\,\bar{M} / \bar{g}(u, v) = 0,\ \forall v \in T_x\,M\}.$$

Let $S(TM)$ be a screen distribution which is a semi-Riemannian complementary distribution of $Rad(TM)$ in TM, i.e., $TM = Rad\,(TM) \perp S(TM)$.

We consider a screen transversal vector bundle $S(TM^{\perp})$, which is a semi-Riemannian complementary vector bundle of $Rad(TM)$ in $TM^{\perp}$. Since, for any local basis $\{\xi_i\}$ of $Rad(TM)$, there exists a local frame $\{N_i\}$ of sections with values in the orthogonal complement of $S(TM^{\perp})$ in $[S(TM)]^{\perp}$ such that $\bar{g}(\xi_i, N_j) = \delta_{ij}$ and $\bar{g}(N_i, N_j) = 0$, it follows that there exists a lightlike transversal vector bundle $ltr(TM)$ locally spanned by $\{N_i\}$ [3, page 144]. Let $tr(TM)$ be complementary (but not orthogonal) vector bundle to TM in $T\bar{M}|_M$. Then

$$\begin{aligned} tr(TM) &= ltr(TM) \perp S(TM^{\perp}), \\ T\bar{M}|_M &= S(TM) \perp [Rad(TM) \oplus ltr(TM)] \perp S(TM^{\perp}). \end{aligned}$$

Although $S(TM)$ is not unique, it is canonically isomorphic to the factor vector bundle $TM/Rad\,TM$ [11]. The following result is important to this paper.

Proposition 2.1. [3, page 157]. *The lightlike second fundamental forms of a lightlike submanifold M do not depend on $S(TM), S(TM^{\perp})$ and $ltr(TM)$.*

We say that a submanifold $(M, g, S(TM), S(TM^{\perp}))$ of $\bar{M}$ is
Case 1: r - lightlike if $r < min\{m, k\}$;
Case 2: Co - isotropic if $r = k < m$; $S(TM^{\perp}) = \{0\}$;
Case 3: Isotropic if $r = m < k$; $S(TM) = \{0\}$;
Case 4: Totally lightlike if $r = m = k$; $S(TM) = \{0\} = S(TM^{\perp})$.

The Gauss and Weingarten equations are:

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y), \quad \forall X, Y \in \Gamma(TM), \tag{2.1}$$

$$\bar{\nabla}_X V = -A_V X + \nabla^t_X V, \quad \forall X \in \Gamma(TM), V \in \Gamma(tr(TM)), \tag{2.2}$$

where $\{\nabla_X Y, A_V X\}$ and $\{h(X,Y), \nabla^t_X V\}$ belong to $\Gamma(TM)$ and $\Gamma(tr(TM))$, respectively. ∇ and ∇^t are linear connections on M and on the vector bundle $tr(TM)$, respectively. Moreover, we have

$$\begin{aligned}
\bar{\nabla}_X Y &= \nabla_X Y + h^l(X,Y) + h^s(X,Y), \quad \forall X, Y \in \Gamma(TM), && (2.3)\\
\bar{\nabla}_X N &= -A_N X + \nabla^l_X(N) + D^s(X,N), \quad N \in \Gamma(ltr(TM)), && (2.4)\\
\bar{\nabla}_X W &= -A_W X + \nabla^s_X(W) + D^l(X,W), \quad W \in \Gamma(S(TM^\perp)). && (2.5)
\end{aligned}$$

Denote the projection of TM on $S(TM)$ by $\bar{P}$. Then, by using (2.1), (2.3)-(2.5) and a metric connection $\bar{\nabla}$, we obtain

$$\begin{aligned}
\bar{g}(h^s(X,Y),W) &+ \bar{g}(Y, D^l(X,W)) = g(A_W X, Y), && (2.6)\\
\bar{g}(D^s(X,N),W) &= \bar{g}(N, A_W X). && (2.7)
\end{aligned}$$

From the decomposition of the tangent bundle of a lightlike submanifold, we have

$$\begin{aligned}
\nabla_X \bar{P}Y &= \nabla^*_X \bar{P}Y + h^*(X, \bar{P}Y), && (2.8)\\
\nabla_X \xi &= -A^*_\xi X + {\nabla^*}^t_X \xi, && (2.9)
\end{aligned}$$

for $X, Y \in \Gamma(TM)$ and $\xi \in \Gamma(RadTM)$. By using above equations, we obtain

$$\begin{aligned}
\bar{g}(h^l(X,\bar{P}Y),\xi) &= g(A^*_\xi X, \bar{P}Y), && (2.10)\\
\bar{g}(h^*(X,\bar{P}Y),N) &= g(A_N X, \bar{P}Y), && (2.11)\\
\bar{g}(h^l(X,\xi),\xi) = 0 \quad &, \quad A^*_\xi \xi = 0. && (2.12)
\end{aligned}$$

In general, the induced connection ∇ on M is not a metric connection. Since $\bar{\nabla}$ is a metric connection, by using (2.3) we get

$$(\nabla_X g)(Y,Z) = \bar{g}(h^l(X,Y),Z) + \bar{g}(h^l(X,Z),Y). \tag{2.13}$$

However, it is important to note that $\nabla^\star$ is a metric connection on $S(TM)$. We recall that the Gauss equation of lightlike submanifolds is given by

$$\begin{aligned}
\bar{R}(X,Y)Z = {} & R(X,Y)Z + A_{h^l(X,Z)}Y - A_{h^l(Y,Z)}X + A_{h^s(X,Z)}Y\\
& - A_{h^s(Y,Z)}X + (\nabla_X h^l)(Y,Z) - (\nabla_Y h^l)(X,Z) \qquad (2.14)\\
& + D^l(X, h^s(Y,Z)) - D^l(Y, h^s(X,Z)) + (\nabla_X h^s)(Y,Z)\\
& - (\nabla_Y h^s)(X,Z) + D^s(X, h^l(Y,Z)) - D^s(Y, h^l(X,Z))
\end{aligned}$$

for $\forall X, Y, Z \in \Gamma(TM)$.

Finally, we recall some basic definitions and results of indefinite Sasakian manifolds. An odd dimensional semi-Riemannian manifold $(\bar{M}, \bar{g})$ is called a *contact metric manifold* [2] if there is a $(1,1)$ tensor field ϕ, a vector field V, called the characteristic vector field and its 1-form η such that

$$\bar{g}(\phi X, \phi Y) = \bar{g}(X,Y) - \epsilon\,\eta(X)\eta(Y),\ \bar{g}(V,V) = \epsilon \tag{2.15}$$
$$\phi^2(X) = -X + \eta(X)V, \qquad \bar{g}(X, V) = \epsilon\eta(X) \tag{2.16}$$
$$d\eta(X, Y) = \bar{g}(X, \phi Y), \qquad \forall X, Y \in \Gamma(TM), \quad \epsilon = \pm 1. \tag{2.17}$$

It follows that

$$\phi V = 0 \tag{2.18}$$
$$\eta \circ \phi = 0, \quad \eta(V) = 1. \tag{2.19}$$

Then $(\phi, V, \eta, \bar{g})$ is called *contact metric structure of* $\bar{M}$. We say that $\bar{M}$ has a normal contact structure if $N_\phi + d\eta \otimes V = 0$, where N_ϕ is the Nijenhuis tensor field of ϕ [15]. A normal contact metric manifold is called a *Sasakian manifold* [15, 14] for which we have

$$\bar{\nabla}_X V = -\phi X, \tag{2.20}$$
$$(\bar{\nabla}_X \phi)Y = \bar{g}(X,Y)V - \epsilon\,\eta(Y)X. \tag{2.21}$$

3. Screen Transversal Lightlike Submanifolds

In this section, we introduce screen transversal, radical screen transversal and screen transversal anti-invariant lightlike submanifolds of indefinite Sasakian manifolds.

Lemma 3.1 *Let M be an r-lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$. Suppose that $\phi RadTM$ is a vector subbundle of $S(TM^\perp)$. Then, $\phi ltr(TM)$ is also vector subbundle of the screen transversal bundle $S(TM^\perp)$. Moreover, $\phi RadTM \cap \phi ltr(TM) = \{0\}$.*

Proof. Let us assume that $ltr(TM)$ is invariant with respect to ϕ, i.e., $\phi(ltr(TM)) = ltr(TM)$. By definition of a lightlike submanifold, there exist vector fields $\xi \in \Gamma(RadTM)$ and $N \in \Gamma(ltr(TM))$ such that $\bar{g}(\xi, N) = 1$. Also, from (2.15) we get

$$\bar{g}(\xi, N) = \bar{g}(\phi\xi, \phi N) = 1.$$

However, if $\phi N \in \Gamma(ltr(TM))$ then by hypothesis, we get $\bar{g}(\phi\xi, \phi N) = 0$. Hence, we obtain a contradiction which implies that ϕNdoes not belong to

$ltr(TM)$. Now, suppose that $\phi N \in \Gamma(S(TM))$. Then, in a similar way, from (2.15) we have

$$1 = \bar{g}(\xi, N) = \bar{g}(\phi\xi, \phi N) = 0$$

since $\phi\xi \in \Gamma(S(TM^{\perp}))$ and $\phi N \in \Gamma(S(TM))$. Thus, ϕN does not belong to $S(TM)$. We can also obtain that ϕN does not belong to $RadTM$. Then, from the decomposition of a lightlike submanifold, we conclude that $\phi N \in \Gamma(S(TM^{\perp}))$. Now, suppose that there exist a vector field $X \in \Gamma(\phi RadTM \cap \phi ltr(TM))$. Then, from (2.15) we get

$$0 \neq \bar{g}(\phi X, N) = -\bar{g}(X, \phi N) = 0$$

which is a contradiction. Thus, we get

$$\phi RadTM \cap \phi ltrTM = \{0\}.$$

Lemma 3.1. enables us to give the following definition.

Definition 3.1. *Let M be an r-lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$. Then, we say that M is a screen transversal lightlike submanifold of $\bar{M}$ if there exists a screen transversal bundle $S(TM^{\perp})$ such that*

$$\phi RadTM \subset S(TM^{\perp}).$$

From the above definition and Lemma 3.1, it follows that $\phi ltr(TM) \subset S(TM^{\perp})$.

Definition 3.2. *Let M be screen transversal lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$.* Then:

1) *We say that M is a radical screen transversal lightlike submanifold if $S(TM)$ is invariant with respect to ϕ.*

2) *We say that M is a screen transversal anti-invariant lightlike submanifold of $\bar{M}$ if $S(TM)$ is screen transversal with respect to ϕ;*

$$\phi(S(TM)) \subset S(TM^{\perp}).$$

From Definition 3.2, if M is screen transversal anti-invariant lightlike submanifold, we have

$$S(TM^{\perp}) = \phi RadTM \oplus \phi ltrTM \perp \phi S(TM) \perp D_o,$$

where D_o is a non-degenerate orthogonal complementary distribution to

$$\phi RadTM \oplus \phi ltrTM \perp \phi S(TM)$$

in $S(TM^{\perp})$. For the distribution D_o, we have the following.

Proposition 3.1. *Let M be a screen transversal anti-invariant lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$. Then, the distribution D_o is invariant with respect to ϕ.*

If M is an isotropic screen transversal lightlike submanifold of an indefinite Sasakian manifold, from Definition 3.2 and Lemma 3.1, we have the following decomposition:

$$TM = RadTM$$

and

$$T\bar{M} = \{TM \oplus ltrTM\} \perp \{\phi RadTM \oplus \phi ltrTM \perp D_o\}.$$

In this paper, we assume that the characteristic vector field V is a spacelike vector field. If V is a timelike vector field then one can obtain similar results. But it is known that V can not be lightlike[2].

From Definition 3.1, we have the following result:

Proposition 3.2. *There exist no co-isotropic or totally lightlike screen transversal lightlike submanifolds of a indefinite Sasakian manifold $\bar{M}$.*

Next result implies that there are many examples of isotropic screen transversal submanifolds.

Proposition 3.3. *A real lightlike curve M of an indefinite Sasakian manifold $\bar{M}$ is an isotropic screen transversal lightlike submanifold.*

Proof. Since M is a real lightlike curve, we have

$$TM = RadTM = span\{\xi\},$$

where ξ is the tangent vector to M. Then, from (2.15) we have $\bar{g}(\phi\xi, \xi) = 0$ which implies that $\phi\xi$ does not belong to $ltr(TM)$. Since $dim(TM) = 1$, $\phi\xi$ and ξ are linearly indepent $\phi\xi$ does not belong to TM. Hence, we conclude that $\phi\xi \in \Gamma(S(TM^{\perp}))$. In a similar way, from (2.15) we get $\bar{g}(\phi N, N) = 0$ which shows that ϕN does not belong to $RadTM$. Also, we have

$$\bar{g}(\phi N, \xi) = -\bar{g}(N, \phi\xi) = 0$$

due to $\phi\xi \in \Gamma(S(TM^{\perp}))$. This implies that ϕN does not belong to $ltr(TM)$. Hence, $\phi N \in \Gamma(S(TM^{\perp}))$. Moreover, from (2.15) we get

$$\bar{g}(\phi N, \phi\xi) = \bar{g}(N, \xi) = 1.$$

Thus, we conclude that $S(TM^{\perp})$ is expressed as follows:

$$S(TM^{\perp}) = \phi RadTM \oplus \phi ltrTM \perp D_o,$$

where D_o is a non-degenerate distribution. Thus, the proof is complete.

4. Screen Transversal Anti-Invariant Lightlike Submanifolds

In this section, we study screen transversal anti-invariant lightlike submanifolds of an indefinite Sasakian manifold. We give an example, investigate the geometry of distributions and obtain necessary and sufficient conditions for the induced connection on this submanifold to be metric connection.

Let M be a screen transversal anti-invariant lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$. Let T_1, T_2, T_3 and T_4 be the projection morphisms on $\phi RadTM$, $\phi S(TM)$, $\phi ltrTM$ and D_o, respectively. Then, for $V \in \Gamma(S(TM^{\perp}))$ we have

$$V = T_1V + T_2V + T_3V + T_4V. \tag{4.1}$$

On the other hand, for $V \in \Gamma(S(TM^{\perp}))$ we write

$$\phi V = BV + CV, \tag{4.2}$$

where BV and CV are tangential and transversal parts of ϕV. Then applying ϕ to (4.1), we get

$$\phi V = \phi T_1V + \phi T_2V + \phi T_3V + \phi T_4V. \tag{4.3}$$

From (4.2) and (4.3), we can write

$$BV = \phi T_1V + \phi T_2V, CV = \phi T_3V + \phi T_4V.$$

Then, if we put $\phi T_1 = B_1$, $\phi T_2 = B_2$, $\phi T_3 = C_1$ and $\phi T_4 = C_2$, we can rewrite (4.3) as follows:

$$\phi V = B_1V + B_2V + C_1V + C_2V, \tag{4.4}$$

where $B_1V \in \Gamma(RadTM)$, $B_2V \in \Gamma(S(TM))$, $C_1V \in \Gamma(ltrTM)$ and $C_2V \in \Gamma(D_o)$.

A plane section p in $T_x\bar{M}$ of a Sasakian manifold $\bar{M}$ is called a ϕ-section if it is spanned by a unit vector X orthogonal to V and ϕX, where X is a non-null vector field on $\bar{M}$. The sectional curvature $K(p)$ with respect to p determined by X is called a ϕ-sectional curvature. If $\bar{M}$ has a ϕ-sectional curvature c which does not depend on the ϕ-section at each point, then c is constant in $\bar{M}$. Then, $\bar{M}$ is called a Sasakian space form, denoted by $\bar{M}(c)$. The curvature tensor $\bar{R}$ of a Sasakian space form $\bar{M}(c)$ is given by [10]

$$\begin{aligned}\bar{R}(X,Y)Z &= \frac{(c+3)}{4}\{\bar{g}(Y,Z)X-\bar{g}(X,Z)Y\}+\frac{(c-1)}{4}\{\epsilon\eta(X)\eta(Z)Y\\ &\quad -\epsilon\eta(Y)\eta(Z)X+\bar{g}(X,Z)\eta(Y)V-\bar{g}(Y,Z)\eta(X)V\\ &\quad +\bar{g}(\phi Y,Z)\phi X+\bar{g}(\phi Z,X)\phi Y-2\bar{g}(\phi X,Y)\phi Z\} \qquad (4.5)\end{aligned}$$

for any X, Y and Z vector fields on $\bar{M}$.

Theorem 4.1. *Let M be a lightlike submanifold of an indefinite Sasakian space form $\bar{M}(c)$, $c \neq 0$ such that $\phi RadTM \subset S(TM^{\perp})$. Then M is a screen transversal anti-invariant lightlike submanifold if and only if for $\forall X,Y \in \Gamma(S(TM))$, $\xi \in \Gamma(RadTM)$ and $N \in \Gamma(ltrTM)$*

$$\bar{g}(\bar{R}(X,Y)\xi,\phi N)=0.$$

Proof. Since $\phi RadTM \subset S(TM^{\perp})$, from Lemma 3.1, we have $\phi ltrTM \subset S(TM^{\perp})$. From (2.15)

$$\begin{aligned} g(\phi X,\xi) &= -g(X,\phi\xi)=0\\ g(\phi X,N) &= -g(X,\phi N)=0 \qquad (4.6)\end{aligned}$$

which imply that $\phi S(TM)\cap RadTM=\{0\}$ and $\phi S(TM)\cap ltrTM=\{0\}$. In a similar way, we have

$$\begin{aligned} g(\phi X,\phi\xi) &= g(X,\xi)=0\\ g(\phi X,\phi N) &= g(X,N)=0 \qquad (4.7)\end{aligned}$$

which imply that $\phi S(TM)\cap\phi RadTM=\{0\}$ and $\phi S(TM)\cap\phi ltrTM=\{0\}$. On the other hand, from (4.5), we have

$$\bar{g}(\bar{R}(X,Y)\xi,\phi N)=\frac{1-c}{2}\bar{g}(\phi X,Y)g(\phi\xi,\phi N). \qquad (4.8)$$

From (2.15), $\bar{g}(\bar{R}(X,Y)\xi, \phi N) = 0$ if and only if $g(\phi X, Y) = 0$, i.e., $\phi S(TM) \perp S(TM)$. Thus, since

$$\phi S(TM) \cap S(TM) = \{0\}$$
$$\phi S(TM) \cap ltrTM = \{0\}$$
$$\phi S(TM) \cap RadTM = \{0\},$$

$\bar{g}(\bar{R}(X,Y)\xi, \phi N) = 0$ if and only if

$$\phi S(TM) \subset S(TM^{\perp}).$$

Thus the proof is complete.

From now on, $(\mathbf{R}_q^{2m+1}, \phi_o, V, \eta, \bar{g})$ will denote the manifold $\mathbf{R}_q^{2m+1}$ with its usual Sasakian structure given by

$$\begin{aligned}
\eta &= \frac{1}{2}(dz - \sum_{i=1}^{m} y^i dx^i), V = 2\partial\, z \\
\bar{g} = \eta \otimes \eta &+ \frac{1}{4}(-\sum_{i=1}^{\frac{q}{2}} dx^i \otimes dx^i + dy^i \otimes dy^i + \sum_{i=q+1}^{m} dx^i \otimes dx^i + dy^i \otimes dy^i) \\
\phi_o(\sum_{i=1}^{m}(X_i \partial\, x^i &+ Y_i\, \partial\, y^i) + Z\, \partial\, z) = \sum_{i=1}^{m}(Y_i \partial\, x^i - X_i\, \partial\, y^i) + \sum_{i=1}^{m} Y_i y^i\, \partial\, z
\end{aligned}$$

where $(x^i; y^i; z)$ are the Cartesian coordinates. Above construction will help in understanding how the contact structure is recovered in next two examples.

Example 1. Let $\bar{M} = (\mathbf{R}_2^9, \bar{g})$ be a semi-Euclidean space, where $\bar{g}$ is of signature $(-,+,+,+,-,+,+,+,+)$ with respect to canonical basis

$$\{\partial x_1, \partial x_2, \partial x_3, \partial x_4, \partial y_1, \partial y_2, \partial y_3, \partial y_4, \partial z\}.$$

Suppose M is a submanifold of $\mathbf{R}_2^9$ defined by

$$x^1 = \cos u_1, x^2 = u_1, x^3 = \sin u_2, x^4 = 0$$

$$y^1 = -\sin u_1, y^2 = 0, y^3 = -\cos u_2, y^4 = u_2.$$

It is easy to see that a local frame of TM is given by

$$\begin{aligned}
Z_1 &= 2(-\sin u_1 \partial\, x_1 + \partial\, x_2 - \cos u_1 \partial\, y_1 + (-\sin u_1 y^1 + y^2)\partial, z) \\
Z_2 &= 2(\cos u_2 \partial\, x_3 + \sin u_2 \partial\, y_3 + \partial\, y_4 + \cos u_2 y^3 \partial\, z) \\
Z_3 &= V = 2\partial\, z.
\end{aligned}$$

Thus, M is a 1-lightlike submanifold with $RadTM = span\{Z_1\}$ and the lightlike transversal bundle $ltr(TM)$ is spanned by

$$N = 2(\sin u_1 \partial x_1 - \cos u_1 \partial y_1 + \sin u_1 y^1 \partial, z).$$

Also, the screen transversal bundle $S(TM^{\perp})$ is spanned by

$$\begin{aligned}
W_1 &= 2(-\cos u_1 \partial x_1 + \sin u_1 \partial y_1 - \partial y_2 - \cos u_1 y^1 \partial, z) \\
W_2 &= 2(\cos u_1 \partial x_1 + \sin u_1 \partial y_1 + \cos u_1 y^1 \partial z) \\
W_3 &= 2(\sin u_2 \partial x_3 + \partial x_4 - \cos u_2 \partial y_3 + (\sin u_2 y^3 + y^4)\partial, z) \\
W_4 &= 2(\cos u_2 \partial x_3 + \sin u_2 \partial y_3 - \partial, y_4 + \cos u_2 y^3 \partial z) \\
W_5 &= 2(\sin u_2 \partial x_3 - \partial, x_4 - \cos u_2 \partial y_3 + (\sin u_2 y^3 - y^4)\partial z).
\end{aligned}$$

Then it is easy to see that $\phi Z_1 = W_1$, $\phi N = W_2$, $\phi Z_2 = W_3$ and $\phi W_4 = W_5$. Thus, M is a screen transversal anti-invariant 1-lightlike submanifold.

Theorem 4.2. *Let M be a screen transversal anti-invariant lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$. Then, the induced connection ∇ on M is a metric connection if and only if $B_2 \nabla^s_X \phi\xi = \eta(\nabla_X \xi)V$ for $X \in \Gamma(TM)$ and $\xi \in \Gamma(RadTM)$.*

Proof. From (2.21), we have

$$\bar{\nabla}_X \phi\xi - \phi \bar{\nabla}_X \xi = 0.$$

Using (2.16) and applying ϕ in this equation, we get

$$\phi \bar{\nabla}_X \phi\xi = -\bar{\nabla}_X \xi + \eta(\bar{\nabla}_X \xi)V.$$

From (2.3), (2.5) and (4.4), we have

$$\begin{aligned}
-\nabla_X \xi - h^l(X,\xi) - h^s(X,\xi) + \eta(\nabla_X \xi)V &= -\phi A_{\phi\xi} X + B_1 \nabla^s_X \phi\xi \\
&\quad + B_2 \nabla^s_X \phi\xi + C_1 \nabla^s_X \phi\xi \\
&\quad + C_2 \nabla^s_X \phi\xi + \phi D^l(X, \phi\xi).
\end{aligned}$$

Then, taking the tangential parts of the above equation, we obtain

$$\nabla_X \xi = -B_1 \nabla^s_X \phi\xi - B_2 \nabla^s_X \phi\xi + \eta(\nabla_X \xi)V \tag{4.9}$$

Thus the proof is complete.

Theorem 4.3. *Let M be a screen transversal anti-invariant lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$. Then the radical distribution is integrable if and only if*

$$g(\nabla^s_X \phi Y - \nabla^s_Y \phi X, \phi Z) = 0$$

for $\forall X, Y \in \Gamma(RadTM)$ *and* $Z \in \Gamma(S(TM))$.

Proof. By the definition of a screen transversal anti-invariant lightlike submanifold, $(RadTM)$ is integrable if and only if $g([X, Y], Z) = 0$ for $X, Y \in \Gamma(RadTM)$, $Z \in \Gamma(S(TM))$. From (2.3) and (2.15), we get

$$g([X, Y], Z) = g(\phi\bar{\nabla}_X Y, \phi Z) + \eta(\bar{\nabla}_X Y)\eta(Z) - g(\phi\bar{\nabla}_Y X, \phi Z) - \eta(\bar{\nabla}_Y X)\eta(Z).$$

Since $\bar{\nabla}$ is metric connection, we have $\eta(\bar{\nabla}_X Y) = 0$. This, using in the above equation, we get

$$g([X, Y], Z) = g(\phi\bar{\nabla}_X Y, \phi Z) - g(\phi\bar{\nabla}_Y X, \phi Z).$$

From (2.5) and (2.21), we obtain

$$g([X, Y], Z) = g(\nabla^s_X \phi Y - \nabla^s_Y \phi X, \phi Z)$$

which completes the proof.

In a similar way, we have the following.

Theorem 4.4. *Let M be a screen transversal anti-invariant lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$. Then screen distribution is integrable if and only if*

$$g(\nabla^s_X \phi Y - \nabla^s_Y \phi X, \phi N) = 0$$

for $\forall X, Y \in \Gamma(S(TM))$ *and* $N \in \Gamma(ltr(TM))$.

5. Radical Screen Transversal Lightlike Submanifolds

In this section, we study radical screen transversal lightlike submanifolds. We first investigate the integrability of distributions, give a necessary and sufficient condition for the induced connection to be a metric connection. We also study the geometry of totally contact umbilical radical screen transversal lightlike submanifolds. We first give an example of such submanifolds.

Example 2. Let $\bar{M} = (\mathbf{R}_2^9, \bar{g})$ be a semi-Euclidean space, where $\bar{g}$ is of signature $(-,+,+,+,-,+,+,+,+)$ with respect to canonical basis

$$\{\partial x_1, \partial x_2, \partial x_3, \partial x_4, \partial y_1, \partial y_2, \partial y_3, \partial y_4, \partial z\}.$$

Suppose M is a submanifold of $\mathbf{R}_2^9$ defined by

$$x^1 = 0, x^2 = u_1, x^3 = u_2, x^4 = u_3$$

$$y^1 = u_1, y^2 = 0, y^3 = -u_3, y^4 = u_2.$$

It is easy to see that a local frame of TM is given by

$$\begin{aligned} Z_1 &= 2(\partial x_2 + \partial y_1 + y^2 \partial z) \\ Z_2 &= 2(\partial x_3 + \partial y_4 + y^3 \partial z) \\ Z_3 &= 2(\partial x_4 - \partial y_3 + y^4 \partial z) \\ Z_4 &= V = 2\partial z. \end{aligned}$$

Thus, M is a 1-lightlike submanifold with $RadTM = span\{Z_1\}$. Also, $\phi Z_2 = Z_3$ implies that $\phi S(TM) = S(TM)$. Lightlike transversal bundle $ltr(TM)$ is spanned by

$$N = 2(-\partial x_2 - 2\partial y_1 - y^2 \partial z).$$

Also, screen transversal bundle $S(TM^\perp)$ is spanned by

$$\begin{aligned} W_1 &= 2(\partial x_1 - \partial y_2 + y^1 \partial z) \\ W_2 &= 2(-2\partial x_1 + \partial y_2 - 2y^1 \partial z) \\ W_3 &= 2(\partial x_2 - \partial x_4 + \partial y_1 - \partial y_3 + (y^2 - y^4)\partial z) \\ W_4 &= 2(\partial x_1 - \partial x_3 - \partial y_2 + \partial y_4 + (y^1 - y^3)\partial z). \end{aligned}$$

It follows that $\phi Z_1 = W_1, \phi N = W_2$ ve $\phi W_3 = W_4$. Hence M is radical screen transversal 1-lightlike submanifold.

Theorem 5.1. *Let M be a radical screen transversal lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$. Then the radical distribution is integrable if and only if*

$$g(A_{\phi\xi_1}\xi_2 - A_{\phi\xi_2}\xi_1, \phi X) = 0$$

for $X \in \Gamma(S(TM) - \{V\})$ and $\xi_1, \xi_2 \in \Gamma(RadTM)$.

Proof. By the definition of a radical screen transversal lightlike submanifold, $(RadTM)$ is integrable if and only if $g([\xi_1, \xi_2], X) = 0$ for $X \in \Gamma(S(TM) - \{V\})$ and $\xi_1, \xi_2 \in \Gamma(RadTM)$. From (2.15) and (2.21), we have

$$\bar{g}([\xi_1, \xi_2], X) = g(\bar{\nabla}_{\xi_1} \phi \xi_2, \phi X) - g(\bar{\nabla}_{\xi_2} \phi \xi_1, \phi X).$$

Using (2.5), we obtain

$$\bar{g}([\xi_1, \xi_2], X) = g(A_{\phi \xi_1} \xi_2 - A_{\phi \xi_2} \xi_1, \phi X).$$

Thus, the proof is complete.

In a similar way, we have the following result.

Theorem 5.2. *Let M be a radical screen transversal lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$. Then the screen distribution is integrable if and only if*

$$g(h^s(X, \phi Y) - h^s(Y, \phi X), \phi N) = 0$$

for $\forall X, Y \in \Gamma(S(TM))$ and $N \in \Gamma(ltr(TM))$.

We also have the following result whose proof is similar to those given in section 3.

Proposition 5.1. *Let M be a radical screen transversal lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$. Then the distribution D_o is invariant with respect to ϕ.*

We now investigate the geometry of leaves of distributions $RadTM$ and $S(TM)$.

Theorem 5.3. *Let M be a radical screen transversal lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$. Then, the screen distribution defines a totally geodesic foliation if and only if $h^s(X, \phi Y)$ has no compenents in $\phi RadTM$ for $X, Y \in \Gamma(S(TM))$.*

Proof. By the definition of radical screen transversal lightlike submanifold $(S(TM))$ is a totally geodesic foliation if and only if $g(\nabla_X Y, N) = 0$ for $X, Y \in \Gamma(S(TM))$, $N \in \Gamma(ltr(TM))$. From (2.3) and (2.15), we have

$$\bar{g}(\nabla_X Y, N) = \bar{g}(\phi \bar{\nabla}_X Y, \phi N).$$

Also, from (2.21) we get

$$\bar{g}(\nabla_X Y, N) = \bar{g}(\bar{\nabla}_X \phi Y, \phi N).$$

Using (2.3), we obtain

$$\bar{g}(\nabla_X Y, N) = \bar{g}(h^s(X, \phi Y), \phi N)$$

which completes the proof.

We also have the following result whose proof is similar to the above theorem.

Theorem 5.4. *Let M be a radical screen transversal lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$. Then, radical distribution defines a totally geodesic foliation on M if and only if $h^s(\xi, \phi X)$ has no compenents in $\phi ltrTM$ for $\xi_1, \xi_2 \in \Gamma(RadTM)$ and $X \in \Gamma(S(TM) - \{V\})$.*

Thus from Theorem 5.3. and Theorem 5.4. we have the following result.

Corollary 5.1. *Let M be an irrotational radical screen transversal lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$. Then, M is a lightlike product manifold if and only if $h^s(X, \phi Y)$ has no components in $\phi ltrTM$ for $\forall X, Y \in \Gamma(S(TM))$.*

In the sequel, we obtain a characterization of the induced connection to be a metric connection.

Theorem 5.4. *Let M be radical screen transversal lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$. Then, the induced connection ∇ on M is a metric connection if and only if $h^s(X, Y)$ has no components in $\phi ltrTM$ for $\forall X, Y \in \Gamma(S(TM))$.*

Proof. For $\xi \in \Gamma(RadTM)$ and $X \in \Gamma(S(TM))$, from (2.21), we have

$$\bar{\nabla}_X \phi\xi - \phi\bar{\nabla}_X \xi = 0.$$

Hence, using (2.3) and (2.5) we get

$$g(A_{\phi\xi} X, Y) = g(\nabla_X \xi, \phi Y).$$

From (2.6), we obtain

$$g(h^s(X, Y), \phi\xi) = g(\nabla_X \xi, \phi Y)$$

which completes the proof.

In the rest of this section, we check the existence (non-existence) of radical screen transversal lightlike submanifolds of indefinite Sasakian manifold $\bar{M}$. First of all, we recall that any totally umbilical lightlike submanifold, tangent to the structure vector field, of an indefinite Sasakian manifold is totally geodesic and invariant [7]. Therefore, the notion of totally umbilical submanifolds [4] of a semi-Riemannian manifolds does not work for lightlike submanifolds of indefinite Sasakian manifold. In [7], the authors introduced the notion of totally contact umbilical submanifolds as follows. According to their definition, a lightlike submanifold of an indefinite Sasakian manifold is totally umbilical if, for $X, Y \in \Gamma(M)$

$$\begin{aligned} h^l(X,Y) &= [g(X,Y) - \eta(X)\eta(Y)]\alpha_L \\ &\quad + \eta(X)h^l(Y,V) + \eta(Y)h^l(X,V) \qquad (5.1) \\ h^s(X,Y) &= [g(X,Y) - \eta(X)\eta(Y)]\alpha_S \\ &\quad + \eta(X)h^s(Y,V) + \eta(Y)h^s(X,V), \qquad (5.2) \end{aligned}$$

where $\alpha_S \in \Gamma(S(TM^{\perp}))$ and $\alpha_L \in \Gamma(ltr(TM))$.

We now give several preparatory lemmas for the proof of Theorem 5.6.

Lemma 5.1. *Let M be a totally contact umbilical radical screen transversal lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$. Then, we have*

$$h^l(X,\xi) = 0, h^s(\xi,V) = -\phi\xi \qquad (5.3)$$

for $X \in \Gamma(S(TM) - \{V\})$ and $\xi \in \Gamma(RadTM)$.

Proof. From (5.1), we get

$$h^l(X,\xi) = \eta(X)h^l(\xi,V). \qquad (5.4)$$

On the other hand, from (2.3) and (2.20), we have

$$-\phi\xi = \nabla_\xi V + h^l(\xi,V) + h^s(\xi,V).$$

Taking transversal parts of the above equation, we obtain

$$-\phi\xi = h^s(\xi,V), h^l(\xi,V) = 0$$

which completes the proof.

In a similar way, we have the following lemmas.

Lemma 5.2. *Let M be a totally contact umbilical radical screen transversal lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$. Then, we have*

$$h^s(X,\xi) = 0 \tag{5.5}$$

for $X \in \Gamma(S(TM) - \{V\})$ and $\xi \in \Gamma(RadTM)$.

Lemma 5.3. *Let M be a radical screen transversal lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$. Then, we have*

$$g(\nabla_X \phi X, V) = g(X,X) \tag{5.6}$$

for $X \in \Gamma(S(TM) - \{V\})$.

By using (2.3), (2.20) and (2.16), we also have the following lemma.

Lemma 5.4. *Let M be a radical screen transversal lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$. Then, we have*

$$g(\nabla_{\phi X} X, V) = -g(X,X) \tag{5.7}$$

for $X \in \Gamma(S(TM) - \{V\})$.

From(2.3) and (2.20), we obtain the following results.

Lemma 5.5. *Let M be a totally contact umbilical radical screen transversal lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$. Then, we have*

$$h^s(X,V) = 0 \tag{5.8}$$

for $X \in \Gamma(S(TM) - \{V\})$.

Lemma 5.6. *Let M be a totally contact umbilical radical transversal lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$. Then, we have*

$$g(X, \nabla_{\phi X}\xi) = -g(h^l(\phi X, X), \xi). \tag{5.9}$$

Lemma 5.7. *Let M be a totally contact umbilical radical transversal lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$. Then, we have*

$$g(\phi X, \nabla_X \xi) = -g(h^l(X, \phi X), \xi). \tag{5.10}$$

Theorem 5.5. *Let M be a totally contact umbilical radical screen transversal lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$. Then, the induced* *connection ∇ on M is a metric connection if and only if $A_{\phi\xi}X = -\eta(X)\xi$ for $X \in \Gamma(TM)$ and $\xi \in \Gamma(RadTM)$.*

Proof. From (2.21), we have

$$\bar{\nabla}_X \phi\xi - \phi\bar{\nabla}_X \xi = 0.$$

Hence, using (2.3) and (2.5) we get

$$-A_{\phi\xi}X + \nabla^s_X \phi\xi + D^l(X, \phi\xi) = \phi\nabla_X \xi + \phi h^l(X, \xi) + \phi h^s(X, \xi).$$

Also, from (2.16), (5.3) and (5.5) we have

$$-A_{\phi\xi}X + \nabla^s_X \phi\xi + D^l(X, \phi\xi) = \phi\nabla_X \xi + \eta(X)\xi.$$

Taking the tangential parts of the above equation, we get

$$\phi\nabla_X \xi = -A_{\phi\xi}X - \eta(X)\xi.$$

Thus, if $\nabla_X \xi \in \Gamma(RadTM)$ then we have $A_{\phi\xi}X = -\eta(X)\xi$.
Let us prove the converse. Suppose that $A_{\phi\xi}X = -\eta(X)\xi$. Then, for $Y \in \Gamma(S(TM))$ we get $g(A_{\phi\xi}X, Y) = 0$. From (2.6), we have $g(h^s(X, Y), \phi\xi) = 0$. Also, using (2.3), we obtain $g(\phi\bar{\nabla}_X Y, \xi) = 0$. On the other hand, from (2.21) we have $g(h^l(X, \phi Y), \xi) = 0$. Thus, from (5.3) and $h^l(X, \phi Y) = 0$, we obtain $h^l(X, \xi) = 0$. Since h^l is vanishes on the $RadTM$, the proof is complete.

Theorem 5.6. *There exist no totally contact umbilical radical screen transversal lightlike submanifolds in an indefinite Sasakian space form $\bar{M}(c)$ with $c \neq -3$.*

Proof. Suppose M is a totally contact umbilical radical screen transversal lightlike submanifold of $\bar{M}(c)$ such that $c \neq -3$. From (4.5), we get

$$\bar{R}(X, \phi X)\xi = \frac{1-c}{2} g(X, X)\phi\xi$$

for $\forall X \in \Gamma(S(TM) - \{V\})$, $\xi \in \Gamma(RadTM)$ and $N \in \Gamma(ltrTM)$. Hence, we get

$$g(\bar{R}(X, \phi X)\xi, \phi N) = \frac{1-c}{2} g(X, X). \tag{5.11}$$

Also, from (2.14) we have

$$g(\bar{R}(X, \phi X)\xi, \phi N) = g((\nabla_X h^s)(\phi X, \xi), \phi N) - g((\nabla_{\phi X} h^s)(X, \xi), \phi N). \tag{5.12}$$

From (5.11) and (5.12), we obtain

$$\frac{1-c}{2}g(X,X) = g((\nabla_X h^s)(\phi X,\xi),\phi N) - g((\nabla_{\phi X} h^s)(X,\xi),\phi N), \quad (5.13)$$

where

$$(\nabla_X h^s)(\phi X,\xi) = \nabla^s_X h^s(\phi X,\xi) - h^s(\nabla_X \phi X,\xi) - h^s(\phi X,\nabla_X \xi) \quad (5.14)$$

and

$$(\nabla_{\phi X} h^s)(X,\xi) = \nabla^s_{\phi X} h^s(X,\xi) - h^s(\nabla_{\phi X} X,\xi) - h^s(X,\nabla_{\phi X}\xi). \quad (5.15)$$

Since M is totally contact umbilical, (5.2) implies that

$$h^s(\phi X,\xi) = 0. \quad (5.16)$$

From (5.2), (5.3) and (5.6) we get

$$h^s(\nabla_X \phi X,\xi) = -g(X,X)\phi\xi. \quad (5.17)$$

Also, from (5.2) and (5.8)

$$h^s(\phi X,\nabla_X \xi) = g(\phi X,\nabla_X \xi)\alpha_S. \quad (5.18)$$

Using (5.16), (5.17) and (5.18) in (5.14), we have

$$(\nabla_X h^s)(\phi X,\xi) = g(X,X)\phi\xi - g(\phi X,\nabla_X \xi)\alpha_S. \quad (5.19)$$

On the other hand, from (5.5), we have

$$h^s(X,\xi) = 0. \quad (5.20)$$

By using (5.2), (5.3) and (5.7), we obtain

$$h^s(\nabla_{\phi X} X,\xi) = g(X,X)\phi\xi. \quad (5.21)$$

From (5.2) and (5.8), we get

$$h^s(X,\nabla_{\phi X}\xi) = g(X,\nabla_{\phi X}\xi)\alpha_S. \quad (5.22)$$

Using (5.20), (5.21), (5.22) in (5.15), we have

$$(\nabla_{\phi X} h^s)(X,\xi) = -g(X,X)\phi\xi - g(X,\nabla_{\phi X}\xi)\alpha_S. \quad (5.23)$$

Hence, using (5.19) and (5.23) in (5.13), we obtain

$$\frac{1-c}{2}g(X,X) = 2g(X,X) + \{g(X,\nabla_{\phi X}\xi) - g(\phi X,\nabla_X \xi)\}g(\alpha_S,\phi N). \quad (5.24)$$

Thus, using (5.9) and (5.10) in (5.24)

$$\frac{1-c}{2}g(X,X) = 2g(X,X). \tag{5.25}$$

Thus, we obtain

$$(c+3)g(X,X) = 0$$

which completes the proof.

6. Isotropic Screen Transversal Lightlike Submanifolds

In this section, we give necessary and sufficient conditions for an isotropic screen transversal lightlike subminifold to be totally geodesic.

Theorem 6.1. *Let M be an isotropic screen transversal lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$. Then, M is totallygeodesic if and only if $D^l(\xi_1, Z) = 0$, $D^l(\xi_1, \phi\xi_2) = 0$ and $D^s(\xi_1, N)$has no components in $\phi ltrTM$ for $\xi_1, \xi_2 \in \Gamma(RadTM)$, $N \in \Gamma(ltrTM)$ and $Z \in \Gamma(D_o)$.*

Proof. From (2.21), we get

$$\bar{\nabla}_{\xi_1}\phi\xi_2 = \phi\bar{\nabla}_{\xi_1}\xi_2. \tag{6.1}$$

By using (6.1) for $\xi \in \Gamma(RadTM)$, we get

$$g(\bar{\nabla}_{\xi_1}\phi\xi_2, \xi) = -g(\bar{\nabla}_{\xi_1}\xi_2, \phi\xi).$$

Thus, using (2.3) and (2.5), we obtain

$$g(D^l(\xi_1, \phi\xi_2), \xi) = -g(h^s(\xi_1, \xi_2), \phi\xi). \tag{6.2}$$

In a similar way, from (6.1), (2.3) and (2.5), we obtain

$$g(D^s(\xi_1, N), \phi\xi_2) = g(h^s(\xi_1, \xi_2), \phi N). \tag{6.3}$$

Also, since $\bar{\nabla}$ is metric connection, we have

$$g(\bar{\nabla}_{\xi_1}\xi_2, Z) = -g(\xi_2, \bar{\nabla}_{\xi_1}Z) = g(h^s(\xi_1, \xi_2), Z).$$

Then, from (2.5), we get

$$g(h^s(\xi_1, \xi_2), Z) = -g(\xi_2, D^l(\xi_1, Z)). \tag{6.4}$$

Thus, proof comes from (6.2), (6.3) and (6.4).

References

[1] C. Călin, *Contributions to geometry of CR-submanifold*, Thesis, University of Iasi, Romania, 1998.

[2] K. L. Duggal, *Spacetime manifolds and contact structures*, Internat. J. Math. & Math. Sci. **13**(1990) 545-554.

[3] K. L. Duggal and A. Bejancu, *Lightlike submanifolds of Semi-Riemannian Manifolds and Applications*, Kluwer Academic Publishers, 1996.

[4] K. L. Duggal and D. H. Jin, *Totally umbilical lightlike submanifolds*, Kodai Math J. **26**(2003) 49-68.

[5] K. L. Duggal and B. Șahin, *Screen Cauchy Riemann lightlike submanifolds*, Acta Math. H. **106**(2005) 137-165.

[6] K. L. Duggal and B. Șahin, *Generalized screen Cauchy Riemann lightlike submanifolds*, Acta Math. Hungar. **112**(2006) 113-136.

[7] K. L. Duggal and B. Șahin, *Lightlike submanifolds of indefinite Sasakian manifolds*,Int.Jour.Math and Math.Sci,DOI=10.1155/2007/57585, 2007.

[8] K. L. Duggal and B. Șahin, *Generalized Cauchy-Rieman lightlike submanifolds of indefinite Sasakian manifolds*, Acta Math. Hungar. **122**(2009) 45-58.

[9] S. W. Hawking and G. F. R.Ellis, *The Large Scale Structure of Spacetime*, Cambridge Univ. Press, Cambridge, 1972.

[10] T. H. Kang, S. D. Jung, B. H. Kim, H. K. Pak And J. S. Pak, *Lightlike hypersurfaces of indefinite Sasakian manifolds*, Indian J. pure appl. Math. **34**(2003) 1369-1380.

[11] D. N. Kupeli, *Singular semi-Riemannian geometry*, Kluwer Academic Publishers. 366, 1996.

[12] B. Șahin, *Transversal lightlike submanifolds of indefinite Kaehler manifolds*, An. Univ. Vest Timiș. Ser. Mat.-Inform. XLIV. **1**(2006) 119-145.

[13] B. Șahin, *Screen transversal lightlike submanifolds of indefinite Kaehler manifolds*, Chaos, solitions and Fractals. **38**(2008) 1439-1448.

[14] S. Tanno, *Sasakian manifolds with constant ϕ-holomorphic sectional curvature*, Tôhoku Math. J. **21**(1969) 501-507.

[15] K. Yano and M. Kon, *Structures on Manifolds*, World Scientific, 1984.

[16] C. Yıldırım and B. Şahin, *Transversal lightlike submanifolds of indefinite Sasakian manifolds*, Turkish J. Math. **34**(2010) 561-584.

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