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GENERAL RESULTS ON THE EXISTENCE AND GLOBAL EXPONENTIAL STABILITY OF PERIODIC SOLUTIONS FOR GENERALIZED SHUNTING INHIBITORY CELLULAR NEURAL NETWORKS WITH DELAYS

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Abstract

In this paper, with the help of the Leray-Schauder fixed point theorem, differential inequality techniques and suitable Lyapunov functional, several novel sufficient conditions on the existence and global exponential stability of periodic solutions for the generalized shunting Inhibitory cellular neural networks are developed, which improve some published results. Particularly, the precise convergence rate index is also obtained. One example with its numerical simulation is employed to illustrate the generalized results.

1 Introduction and preliminaries

In recent years, the shunting inhibitory cellular neural networks (SICNNs) have been extensively studied and found many important applications in different

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areas such as psychophysics, speech, perception, robotics, adaptive pattern recognition, vision, and image processing. To date, many interesting results on stability of SICNNs have been obtained. In particular, some results on the existence and exponential stability of (almost) periodic solutions for SICNNs with delays have been reported in [1, 3, 4, 5, 6, 7, 8]. However, SICNNs in [1, 3, 4, 5, 6, 7, 8] are all with constant coefficients. Non-autonomous phenomena often occur in many realistic systems, hence, it is of prime importance to study the existence and exponential stability of periodic solutions and almost periodic solutions of SICNNs with variable coefficients. In this paper, we consider the following general SICNNs with delays, which includes SICNNs in [1, 3, 4, 6] as special cases.

$$\begin{aligned} x'_{ij}(t) &= -a_{ij}(t, x_{ij}(t)) + \sum_{B^{kl}\in N_r(i,j)} B^{kl}_{ij}(t) f_{ij}(t, x_{kl}(t)) x_{ij}(t) \\ &\quad + \sum_{C^{kl}\in N_r(i,j)} C^{kl}_{ij}(t) g_{ij}(t, x_{kl}(t-\tau_{kl}(t))) x_{ij}(t) + I_{ij}(t), \quad (1.1) \end{aligned}$$

where $i = 1, 2, \cdots, n, j = 1, 2, \cdots, m$. $\tau_{ij}(t)$ represents axonal signal transmission delays and continuous with $0 \le \tau_{kl}(t) \le \tau$; $C_{ij}(t)$ denotes the cell at the (i, j) position of the lattice at the t, the r-neighborhood $N_r(i, j)$ of $C_{ij}(t)$ is

$$N_r(i, j) = \{C^{kl}(t) : \max(|k - i|, |l - j|) \le r, 1 \le k \le m, 1 \le l \le n\}.$$

$x_{ij}(t)$ is the activity of the cell $C_{ij}(t)$, $I_{ij}(t)$ is the external input to $C_{ij}(t)$, $a_{ij}(t, x_{ij}(t))$ represents the passive decay function of the cell activity; $C^{kl}_{ij}(t)$ is the connection or coupling strength of postsynaptic activity of the cell transmitted to the cell $C_{ij}(t)$, the activity function $f_{ij}(t, \cdot)$ is a continuous function representing the output or firing rate of the cell $C^{kl}(t)$; $\varphi_{ij}(t)$ is the initial function, and is assumed to be bounded and continuous on $[-\tau, 0]$. $a_{ij}(t), C^{kl}_{ij}(t), f_{ij}(t, \cdot), I_{ij}(t), \varphi_{ij}(t)$ are all continuous periodic functions.

For convenience, we introduce the notations: $\tau = \max_{(i,j)}\{\tau_{ij}(t)|t \in [0, \omega]\}$, $\overline{I}_{ij} = \max_{t\in[0,\omega]} |I_{ij}(t)|$, $\underline{\mu}_{ij} = \min_{t\in[0,\omega]} \mu_{ij}(t)$,

$$x = (x_{11}, x_{12}, \cdots, x_{1m}, \cdots, x_{n1}, x_{n2}, \cdots, x_{nm})^T$$

be a column vector, in which the symbol T denotes the transpose of a vector.

The initial condition $\phi = (\phi_{11}, \cdots, \phi_{1m}, \cdots, \phi_{n1}, \cdots, \phi_{nm})^T$ of (1.1) is of the form

$$x_{ij}(s) = \phi_{ij}(s), \quad s \in (-\tau, 0],$$

where $\phi_{ij}(s), i = 1, 2, \cdots, n, j = 1, \cdots, m$, are continuous ω-periodic solutions.

Definition 1.1. Let $x^*(t)$ be an ω-periodic solution of (1.1) with initial value ϕ^*. If there exist constants $\alpha > 0$ and $P \geq 1$ such that for every solution $x(t)$ of (1.1) with initial value ϕ,

$$|x_{ij}(t) - x_{ij}^*(t)| \leq P\|\phi - \phi^*\|e^{-\alpha t}, \quad \forall t > 0,\ i = 1, 2, \cdots, n, j = 1, 2, \cdots, m,$$

where $\|\phi - \phi^*\| = \max_{(i,j)} \sup_{-\tau \leq s \leq 0}\{|\phi_{ij}(s) - \phi_{ij}^*(s)|\}$. Then $x^*(t)$ is said to be *globally exponentially stable.*

Lemma 1.1(Leray-Schauder). Let E be a Banach space, and let the operator $A : E \to E$ be completely continuous. If the set $\{\|x\||x \in E, x = \lambda Ax, 0 < \lambda < 1\}$ is bounded, then A has a fixed point in T, where

$$T = \{x|x \in E, \|x\| \leq R\},\ R = \sup\{\|x\||x = \lambda Ax, 0 < \lambda < 1\}.$$

Lemma 1.2 [2]. For any $x, y \geq 0, q > 0.5$, the inequality

$$x^{2q-1}y \leq \frac{2q-1}{2q}x^{2q} + \frac{1}{2q}y^{2q}$$

holds.

Obviously, when $q = 0.5$, the above inequality also holds. Hence, for any $x, y \geq 0, q \geq 1$, we have

$$x^{q-1}y \leq \frac{q-1}{q}x^q + \frac{1}{q}y^q. \tag{1.2}$$

The main purpose of this paper is to obtain sufficient conditions for the existence and global exponential stability of periodic solutions for (1.1). The main methods used in this paper are Leray-Schauder's fixed point theorem, differential inequality techniques and Lyapunov functional. The results of this paper generalize and complement some published results. One example is employed to illustrate our feasible results.

Throughout this paper, we assume that

- (H_1) $B_{ij}^{kl}(t), C_{ij}^{kl}(t), \tau_{ij}(t) \geq 0, I_{ij}(t)$ are continuous ω-periodic functions. $\omega > 0$ is a constant, $i = 1, 2, \cdots, n$, $j = 1, 2, \cdots, m$;
- (H_2) $a_{ij}(t, u) \in C(R^2, R)$ are ω-periodic about the first argument, $a_{ij}(t, 0) = 0$ and there are positive continuous ω-periodic functions $\mu_{ij}(t)$ such that $\frac{\partial a_{ij}(t,u)}{\partial u} \geq \mu_{ij}(t)$, $\frac{\partial a_{ij}(t,u)}{\partial u}$ are bounded, $i = 1, 2, \cdots, n$, $j = 1, 2, \cdots, m$;
- (H_3) $f_{ij}(t, u), g_{ij}(t, u) \in C(R^2, R)$ are ω-periodic about the first argument. There are continuous ω-periodic solutions $\delta_{ij}(t)$ and $\gamma_{ij}(t)$ such that $\delta_{ij}(t) = \sup_{u\in R} |f_{ij}(t, u)|$, $\gamma_{ij}(t) = \sup_{u\in R} |g_{ij}(t, u)|$, $i = 1, 2, \cdots, n$, $j = 1, 2, \cdots, m$;

(H$_4$) $\max_{(i,j)} \sup_{0\le t\le\omega} \left\{ \frac{\delta_{ij}(t)\sum_{B^{kl}\in N_r(i,j)} |B_{ij}^{kl}(t)| + \gamma_{ij}(t)\sum_{C^{kl}\in N_r(i,j)} |C_{ij}^{kl}(t)|}{\mu_{ij}(t)} \right\} = \theta$ and $\theta < 1$;

(H$_5$) There are non-negative continuous ω-periodic solutions $\alpha_{ij}(t)$ and $\beta_{ij}(t)$ such that $\alpha_{ij}(t) = \sup_{u\neq v} \left| \frac{f_{ij}(t,u)-f_{ij}(t,v)}{u-v} \right|$, $\beta_{ij}(t) = \sup_{u\neq v} \left| \frac{g_{ij}(t,u)-g_{ij}(t,v)}{u-v} \right|$ for all $u, v \in R$, $u \neq v$, $i = 1, 2, \cdots, n$ $j = 1, 2, \cdots, m$.

The organization of this paper is as follows. In Section 2, we study the existence of periodic solutions of system (1.1) by using the Leray-Schauder's fixed point theorem. In Section 3, by constructing Lyapunov functional, we shall derive new sufficient conditions for the global exponential stability of the periodic solution of system (1.1). Moreover, we compare our results with some of previously know results. At last, an example is employed to illustrate the feasible results of this paper.

2 Existence of periodic solutions

Let $\xi_{ij}, i = 1, 2, \cdots, n, j = 1, 2, \cdots, m$ be positive constants. Make the change of variables

$$x_{ij} = \xi_{ij} y_{ij}(t),\ i = 1, 2, \cdots, n, j = 1, 2, \cdots, m, \tag{2.1}$$

then (1.1) can be reformulated as

$$\begin{aligned} y'_{ij}(t) = &-\xi_{ij}^{-1} a_{ij}(t, \xi_{ij} y_{ij}(t)) + \sum_{B^{kl}\in N_r(i,j)} B_{ij}^{kl}(t) f_{ij}(t, \xi_{kl} y_{kl}(t)) y_{ij}(t) \\ &+ \sum_{C^{kl}\in N_r(i,j)} C_{ij}^{kl}(t) g_{ij}(t, \xi_{kl} y_{kl}(t-\tau_{kl}(t))) y_{ij}(t) + \xi_{ij}^{-1} I_{ij}(t). \end{aligned} \tag{2.2}$$

System (2.2) can be rewritten as

$$\begin{aligned} y'_{ij}(t) = &-d_{ij}(t, y_{ij}(t)) y_{ij}(t) + \sum_{B^{kl}\in N_r(i,j)} B_{ij}^{kl}(t) f_{ij}(t, \xi_{kl} y_{kl}(t)) y_{ij}(t) \\ &+ \sum_{C^{kl}\in N_r(i,j)} C_{ij}^{kl}(t) g_{ij}(t, \xi_{kl} y_{kl}(t-\tau_{kl}(t))) y_{ij}(t) + \xi_{ij}^{-1} I_{ij}(t), \end{aligned} \tag{2.3}$$

where $d_{ij}(t, y_{ij}(t)) \doteq \frac{\partial a_{ij}(t,u)}{\partial u}|_{u=d_{ij}}$, d_{ij} is between 0 and $\xi_{ij} y_{ij}(t)$, $d_{ij} \in R$.

By (H$_2$), we obtain $a_{ij}(t, \xi_{ij} y_{ij})$ is strictly monotone increasing about y_{ij}. Hence, $d_{ij}(t, y_{ij}(t))$ is unique for any $y_{ij}(t)$. Obviously, $d_{ij}(t, y_{ij}(t))$ is continuous ω-periodic about the first argument and $d_{ij}(t, y_{ij}(t)) \geq \mu_{ij}(t)$.

Lemma 2.1. Suppose that (H_1)-(H_3) hold and let $x(t)$ be an ω-periodic solution of (1.1). Then,

$$\begin{aligned} y_{ij}(t) &= \int_0^\omega H_{ij}^y(t,s)\Big[\sum_{B^{kl}\in N_r(i,j)} B_{ij}^{kl}(s) f_{ij}(s,\xi_{kl}y_{kl}(s))y_{ij}(s) \\ &\quad + \sum_{C^{kl}\in N_r(i,j)} C_{ij}^{kl}(s) g_{ij}(s,\xi_{kl}y_{kl}(s-\tau_{kl}(s)))y_{ij}(s) + \xi_{ij}^{-1}I_{ij}(s)\Big]\mathrm{d}s, \\ &\quad t\in[0,\omega], i=1,2,\cdots,n, j=1,2,\cdots,m, \end{aligned} \tag{2.4}$$

where,

$$H_{ij}^y(t,s)=\begin{cases} \frac{e^{-\int_s^t d_{ij}(v,y_{ij}(v))\mathrm{d}v}}{1-e^{-\int_0^\omega d_{ij}(v,y_{ij}(v))\mathrm{d}v}}, & 0\le s\le t\le\omega, \\ \frac{e^{-(\int_0^\omega d_{ij}(v,y_{ij}(v))\mathrm{d}v-\int_t^s d_{ij}(v,y_{ij}(v))\mathrm{d}v)}}{1-e^{-\int_0^\omega d_{ij}(v,y_{ij}(v))\mathrm{d}v}}, & 0\le t< s\le\omega. \end{cases}$$

Proof. From the system (2.3), we have

$$\Big(y_{ij}(t)e^{\int_0^t d_{ij}(s,y_{ij}(s))\mathrm{d}s}\Big)' = \Big[\sum_{B^{kl}\in N_r(i,j)} B_{ij}^{kl}(t) f_{ij}(t,\xi_{kl}y_{kl}(t))y_{ij}(t) + \sum_{C^{kl}\in N_r(i,j)} C_{ij}^{kl}(t) g_{ij}(t,\xi_{kl}y_{kl}(t-\tau_{kl}(t)))y_{ij}(t) + \xi_{ij}^{-1}I_{ij}(t)\Big]e^{\int_0^t d_{ij}(s,y_{ij}(s))\mathrm{d}s}. \tag{2.5}$$

Integrating (2.5) from 0 to t, we have

$$y_{ij}(t) = e^{-\int_0^t d_{ij}(s,y_{ij}(s))\mathrm{d}s}y_{ij}(0) + \int_0^t \Big[\sum_{B^{kl}\in N_r(i,j)} B_{ij}^{kl}(s) f_{ij}(s,\xi_{kl}y_{kl}(s))y_{ij}(s) + \sum_{C^{kl}\in N_r(i,j)} C_{ij}^{kl}(s) g_{ij}(s,\xi_{kl}y_{kl}(s-\tau_{kl}(s)))y_{ij}(s) + \xi_{ij}^{-1}I_{ij}(s)\Big]e^{-\int_s^t d_{ij}(v,y_{ij}(v))\mathrm{d}v}\mathrm{d}s. \tag{2.6}$$

From $x_{ij}(\omega)=x_{ij}(0)$ and (2.1), we have $y_{ij}(\omega)=y_{ij}(0)$. By (2.6), we obtain

$$y_{ij}(0) = \frac{1}{1-e^{-\int_0^\omega d_{ij}(s,y_{ij}(s))\mathrm{d}s}} \int_0^\omega \Big[\sum_{B^{kl}\in N_r(i,j)} B_{ij}^{kl}(s) f_{ij}(s,\xi_{kl}y_{kl}(s))y_{ij}(s) + \sum_{C^{kl}\in N_r(i,j)} C_{ij}^{kl}(s) g_{ij}(s,\xi_{kl}y_{kl}(s-\tau_{kl}(s)))y_{ij}(s) + \xi_{ij}^{-1}I_{ij}(s)\Big]e^{-\int_s^\omega d_{ij}(v,y_{ij}(v))\mathrm{d}v}\mathrm{d}s. \tag{2.7}$$

Substituting (2.7) into (2.6), we obtain

$$y_{ij}(t) = \frac{e^{-\int_0^t d_{ij}(s,y_{ij}(s))\mathrm{d}s}}{1-e^{-\int_0^\omega d_{ij}(v,y_{ij}(v))\mathrm{d}v}} \int_0^\omega \Big[\sum_{B^{kl}\in N_r(i,j)} B_{ij}^{kl}(s) f_{ij}(s,\xi_{kl}y_{kl}(s))y_{ij}(s) +$$
$$\sum_{C^{kl}\in N_r(i,j)} C_{ij}^{kl}(s) g_{ij}(s,\xi_{kl}y_{kl}(s-\tau_{kl}(s)))y_{ij}(s) + \xi_{ij}^{-1} I_{ij}(s)\Big] e^{-\int_s^\omega d_{ij}(v,y_{ij}(v))\mathrm{d}v}\mathrm{d}s$$
$$+ \int_0^t \Big[\sum_{B^{kl}\in N_r(i,j)} B_{ij}^{kl}(s) f_{ij}(s,\xi_{kl}y_{kl}(s))y_{ij}(s) +$$
$$\sum_{C^{kl}\in N_r(i,j)} C_{ij}^{kl}(s) g_{ij}(s,\xi_{kl}y_{kl}(s-\tau_{kl}(s)))y_{ij}(s) + \xi_{ij}^{-1} I_{ij}(s)\Big] e^{-\int_s^t d_{ij}(v,y_{ij}(v))\mathrm{d}v}\mathrm{d}s$$
$$= \int_0^\omega H_{ij}^y(t,s)\Big[\sum_{B^{kl}\in N_r(i,j)} B_{ij}^{kl}(s) f_{ij}(s,\xi_{kl}y_{kl}(s))y_{ij}(s) +$$
$$\sum_{C^{kl}\in N_r(i,j)} C_{ij}^{kl}(s) g_{ij}(s,\xi_{kl}y_{kl}(s-\tau_{kl}(s)))y_{ij}(s) + \xi_{ij}^{-1} I_{ij}(s)\Big]\mathrm{d}s.$$

This completes the proof. ∎

In order to use Lemma 1.1, we take $X = \{y | y \in C([0,\omega], R^{nm})\}$. Then X is a Banach space with the norm

$$\|y\| = \max_{(i,j)}\{|y_{ij}|_0\},\ |y_{ij}|_0 = \sup_{0\le t\le\omega} |y_{ij}(t)|, i = 1,\cdots,n, j = 1,\cdots,m.$$

Take a mapping $\Phi : X \to X$ by setting

$$(\Phi y)(t) = ((\Phi y)_{11}(t), (\Phi y)_{12}(t), \cdots, (\Phi y)_{nm}(t))^T,$$

where

$$(\Phi y)_{ij}(t) = \int_0^\omega H_{ij}^y(t,s)\Big[\sum_{B^{kl}\in N_r(i,j)} B_{ij}^{kl}(s) f_{ij}(s,\xi_{kl}y_{kl}(s))y_{ij}(s) +$$
$$\sum_{C^{kl}\in N_r(i,j)} C_{ij}^{kl}(s) g_{ij}(s,\xi_{kl}y_{kl}(s-\tau_{kl}(s)))y_{ij}(s) + \xi_{ij}^{-1} I_{ij}(s)\Big]\mathrm{d}s,$$
$$i = 1,2,\cdots,n, j = 1,2,\cdots,m.$$

It is easy to know the fact that the existence of ω-periodic solution of (1.1) is equivalent to the existence of fixed point of the mapping Φ in X.

Lemma 2.2. Suppose that (H_1)-(H_4) hold. Then $\Phi : X \to X$ is completely continuous.

Proof. Under our assumptions, it is clear that the operator Φ is continuous. Next, we show that Φ is compact.

Since $\mu_{ij}(t), i = 1, 2, \cdots, n, j = 1, 2, \cdots, m$ are positive ω-periodic solutions, $\underline{\mu}_{ij} > 0$. For any constant $D > 0$, let $\Omega = \{y | y \in X, \|y\| < D\}$. For any $y \in \Omega$, we have

$$\begin{aligned}
\|\Phi y\| &= \max_{(i,j)} \sup_{0\le t\le\omega} \Bigg\{ \Bigg| \int_0^\omega H_{ij}^y(t,s) \Bigg[\sum_{B^{kl}\in N_r(i,j)} B_{ij}^{kl}(s) f_{ij}(s, \xi_{kl} y_{kl}(s)) y_{ij}(s) \\
&\quad + \sum_{C^{kl}\in N_r(i,j)} C_{ij}^{kl}(s) g_{ij}(s, \xi_{kl} y_{kl}(s - \tau_{kl}(s))) y_{ij}(s) + \xi_{ij}^{-1} I_{ij}(s) \Bigg] \mathrm{d}s \Bigg| \Bigg\} \\
&\le \max_{(i,j)} \sup_{0\le t\le\omega} \Bigg\{ \int_0^\omega H_{ij}^y(t,s) \Bigg[\Bigg(\delta_{ij}(s) \sum_{B^{kl}\in N_r(i,j)} |B_{ij}^{kl}(s)| \\
&\quad + \gamma_{ij}(s) \sum_{C^{kl}\in N_r(i,j)} |C_{ij}^{kl}(s)| \Bigg) y_{ij}(s) + \xi_{ij}^{-1} I_{ij}(s) \Bigg] \mathrm{d}s \Bigg\} \\
&\le \max_{(i,j)} \sup_{0\le t\le\omega} \Bigg\{ \int_0^\omega H_{ij}^y(t,s) \mu_{ij}(s) \theta \mathrm{d}s \Bigg\} \|y\| + \max_{(i,j)} \Bigg\{ \frac{\overline{I}_{ij}}{\xi_{ij} \underline{\mu}_{ij}} \Bigg\} \\
&< \theta D + \max_{(i,j)} \Bigg\{ \frac{\overline{I}_{ij}}{\xi_{ij} \underline{\mu}_{ij}} \Bigg\},
\end{aligned}$$

this implies that $\Phi(\Omega)$ is uniformly bounded, where,

$$\begin{aligned}
&\int_0^\omega H_{ij}^y(t,s)\mu_{ij}(s)\mathrm{d}s = \frac{1}{1 - e^{-\int_0^\omega d_{ij}(v, y_{ij}(v))\mathrm{d}v}} \Bigg\{ \int_0^t e^{-\int_s^t d_{ij}(v, y_{ij}(v))\mathrm{d}v} \mu_{ij}(s) \mathrm{d}s \\
&+ e^{-\int_0^\omega d_{ij}(v, y_{ij}(v))\mathrm{d}v} \int_t^\omega e^{\int_t^s d_{ij}(v, y_{ij}(v)\mathrm{d}v)} \mu_{ij}(s) \mathrm{d}s \Bigg\} \\
\le\ & \frac{1}{1 - e^{-\int_0^\omega d_{ij}(v, y_{ij}(v))\mathrm{d}v}} \Bigg\{ \int_0^t e^{-\int_s^t \mu_{ij}(v)\mathrm{d}v} \mu_{ij}(s) \mathrm{d}s \\
&+ e^{-\int_0^\omega d_{ij}(v, y_{ij}(v))\mathrm{d}v} \int_t^\omega e^{\int_t^s d_{ij}(v, y_{ij}(v))\mathrm{d}v} d_{ij}(s, y_{ij}(s)) \mathrm{d}s \Bigg\} \\
=\ & \frac{\Big\{ 1 - e^{-\int_0^t \mu_{ij}(s)\mathrm{d}v} + e^{-\int_0^\omega d_{ij}(v, y_{ij}(v))\mathrm{d}v} \Big(e^{\int_t^\omega d_{ij}(v, y_{ij}(v)\mathrm{d}v)} - 1 \Big) \Big\}}{1 - e^{-\int_0^\omega d_{ij}(v, y_{ij}(v))\mathrm{d}v}}
\end{aligned}$$

$$= \frac{\left\{1 - e^{-\int_0^t \mu_{ij}(s)\mathrm{d}v} + e^{-\int_0^t d_{ij}(v,y_{ij}(v))\mathrm{d}v} - e^{-\int_0^\omega d_{ij}(v,y_{ij}(v))\mathrm{d}v}\right\}}{1 - e^{-\int_0^\omega d_{ij}(v,y_{ij}(v))\mathrm{d}v}} \leq 1.$$

By (H_2), there exists a constant $M > 0$ such that

$$|d_{ij}(t, y_{ij}(t))| \leq M, \quad \text{for}\, t \in [0,\omega] \times \Omega, i = 1, 2, \cdots, n, j = 1, 2, \cdots, m.$$

In view of the definition of Φ, we have

$$\begin{aligned}(\Phi y)'_{ij}(t) &= \frac{\mathrm{d}}{\mathrm{d}t}\bigg(\int_0^\omega H^y_{ij}(t,s)\bigg[\sum_{B^{kl}\in N_r(i,j)} B^{kl}_{ij}(s) f_{ij}(s, \xi_{kl} y_{kl}(s)) y_{ij}(s) + \\ &\quad \sum_{C^{kl}\in N_r(i,j)} C^{kl}_{ij}(s) g_{ij}(s, \xi_{kl} y_{kl}(s - \tau_{kl}(s))) y_{ij}(s) + \xi_{ij}^{-1} I_{ij}(s)\bigg]\mathrm{d}s\bigg) \\ &= -d_{ij}(t, y_{ij}(t))(\Phi y)_{ij}(t) + \sum_{B^{kl}\in N_r(i,j)} B^{kl}_{ij}(s) f_{ij}(t, \xi_{kl} y_{kl}(t)) y_{ij}(t) + \\ &\quad \sum_{C^{kl}\in N_r(i,j)} C^{kl}_{ij}(t) g_{ij}(t, \xi_{kl} y_{kl}(t - \tau_{kl}(t))) y_{ij}(t) + \xi_{ij}^{-1} I_{ij}(t).\end{aligned}$$

Hence,

$$\begin{aligned}&\left|(\Phi y)'_{ij}(t)\right| \leq M\left(\theta D + \max_{(i,j)}\left\{\frac{\overline{I}_{ij}}{\xi_{ij}\underline{\mu}_{ij}}\right\}\right) + \\ &\max_{(i,j)}\left\{\sum_{B^{kl}\in N_r(i,j)} \overline{B}^{kl}_{ij}\overline{\delta}_{ij} D + \sum_{C^{kl}\in N_r(i,j)} \overline{C}^{kl}_{ij}\overline{\gamma}_{ij} D + \frac{\overline{I}_{ij}}{\xi_{ij}}\right\},\end{aligned}$$

where $\overline{B}^{kl}_{ij} = \sup_{t\in[0,\omega]} |B^{kl}_{ij}(t)|$, $\overline{\delta}_{ij} = \sup_{t\in[0,\omega]} \delta_{ij}(t)$ $\overline{C}^{kl}_{ij} = \sup_{t\in[0,\omega]} |C^{kl}_{ij}(t)|$, $\overline{\gamma}_{ij} = \sup_{t\in[0,\omega]} \gamma_{ij}(t)$. So, $\Phi(\Omega) \subseteq X$ is a family of uniformly bounded and equi-continuous subsets. By using the Arzela-Ascoli Theorem, $\Phi : X \to X$ is compact. Therefore, $\Phi : X \to X$ is completely continuous. This completes the proof. ∎

Theorem 2.1. Suppose that (H_1)-(H_4) hold. Let $\xi_{ij}, i = 1, 2, \cdots, n, j = 1, 2, \cdots, m$ be positive constants. Then the system (1.1) has an ω-periodic solution $x^*(t)$ with $\|x^*\| \leq \max_{(i,j)}\{\xi_{ij}\}\widetilde{R} \doteq R_0$, where

$$\widetilde{R} = \frac{\max_{(i,j)}\left\{\frac{\overline{I}_{ij}}{\xi_{ij}\underline{\mu}_{ij}}\right\}}{1 - \theta}.$$

Proof. Let $y \in X$, $t \in [0, \omega]$. We consider the operator equation

$$y = \lambda \Phi y,\ \lambda \in (0,1). \tag{2.8}$$

If y is a solution of (2.8), for $t \in [0, \omega]$, we obtain

$$\|y\| \le \|\Phi y\| \le \theta \|y\| + \max_{(i,j)} \left\{ \frac{\overline{I}_{ij}}{\xi_{ij} \underline{\mu}_{ij}} \right\}.$$

This and (H_4) imply that

$$\|y\| \le \widetilde{R}.$$

In view of Lemma 1.1, we obtain that Φ has a fixed point $y^*(t)$ with $\|y^*\| \le \widetilde{R}$. Hence, system (2.3) has one ω-periodic solution

$$y^*(t) = (y_{11}^*, y_{12}^*, \cdots, y_{1m}^*, \cdots, y_{n1}^*, y_{n2}^*, \cdots, y_{nm}^*)^T$$

with $\|y^*\| \le \widetilde{R}$. It follows from (2.1) that

$$x^*(t) = (x_{11}^*(t), x_{12}^*(t), \cdots, x_{1m}^*(t), \cdots, x_{n1}^*(t), x_{n2}^*(t), \cdots, x_{nm}^*(t))^T =$$

$$= (\xi_{11} y_{11}^*, \xi_{12} y_{12}^*, \cdots, \xi_{1m} y_{1m}^*, \cdots, \xi_{n1} y_{n1}^*, \xi_{n2} y_{n2}^*, \cdots, \xi_{nm} y_{nm}^*)^T$$

is one ω-periodic solution of (1.1) with

$$\|x^*\| \le \max_{(i,j)} \{\xi_{ij}\} \widetilde{R} \doteq R_0.$$

This completes the proof. ∎

3 Global exponential stability of periodic solution

In this Section, we shall construct a suitable Lyapunov functional to derive sufficient conditions ensuring that (1.1) has a unique ω-periodic solution and all solutions of (1.1) exponentially converge to its unique ω-periodic solution.

Theorem 3.1. Assume (H_1)-(H_3), (H_5) and (H_6); where (H_6) is

(H_6) There are positive constants $p \ge 1$ and ξ_{ij}, $i = 1, 2, \cdots, n$, $j = 1, 2, \cdots, m$, such that

$$\max_{(i,j)} \sup_{0 \le t \le \omega} \left\{ \frac{p A_{ij}^{kl}(t) \xi_{ij} + R_0 (p-1) D_{ij}^{kl}(t) \xi_{ij} + R_0 F_{ij}^{kl}(\xi, p, t)}{p \mu_{ij}(t) \xi_{ij}} \right\} < 1,$$

where,

$$A_{ij}^{kl}(t) = \delta_{ij}(t) \sum_{B^{kl}\in N_r(i,j)} |B_{ij}^{kl}(t)| + \gamma_{ij}(t) \sum_{C^{kl}\in N_r(i,j)} |C_{ij}^{kl}(t)|,$$
$$D_{ij}^{kl}(t) = \sum_{B^{kl}\in N_r(i,j)} |B_{ij}^{kl}(t)| + \sum_{C^{kl}\in N_r(i,j)} |C_{ij}^{kl}(t)|,$$
$$F_{ij}^{kl}(\xi,p,t) = \sum_{B^{kl}\in N_r(i,j)} |B_{ij}^{kl}(t)|\xi_{kl}\alpha_{ij}^{p}(t) + \sum_{C^{kl}\in N_r(i,j)} |C_{ij}^{kl}(t)|\xi_{kl}\beta_{ij}^{p}(t).$$

Then the system (1.1) has exactly one ω-periodic solution, which is globally exponentially stable.

Proof. Obviously that (H_6) implies (H_4). By Theorem 2.1, there exists an ω-periodic solution $x^*(t)$ of (1.2) with initial value

$$\phi^*(t) = (\phi_{11}^*(t), \cdots, \phi_{1m}^*(t), \cdots, \phi_{n1}^*(t), \cdots, \phi_{nm}^*(t))^T$$

and $\|x^*\| \le R_0$. Suppose that $x(t)$ is an arbitrary solution of system (1.1) with initial value $\phi(t) = (\phi_{11}(t), \cdots, \phi_{1m}(t), \cdots, \phi_{n1}(t), \cdots, \phi_{nm}(t))^T$. Set $z(t) = (z_{11}(t), \cdots, z_{1m}(t), \cdots, z_{n1}(t), \cdots, z_{nm}(t))^T = x(t) - x^*(t)$. Then, from system (1.1) we have

$$\begin{aligned} z_{ij}'(t) &= -[a_{ij}(t,x_{ij}(t)) - a_{ij}(t,x_{ij}^*(t))] \\ &+ \sum_{B^{kl}\in N_r(i,j)} B_{ij}^{kl}(t)[f_{ij}(t,x_{kl}(t))x_{ij}(t) - f_{ij}(t,x_{kl}^*(t))x_{ij}^*(t)] \\ &+ \sum_{C^{kl}\in N_r(i,j)} C_{ij}^{kl}(t)[g_{ij}(t,x_{kl}(t-\tau_{kl}(t)))x_{ij}(t) - g_{ij}(t,x_{kl}^*(t-\tau_{kl}(t)))x_{ij}^*(t)]. \end{aligned} \tag{3.1}$$

From (H_6) we have

$$-p\mu_{ij}(t)\xi_{ij} + pA_{ij}^{kl}(t)\xi_{ij} + R_0(p-1)D_{ij}^{kl}(t)\xi_{ij} + R_0F_{ij}^{kl}(\xi,p,t) < 0,$$
$$i = 1,2,\cdots,n, j = 1,2,\cdots,m. \tag{3.2}$$

Set

$$\begin{aligned} h_{ij}(\lambda) \;=\; & \lambda\xi_{ij} - p\mu_{ij}(t)\xi_{ij} + pA_{ij}^{kl}(t)\xi_{ij} + R_0(p-1)D_{ij}^{kl}(t)\xi_{ij} \\ & +R_0\Bigg(\sum_{B^{kl}\in N_r(i,j)} |B_{ij}^{kl}(t)|\xi_{kl}\alpha_{ij}^{p}(t) + \sum_{C^{kl}\in N_r(i,j)} |C_{ij}^{kl}(t)|\xi_{kl}\beta_{ij}^{p}(t)e^{\lambda\tau}\Bigg). \end{aligned}$$

Clearly, $h_{ij}(\lambda), i = 1,2,\cdots,n, j = 1,2,\cdots,m$, are continuous functions on R. Since $h_{ij}(0) < 0$,

$$\frac{\mathrm{d}h_{ij}(\lambda)}{\mathrm{d}\lambda} = \xi_{ij} + \lambda R_0 \sum_{C^{kl}\in N_r(i,j)} |C_{ij}^{kl}(t)|\beta_{ij}^{p}(t)e^{\lambda\tau}\xi_{kl} > 0,$$

and $h_{ij}(+\infty) = +\infty$, hence $h_{ij}(\lambda), i = 1, 2, \cdots, n, j = 1, 2, \cdots, m$, are strictly monotone increasing functions. Therefore, for any $i \in \{1, 2, \cdots, n\}$, $j \in \{1, 2, \cdots, m\}$ and $t \geq 0$, there is unique $\lambda(t)$ such that

$$\lambda(t)\xi_{ij} - p\mu_{ij}(t)\xi_{ij} + pA_{ij}^{kl}(t)\xi_{ij} + R_0(p-1)D_{ij}^{kl}(t)\xi_{ij}$$
$$+R_0\Bigg(\sum_{B^{kl}\in N_r(i,j)} |B_{ij}^{kl}(t)|\xi_{kl}\alpha_{ij}^p(t) + \sum_{C^{kl}\in N_r(i,j)} |C_{ij}^{kl}(t)|\xi_{kl}\beta_{ij}^p(t)e^{\lambda(t)\tau}\Bigg) = 0.$$

Let

$$\lambda_{ij}^* = \inf_{t\geq 0}\Bigg\{\lambda(t)\Bigg|\lambda(t)\xi_{ij} - p\mu_{ij}(t)\xi_{ij} + pA_{ij}^{kl}(t)\xi_{ij} + R_0(p-1)D_{ij}^{kl}(t)\xi_{ij}$$
$$+R_0\Bigg(\sum_{B^{kl}\in N_r(i,j)} |B_{ij}^{kl}(t)|\xi_{kl}\alpha_{ij}^p(t) + \sum_{C^{kl}\in N_r(i,j)} |C_{ij}^{kl}(t)|\xi_{kl}\beta_{ij}^p(t)e^{\lambda(t)\tau}\Bigg) = 0\Bigg\}.$$

Obviously, $\lambda_{ij}^* \geq 0, i = 1, 2, \cdots, n, j = 1, 2, \cdots, m$. Now, we shall prove that $\lambda_{ij}^* > 0$. Suppose this is not true. From (3.2), there exists a positive constant η such that

$$\inf_{t\geq 0,(i,j)}\left\{\frac{p\mu_{ij}(t_0)\xi_{ij} - pA_{ij}^{kl}(t_0)\xi_{ij} - R_0(p-1)D_{ij}^{kl}(t_0)\xi_{ij} - R_0F_{ij}^{kl}(\xi,p,t_0)}{\xi_{ij} + 1.5\tau R_0\beta_{ij}^p(t)\sum_{C^{kl}\in N_r(i,j)}|C_{ij}^{kl}(t)|\xi_{kl}}\right\} \geq \eta.$$

Pick small $\varepsilon > 0$, then there exists $t_0 \geq 0$ such that

$$0 < \lambda_{ij}^*(t_0) < \varepsilon < \eta.$$

Let us recall the inequality $e^x < 1 + 1.5x$ for sufficiently small $x > 0$. Then we obtain

$$\begin{aligned}
0 &= \lambda(t_0)\xi_{ij} - p\mu_{ij}(t_0)\xi_{ij} + pA_{ij}^{kl}(t_0)\xi_{ij} + R_0(p-1)D_{ij}^{kl}(t_0)\xi_{ij} \\
&\quad +R_0\Bigg(\sum_{B^{kl}\in N_r(i,j)} |B_{ij}^{kl}(t_0)|\xi_{kl}\alpha_{ij}^p(t_0) + \sum_{C^{kl}\in N_r(i,j)} |C_{ij}^{kl}(t_0)|\xi_{kl}\beta_{ij}^p(t_0)e^{\lambda(t_0)\tau}\Bigg) \\
&< \varepsilon\xi_{ij} - p\mu_{ij}(t_0)\xi_{ij} + pA_{ij}^{kl}(t_0)\xi_{ij} + R_0(p-1)D_{ij}^{kl}(t_0)\xi_{ij} \\
&\quad +R_0\Bigg(\sum_{B^{kl}\in N_r(i,j)} |B_{ij}^{kl}(t_0)|\xi_{kl}\alpha_{ij}^p(t_0) + \sum_{C^{kl}\in N_r(i,j)} |C_{ij}^{kl}(t_0)|\xi_{kl}\beta_{ij}^p(t_0)e^{\varepsilon\tau}\Bigg) \\
&< \eta\xi_{ij} - p\mu_{ij}(t_0)\xi_{ij} + pA_{ij}^{kl}(t_0)\xi_{ij} + R_0(p-1)D_{ij}^{kl}(t_0)\xi_{ij} \\
&\quad +R_0\Bigg(\sum_{B^{kl}\in N_r(i,j)} |B_{ij}^{kl}(t_0)|\xi_{kl}\alpha_{ij}^p(t_0) + \sum_{C^{kl}\in N_r(i,j)} |C_{ij}^{kl}(t_0)|\xi_{kl}\beta_{ij}^p(t_0)(1+1.5\eta\tau)\Bigg) \\
&= -p\mu_{ij}(t_0)\xi_{ij} + pA_{ij}^{kl}(t_0)\xi_{ij} + R_0(p-1)D_{ij}^{kl}(t_0)\xi_{ij} + R_0F_{ij}^{kl}(\xi,p,t_0)
\end{aligned}$$

$$+\eta\Bigg(\xi_{ij}+1.5\tau R_0\beta_{ij}^p(t_0)\sum_{C^{kl}\in N_r(i,j)}|C_{ij}^{kl}(t_0)|\xi_{kl}\Bigg)$$
$$\le -p\mu_{ij}(t_0)\xi_{ij}+pA_{ij}^{kl}(t_0)\xi_{ij}+R_0(p-1)D_{ij}^{kl}(t_0)\xi_{ij}+R_0F_{ij}^{kl}(\xi,p,t_0)$$
$$+\frac{p\mu_{ij}(t_0)\xi_{ij}-pA_{ij}^{kl}(t_0)\xi_{ij}-R_0(p-1)D_{ij}^{kl}(t_0)\xi_{ij}-R_0F_{ij}^{kl}(\xi,p,t_0)}{\xi_{ij}+1.5\tau R_0\beta_{ij}^p(t_0)\sum_{C^{kl}\in N_r(i,j)}|C_{ij}^{kl}(t_0)|\xi_{kl}}\times$$
$$\Bigg(\xi_{ij}+1.5\tau R_0\beta_{ij}^p(t_0)\sum_{C^{kl}\in N_r(i,j)}|C_{ij}^{kl}(t_0)|\xi_{kl}\Bigg)=0,$$

which is a contradiction, and hence $\lambda_{ij}^*>0, i=1,2,\cdots,n, j=1,2,\cdots,m$.

Let $\epsilon=\min_{(i,j)}\{\lambda_{ij}^*\}$. Obviously,

$$h_{ij}(\epsilon)=\epsilon\xi_{ij}-p\mu_{ij}(t)\xi_{ij}+pA_{ij}^{kl}(t)\xi_{ij}+R_0(p-1)D_{ij}^{kl}(t)\xi_{ij}$$
$$+R_0\Bigg(\sum_{B^{kl}\in N_r(i,j)}|B_{ij}^{kl}(t)|\xi_{kl}\alpha_{ij}^p(t)+\sum_{C^{kl}\in N_r(i,j)}|C_{ij}^{kl}(t)|\xi_{kl}\beta_{ij}^p(t)e^{\epsilon\tau}\Bigg)\le 0,$$
$$i=1,2,\cdots,n, j=1,2,\cdots,m. \quad (3.3)$$

We choose a constant $d>1$ such that

$$pd\xi_{ij}e^{-\epsilon t}\ge 1,\ \text{for } t\in(-\tau,0],\ i=1,2,\cdots,n, j=1,2,\cdots,m.$$

It is obvious that

$$|z_{ij}(t)|^p\le\|\phi-\phi^*\|^p\le pd\xi_{ij}\|\phi-\phi^*\|^p e^{-\epsilon t},\ \text{for } t\in(-\tau,0] \text{ and } i=1,2,\cdots,n,$$

$j=1,2,\cdots,m$, where $\|\phi-\phi^*\|$ is defined as that in Definition 1.1.
Define a Lyapunov functional $V(t)=(V_{11}(t),\cdots,V_{1m}(t),\cdots,V_{n1}(t),\cdots,V_{nm}(t),)^T$ by $V_{ij}(t)=\frac{1}{p}e^{\epsilon t}|z_{ij}(t)|^p, i=1,2,\cdots,n, j=1,2,\cdots,m$. In view of (1.2) and (3.1), we obtain

$$\frac{\mathrm{d}^+V_{ij}(t)}{\mathrm{d}t}=|z_{ij}(t)|^{p-1}e^{\epsilon t}\operatorname{sgn} z_{ij}\Bigg\{-[a_{ij}(t,x_{ij}(t))-a_{ij}(t,x_{ij}^*(t))]$$
$$+\sum_{B^{kl}\in N_r(i,j)}B_{ij}^{kl}(t)[f_{ij}(t,x_{kl}(t))x_{ij}(t)-f_{ij}(t,x_{kl}^*(t))x_{ij}^*(t)]$$
$$+\sum_{C^{kl}\in N_r(i,j)}C_{ij}^{kl}(t)[g_{ij}(t,x_{kl}(t-\tau_{kl}(t)))x_{ij}(t)-g_{ij}(t,x_{kl}^*(t-\tau_{kl}(t)))x_{ij}^*(t)]\Bigg\}$$
$$+\frac{\epsilon}{p}e^{\epsilon t}|z_{ij}(t)|^p$$
$$\le|z_{ij}(t)|^{p-1}e^{\epsilon t}\Bigg\{-\mu_{ij}(t)|z_{ij}(t)|+\sum_{B^{kl}\in N_r(i,j)}|B_{ij}^{kl}(t)|\big[|f_{ij}(t,x_{kl}(t))||z_{ij}(t)|$$

$$
\begin{aligned}
&+|f_{ij}(t,x_{kl}(t))-f_{ij}(t,x^*_{kl}(t))||x^*_{ij}(t)|\big] \\
&+\sum_{C^{kl}\in N_r(i,j)}|C^{kl}_{ij}(t)|\big[|g_{ij}(t,x_{kl}(t-\tau_{kl}(t)))||z_{ij}(t)| \\
&+|g_{ij}(t,x_{kl}(t-\tau_{kl}(t)))-g_{ij}(t,x^*_{kl}(t-\tau_{kl}(t)))||x^*_{ij}(t)|\big]\Big\}+\frac{\epsilon}{p}e^{\epsilon t}|z_{ij}(t)|^p \\
&\le e^{\epsilon t}\Big\{\Big(\frac{\epsilon}{p}-\mu_{ij}(t)+\delta_{ij}(t)\sum_{B^{kl}\in N_r(i,j)}|B^{kl}_{ij}(t)|+\gamma_{ij}(t)\sum_{C^{kl}\in N_r(i,j)}|C^{kl}_{ij}(t)|\Big)|z_{ij}(t)|^p \\
&+R_0\sum_{B^{kl}\in N_r(i,j)}|B^{kl}_{ij}(t)||z_{ij}(t)|^{p-1}\alpha_{ij}(t)|z_{kl}(t)| \\
&+R_0\sum_{C^{kl}\in N_r(i,j)}|C^{kl}_{ij}(t)||z_{ij}(t)|^{p-1}\beta_{ij}(t)|z_{kl}(t-\tau_{kl}(t))|\Big\} \\
&\le\Big(\epsilon-p\mu_{ij}(t)+pA^{kl}_{ij}(t)+R_0(p-1)D^{kl}_{ij}(t)\Big)V_{ij}(t)+ \\
&+R_0\Big(\sum_{B^{kl}\in N_r(i,j)}|B^{kl}_{ij}(t)|\alpha^p_{ij}(t)V_{kl}(t)+\sum_{C^{kl}\in N_r(i,j)}|C^{kl}_{ij}(t)|\beta^p_{ij}(t)e^{\epsilon\tau}V_{kl}(t-\tau_{kl}(t))\Big). \qquad (3.4)
\end{aligned}
$$

We claim that

$$
V_{ij}(t)=\frac{1}{p}e^{\epsilon t}|z_{ij}(t)|^p\le d\xi_{ij}\|\phi-\phi^*\|^p,\quad i=1,2,\cdots,n, j=1,2,\cdots,m, \text{ for all } t>0. \qquad (3.5)
$$

Contrarily, there must exists some some $i\in\{1,2,\cdots,n\}$, $j\in\{1,2,\cdots,m\}$ and $\widetilde{t}>0$ such that

$$
V_{ij}(\widetilde{t})=d\xi_{ij}\|\phi-\phi^*\|^p,\ \frac{\mathrm{d}^+V_{ij}(\widetilde{t})}{\mathrm{d}t}>0 \text{ and } V_{ij}(t)\le d\xi_{ij}\|\phi-\phi^*\|^p, \qquad (3.6)
$$

$\forall t\in(-\tau,\widetilde{t}], i=1,2,\cdots,n, j=1,2,\cdots,m$. Together with (3.4) and (3.6), we obtain

$$
\begin{aligned}
0<\frac{\mathrm{d}^+V_{ij}(\widetilde{t})}{\mathrm{d}t}\le d\|\phi-\phi^*\|^p\Big\{&\epsilon\xi_{ij}-p\mu_{ij}(t)\xi_{ij}+pA^{kl}_{ij}(t)\xi_{ij}+R_0(p-1)D^{kl}_{ij}(t)\xi_{ij} \\
&+R_0\Big(\sum_{B^{kl}\in N_r(i,j)}|B^{kl}_{ij}(t)|\xi_{kl}\alpha^p_{ij}(t)+\sum_{C^{kl}\in N_r(i,j)}|C^{kl}_{ij}(t)|\xi_{kl}\beta^p_{ij}(t)e^{\epsilon\tau}\Big)\Big\}.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
0 \ <\ &\epsilon\xi_{ij}-p\mu_{ij}(t)\xi_{ij}+pA^{kl}_{ij}(t)\xi_{ij}+R_0(p-1)D^{kl}_{ij}(t)\xi_{ij} \\
&+R_0\Big(\sum_{B^{kl}\in N_r(i,j)}|B^{kl}_{ij}(t)|\xi_{kl}\alpha^p_{ij}(t)+\sum_{C^{kl}\in N_r(i,j)}|C^{kl}_{ij}(t)|\xi_{kl}\beta^p_{ij}(t)e^{\epsilon\tau}\Big),
\end{aligned}
$$

which contradicts (3.3). Hence, (3.5) holds. It follows that

$$|x_{ij}(t) - x^*_{ij}(t)| = |z_{ij}(t)| \le (pd\xi_{ij})^{\frac{1}{p}} \|\phi - \phi^*\| e^{-\frac{\epsilon}{p}t}, \quad \forall t > 0,\ i = 1, 2, \cdots, n,$$

$j = 1, 2, \cdots, m$. Let $M = \max_{(i,j)}\{(pd\xi_{ij})^{\frac{1}{p}} + 1\}$. Then, we have

$$|x_{ij}(t) - x^*_{ij}(t)| \le M\|\phi - \phi^*\| e^{-\frac{\epsilon}{p}t}, \quad \forall t > 0,\ i = 1, 2, \cdots, n, j = 1, 2, \cdots, m.$$

In view of Definition 1.1, the ω-periodic solution $x^*(t)$ of the system (1.1) is globally exponentially stable. This completes the proof. ∎

Taking $p = 1$ in Theorem 3.1, we have the following corollary.

Corollary 3.1. Assume (H_1)-(H_3), (H_5) and (H_6) hold, where

(H'_6) There are positive constants ξ_{ij}, $i = 1, 2, \cdots, n$, $j = 1, 2, \cdots, m$, such that

$$\max_{(i,j)} \sup_{0\le t\le\omega} \left\{ \frac{A^{kl}_{ij}(t)\xi_{ij} + R_0 E^{kl}_{ij}(\xi, t)}{\mu_{ij}(t)\xi_{ij}} \right\} < 1,$$

where, $E^{kl}_{ij}(\xi, t) = \sum_{B^{kl}\in N_r(i,j)} |B^{kl}_{ij}(t)|\xi_{kl}\alpha_{ij}(t) + \sum_{C^{kl}\in N_r(i,j)} |C^{kl}_{ij}(t)|\xi_{kl}\beta_{ij}(t)$.

Then the system (1.1) has exactly one ω-periodic solution, which is globally exponentially stable.

Remark 1. In [4], the authors studied the existence and exponential stability of the following SICNNs with delays and variable coefficients

$$\begin{aligned} x'_{ij}(t) &= -h_{ij}(t)x_{ij}(t) - \sum_{J^{kl}\in N_r(i,j)} J^{kl}_{ij}(t)e_{ij}(x_{kl}(t))x_{ij}(t) \\ &\quad - \sum_{W^{kl}\in N_r(i,j)} W^{kl}_{ij}(t)q_{ij}(x_{kl}(t - \tau_{kl}(t)))x_{ij}(t) + I_{ij}(t), \end{aligned} \quad (3.7)$$

where $h_{ij}(t) > 0$, $J^{kl}_{ij}(t) \ge 0$, $W^{kl}_{ij}(t) \ge 0$ and $I_{ij}(t)$ are all continuous ω-periodic solutions.

Theorem $*$ (Li[4]). Assume that the following conditions are satisfied:

(F_1) $e_{ij}, q_{ij} \in C(R, R)$ are bounded on R, $i = 1, 2, \cdots, n$, $j = 1, 2, \cdots, m$;

(F_2) there exist positive numbers μ_{ij} and ν_{ij} such that $|e_{ij}(x) - e_{ij}(y)| \le \mu_{ij}|x - y|$, $|q_{ij}(x) - e_{ij}(y)| \le \nu_{ij}|x - y|$ for all $x, y \in R$, $i = 1, 2, \cdots, n$, $j = 1, 2, \cdots, m$;

(F_3) for each $i = 1, 2, \cdots, n,\ j = 1, 2, \cdots, m$

$$\underline{h}_{ij} - M_{ij} \sum_{J^{kl} \in N_r(i,j)} \overline{J}_{ij}^{kl} - N_{ij} \sum_{W^{kl} \in N_r(i,j)} \overline{W}_{ij}^{kl} - \frac{\overline{I}_{ij}}{\Gamma_{ij}} \left[\mu_{ij} \sum_{J^{kl} \in N_r(i,j)} \overline{J}_{ij}^{kl} + \nu_{ij} \sum_{W^{kl} \in N_r(i,j)} \overline{W}_{ij}^{kl} \right] > 0.$$

Then the system (3.7) has a unique ω-periodic solution which is globally exponentially stable, where
$\underline{h}_{ij} = \min_{t\in[0,\omega]}\{h_{ij}(t)\}$, $\overline{J}_{ij}^{kl} = \max_{t\in[0,\omega]}\{J_{ij}^{kl}(t)\}$, $\overline{W}_{ij}^{kl} = \max_{t\in[0,\omega]}\{W_{ij}^{kl}(t)\}$, $\overline{I}_{ij} = \max_{t\in[0,\omega]}\{|I_{ij}(t)|\}$, $M_{ij} = \sup_{u\in R}\{|e_{ij}(u)|\}$, $N_{ij} = \sup_{u\in R}\{|q_{ij}(u)|\}$.

One can easily sees that (F_1)-(F_3) are special cases of (H_3), (H_5) and (H_6'), respectively. Taking $\xi_{ij} = 1$, $i = 1, 2, \cdots, n$, $j = 1, 2, \cdots, m$ in Corollary 3.1, it is obvious that Theorem 3.2 is direct corollary of Corollary 3.1.

Remark 2. It is worth noting that our results Theorem 2.1, Theorem 3.1 and Corollary 3.1 are also applicable to SCINNs with distributed delays [5, 7, 8]. On the other hand, the results of this paper are also applicable to studying the existence and exponential stability of almost periodic solution for SCINNs with time-varying coefficients and delays. Consider the following SCINNs with time-varying coefficients and delays:

$$\begin{cases} x_{ij}'(t) = & -\widetilde{a}_{ij}(t)x_{ij}(t) - \sum_{\widetilde{C}^{kl}} \widetilde{C}_{ij}^{kl}(t)\widetilde{g}_{ij}(t, x_{kl}(t - \widetilde{\tau}_{kl}(t)))x_{ij}(t) + \widetilde{I}_{ij}(t), \\ x_{ij}(t) = & \varphi_{ij}(t),\ t \in [0,\omega]\ i = 1, 2, \cdots, n,\ j = 1, 2, \cdots, m. \end{cases} \tag{3.8}$$

where $t \geq 0, \widetilde{a}_{ij}(t) > 0$, $\widetilde{C}_{ij}^{kl}(t) \geq 0$, $\widetilde{g}_{ij}(t, \cdot)$, $\widetilde{\tau}_{ij}(t) \geq 0$, $\varphi_{ij}(t)$, $\widetilde{I}_{ij}(t)$ are all continuous almost periodic functions.

We further assume that

(F_1') $\widetilde{g}_{ij}(t, u) \in C(R^2, R)$ are almost periodic about the first argument. There are continuous almost periodic solutions $\widetilde{\gamma}_{ij}(t)$ such that $\widetilde{\gamma}_{ij}(t) = \sup_{u\in R} |\widetilde{g}_{ij}(t, u)|$, $i = 1, 2, \cdots, n$, $j = 1, 2, \cdots, m$;

(F_2') there are non-negative continuous almost periodic solutions $\widetilde{\beta}_{ij}(t)$ such that $\widetilde{\beta}_{ij}(t) = \sup_{u\neq v} \left| \frac{\widetilde{g}_{ij}(t,u) - \widetilde{g}_{ij}(t,v)}{u-v} \right|$, for all $u, v \in R, u \neq v, i = 1, 2, \cdots, n$ $j = 1, 2, \cdots, m$;

(F_3') There are positive constants $p \geq 1$ and ξ_{ij}, $i = 1, 2, \cdots, n$, $j = 1, 2, \cdots, m$, such that

$$\max_{(i,j)} \sup_{t\in R} \left\{ \frac{\sum_{\widetilde{C}^{kl}\in N_r(i,j)} \widetilde{C}_{ij}^{kl}(t)(\widetilde{\gamma}_{ij}(t)\xi_{ij} + \widetilde{R}_0(p-1)\xi_{ij} + \widetilde{R}_0\xi_{kl}\widetilde{\beta}_{ij}^p(t))}{\widetilde{a}_{ij}(t)\xi_{ij}} \right\} < 1,$$

where $\widetilde{R}_0 = \max_{(i,j)}\{\xi_{ij}\}\widetilde{R}$, $\widetilde{R} = \max_{(i,j)} \left\{ \frac{\overline{I}_{ij}}{\xi_{ij}\underline{\widetilde{a}}_{ij}} \right\} \Big/ (1-\theta)$, $\overline{I}_{ij} = \max_{t\in R} |I_{ij}(t)|$, $\underline{\widetilde{a}}_{ij} = \min_{t\in R} \widetilde{a}_{ij}(t)$,
$\max_{(i,j)} \sup_{t\in R} \left\{ \frac{\widetilde{\gamma}_{ij}(t) \sum_{\widetilde{C}^{kl}\in N_r(i,j)} |\widetilde{C}_{ij}^{kl}(t)|}{\widetilde{a}_{ij}(t)} \right\} = \theta < 1.$

Let $\xi_{ij}, i = 1, 2, \cdots, n, j = 1, 2, \cdots, m$ be positive constants, $\psi(t)$ be continuous almost periodic solution. Make the change of variables as that of (1.2). We have the unique almost periodic solution from system (3.8)

$$y^{\psi(t)}(t) = \left\{ \int_{-\infty}^{t} e^{-\int_s^t d_{ij}(u,\psi_{ij}(u))\mathrm{d}u} - \right.$$
$$\left. -\left[\sum_{C^{kl}\in N_r(i,j)} C_{ij}^{kl}(s) f_{ij}(s, \xi_{kl}\psi_{kl}(s-\tau_{kl}(s)))\psi_{ij}(s) + \xi_{ij}^{-1} I_{ij}(s) \right] \mathrm{d}s \right\}.$$

Similar to the proof of Theorem 2.1 and Theorem 3.1 of this paper, we obtain the following theorem.

Theorem 3.2. Assume (F_1')-(F_3') hold. Then the system (3.8) has exactly one almost periodic solution, which is globally exponentially stable.

Take $p = 1$ in Theorem 3.3 and obtain the following corollary:

Corollary 3.2. Assume (F_1'), (F_2') hold and

(F_3'') There are positive constants ξ_{ij}, $i = 1, 2, \cdots, n$, $j = 1, 2, \cdots, m$, such that

$$\max_{(i,j)} \sup_{t\in R} \left\{ \frac{\sum_{\widetilde{C}^{kl}\in N_r(i,j)} \widetilde{C}_{ij}^{kl}(t)(\widetilde{\gamma}_{ij}(t)\xi_{ij} + \widetilde{R}_0\xi_{kl}\widetilde{\beta}_{ij}(t))}{\widetilde{a}_{ij}(t)\xi_{ij}} \right\} < 1,$$

where $\widetilde{R}_0$ is the same as that in (F_3').

Then the system (3.8) has exactly one almost periodic solution, which is globally exponentially stable.

Remark 3. From (3.3), it is obvious that the convergent index ϵ is more precise than the estimation of convergent index in [1, 3, 4, 5, 6, 7, 8].

4 Application

In this Section, we give an example to illustrate the obtained results. Consider the following generalized SICNNs with time-varying delays and coefficients

$$x'_{ij}(t) = -a_{ij}(t, x_{ij}(t)) + \sum_{C^{kl}\in N_r(i,j)} C^{kl}_{ij}(t) g_{ij}(t, x_{kl}(t - \tau_{kl}(t))) x_{ij}(t) + I_{ij}(t), \quad (4.1)$$

where $r = 1$. Take $a_{11}(t, x) = a_{13}(t, x) = a_{21}(t, x) = a_{32}(t, x) = 4x + \sin x + x \sin t$, $a_{12}(t, x) = a_{23}(t, x) = a_{31}(t, x) = a_{33}(t, x) = 5x - \sin x + x \cos t$, $a_{22}(t, x) = 3x + \cos x - x \sin t$, $g_{ij}(t, x) = 0.1 \sin x$, $\tau_{ij}(t) = (\cos t)^2$, and

$$\begin{bmatrix} C_{11}(t) & C_{12}(t) & C_{13}(t) \\ C_{21}(t) & C_{22}(t) & C_{23}(t) \\ C_{31}(t) & C_{32}(t) & C_{33}(t) \end{bmatrix} = \begin{bmatrix} 0.2\cos t & 0.4\sin t & 0.3\cos t \\ 0.6\cos t & 0 & 0.5\sin t \\ 0.5\sin t & 0.6\cos t & 0.5\sin t \end{bmatrix},$$

$$\begin{bmatrix} I_{11}(t) & I_{12}(t) & I_{13}(t) \\ I_{21}(t) & I_{22}(t) & I_{23}(t) \\ I_{31}(t) & I_{32}(t) & I_{33}(t) \end{bmatrix} = \begin{bmatrix} 1.5\cos t & 2\sin t & 2\cos t \\ 3.5\cos t & 4\cos t & 2\sin t \\ 5\sin t & 3\cos t & 4\sin t \end{bmatrix}.$$

Then $\omega = 2\pi$, $\mu_{11}(t) = \mu_{13}(t) = \mu_{21}(t) = \mu_{32}(t) = 3 + \sin t$, $\mu_{12}(t) = \mu_{23}(t) = \mu_{31}(t) = \mu_{33}(t) = 4 + \cos t$, $\mu_{22}(t) = 2 - \sin t$, $\gamma_{ij}(t) = \beta_{ij}(t) = 0.1$, $\sum_{C^{kl}\in N_r(1,1)} |C^{kl}_{11}(t)| = 0.8|\cos t| + 0.4|\sin t|$, $\sum_{C^{kl}\in N_r(1,2)} C^{kl}_{12}(t) = 1.1|\cos t| + 0.9|\sin t|$, $\sum_{C^{kl}\in N_r(1,3)} |C^{kl}_{13}(t)| = 0.3|\cos t| + 0.9|\sin t|$, $\sum_{C^{kl}\in N_r(2,1)} |C^{kl}_{21}(t)| = 1.4|\cos t| + 0.9|\sin t|$, $\sum_{C^{kl}\in N_r(2,2)} |C^{kl}_{22}(t)| = 1.7|\cos t| + 1.9|\sin t|$, $\sum_{C^{kl}\in N_r(2,3)} |C^{kl}_{23}(t)| = 0.9|\cos t| + 1.4|\sin t|$, $\sum_{C^{kl}\in N_r(3,1)} |C^{kl}_{31}(t)| = 1.2|\cos t| + 0.5|\sin t|$, $\sum_{C^{kl}\in N_r(3,2)} |C^{kl}_{32}(t)| = 1.2|\cos t| + 1.5|\sin t|$, $\sum_{C^{kl}\in N_r(3,3)} |C^{kl}_{33}(t)| = 0.6|\cos t| + |\sin t|$.

Take $\xi_{ij} = 1$, $i, j = 1, 2, 3$. Computing by MATLAB, we have $\theta \approx 0.22358 < 1$, $R_0 \approx 2.1466$, and $\kappa \approx 0.7035 < 1$. It's easy to check that all the conditions in Corollary 3.1 are hold. Therefore, system (4.1) has a unique 2π-periodic solution $x^*(t)$ with $\|x^*\| \leq R_0$, which is globally exponentially stable. The numerical simulations with the following initial conditions:

$$(x_{11}(s), x_{12}(s), x_{13}(s), x_{21}(s), x_{22}(s), x_{23}(s), x_{31}(s), x_{32}(s), x_{33}(s))^T = (1, 2, 3, 4, 5, 6, 7, 8, 9)^T$$

and

$$(x_{11}(s), x_{12}(s), x_{13}(s), x_{21}(s), x_{22}(s), x_{23}(s), x_{31}(s), x_{32}(s), x_{33}(s))^T = (-1, -2, -3, -4, -5, -6, -7, -8, -9)^T$$

for $s \in [-1, 0]$. For the trajectories of $x_{ij}(t), i, j = 1, 2, 3$, please see the below Figure.

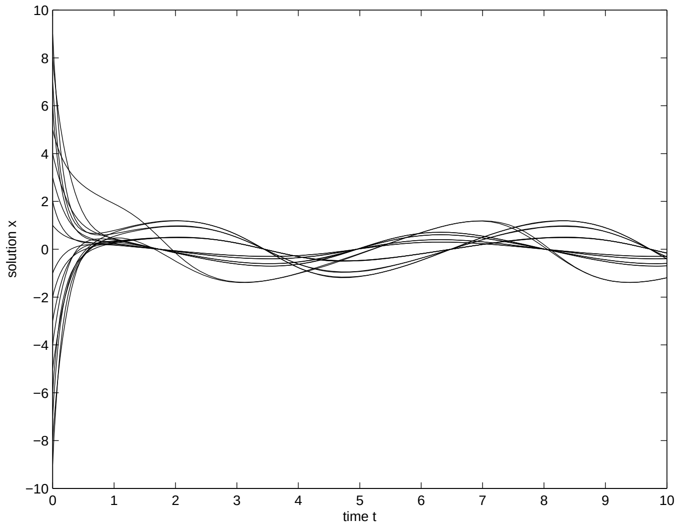

Fig. : Trajectories of $x_{ij}(t), i, j = 1, 2, 3$ for system (4.1).

Remark 4. The system (4.1) is a simple generalized SICNN with delays and time-varying coefficients. Obviously, $a_{ij}(t, x), i, j = 1, 2, 3$, are non-linear about x, hence none of the the results in [1, 3, 4, 5, 6, 7, 8] and references cited therein can be applied to (4.1). Moreover, the periodic solution $x^*(t)$ satisfies $\|x^*\| \leq R_0$, which has nothing to do with the initial value of (4.1). Hence, the results of this paper generalize and complement some previously known results.

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