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STRONG CONVERGENCE OF AN IMPLICIT ITERATION PROCESS FOR TWO ASYMPTOTICALLY NONEXPANSIVE MAPPINGS IN BANACH SPACES

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Abstract

The purpose of this paper is to introduce an implicit iteration process for approximating common fixed points of two asymptotically nonexpansive mappings and to prove strong convergence theorems in uniformly convex Banach spaces.

1. Introduction

Let K be a nonempty closed convex subset of a real normed linear space E, and $T : K \to K$ be a mapping. T is said to be *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$, for all $x, y \in K$; T is said to be *asymptotically nonexpansive* if there exists a real sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ such that $\|T^n x - T^n y\| \leq k_n \|x - y\|$ for all $x, y \in K$ and all positive integer $n \geq 1$. Denote by $F(T)$ the set of fixed points of T, that is, $F(T) = \{x \in K : Tx = x\}$. Throughout this paper, we always assume that $F(T) \neq \phi$.

In 1972, the class of asymptotically nonexpansive mappings was introduced by Goebel and Kirk [3] as an important generalization of the class of nonexpansive mappings. Since then, many authors used different iteration processes to approximate the fixed points of asymptotically nonexpansive mappings, such

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as Mann and Ishikawa iteration processes, CQ method, Viscosity approximation method and some implicit or explicit iteration methods [1, 4, 5, 7, 10].

In 2001, Xu and Ori [11] introduced the following implicit iteration process for a finite family of nonexpansive mappings $\{T_j : j \in J\}$ (here $J = \{1, 2, \cdots, N\}$). From an initial point $x_0 \in K$, $\{x_n\}$ is define as follows:

$$x_n = \alpha_n x_{n-1} + (1 - \alpha_n) T_n x_n, \quad n \geq 1, \tag{1.1}$$

where $T_n = T_{(\mod N)}$ (here the $\mod N$ function takes values in J), $\{\alpha_n\}$ is a real sequence in (0,1).

In 2004, Sun [9] extended the process (1.1) to a process for a finite family of asymptotically quasi-nonexpansive mappings $\{T_j : j \in J\}$, and an initial point $x_0 \in K$, which is defined as follows:

$$x_n = \alpha_n x_{n-1} + (1 - \alpha_n) T_i^k x_n, \quad n \geq 1, \tag{1.2}$$

where $n = (k - 1)N + i$, $1 \leq i \leq N$, $\{\alpha_n\}$ is a real sequence in (0,1). In addition, Zhao et al. [12] introduced a new implicit iteration scheme:

$$x_n = \alpha_n x_{n-1} + \beta_n T_n x_{n-1} + \gamma_n T x_n, \quad n \geq 1, \tag{1.3}$$

for fixed points of a nonexpansive mapping T in Banach space.

Recently, Zhao and Wang [13] introduced the following implicit iteration scheme for fixed points of an asymptotically nonexpansive mapping T in Banach spaces. For arbitrarily chosen $x_0 \in K$, $\{x_n\}$ is define as follows:

$$x_n = \alpha_n x_{n-1} + \beta_n T_n^{n-1} x_{n-1} + \gamma_n T^n x_n, \quad n \geq 1, \tag{1.4}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ are three real sequences in [0,1] satisfying $\alpha_n + \beta_n + \gamma_n = 1$ for $n \geq 1$. And they obtained the following strong convergence theorems.

Theorem 1.1. Let E be a real uniformly convex Banach space, K be a nonempty closed convex subset of E. Suppose that $T : K \to K$ is an asymptotically nonexpansive mapping with sequence $\{k_n\} \subset [1, \infty)$ such that $\lim_{n\to\infty} k_n = 1$, $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. Let $\{x_n\}$ be generated by (1.4) and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ be three real sequences in [0,1] satisfying $\alpha_n + \beta_n + \gamma_n = 1$, $\gamma_n k_n < 1$ for each integer $n \geq 1$ and $s \leq \gamma_n \leq 1 - s$ for some $s \in (0, 1)$. If T satisfies condition (A) and $F(T) = \{x \in K : Tx = x\} \neq \phi$, then $\{x_n\}$ converges strongly to a fixed point of T.

Theorem 1.2. Let E be a real uniformly convex Banach space, K be a nonempty closed convex subset of E. Suppose that $T : K \to K$ is an asymptotically nonexpansive mapping with sequence $\{k_n\} \subset [1, \infty)$ such that $\lim_{n\to\infty} k_n = 1$, $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. Let $\{x_n\}$ be generated by (1.4) and

$\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ be three real sequences in [0,1] satisfying $\alpha_n+\beta_n+\gamma_n=1$, $\gamma_n k_n<1$ for each integer $n\geq 1$ and $s\leq\gamma_n\leq 1-s$ for some $s\in(0,1)$. If T is semi-compact and $F(T)=\{x\in K: Tx=x\}\neq\phi$, then $\{x_n\}$ converges strongly to a fixed point of T.

Now, we introduce an implicit iteration process which can be viewed as an extension for two asymptotically nonexpansive mappings of implicit iteration process of Zhao and Wang [13]. This implicit iteration process is defined as follows:

Let E be a Banach space, K be a nonempty closed convex subset of E and $T,S: K\to K$ be two asymptotically nonexpansive mappings. Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\alpha'_n\}$, $\{\beta'_n\}$, $\{\gamma'_n\}$ be real sequences in [0,1) satisfying $\alpha_n+\beta_n+\gamma_n=\alpha'_n+\beta'_n+\gamma'_n=1$. We have the following iteration process: for arbitrarily chosen $x_0\in K$,

$$x_n=\alpha_n x_{n-1}+\beta_n T^{n-1}x_{n-1}+\gamma_n T^n[\alpha'_n x_n+\beta'_n S^{n-1}x_{n-1}+\gamma'_n S^n x_n],\quad n\geq 1.$$

Putting $y_n=\alpha'_n x_n+\beta'_n S^{n-1}x_{n-1}+\gamma'_n S^n x_n$, we have the following composite iterative scheme:

$$\begin{aligned} x_n &= \alpha_n x_{n-1}+\beta_n T^{n-1}x_{n-1}+\gamma_n T^n y_n, \\ y_n &= \alpha'_n x_n+\beta'_n S^{n-1}x_{n-1}+\gamma'_n S^n x_n,\quad n\geq 1. \end{aligned} \tag{1.5}$$

We remark that the implicit iterative process (1.5) is more general than the algorithms (1.3) and (1.4), and includes the algorithms (1.3) and (1.4) as the special cases.

The purpose of this paper is to establish strong convergence theorems of the implicit iteration process (1.5) for two asymptotically nonexpansive mappings in uniformly convex Banach spaces. Our results improve and extend the corresponding ones announced by Zhao et al. [12] and Zhao and Wang [13].

2. Preliminaries

Let E be a Banach space, K be a nonempty closed convex subset of E and $T,S: K\to K$ be two asymptotically nonexpansive mappings with sequences $\{k_n\}$, $\{r_n\}\subset[1,\infty)$ and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\alpha'_n\}$, $\{\beta'_n\}$, $\{\gamma'_n\}$ be numbers in [0,1] satisfying $\alpha_n+\beta_n+\gamma_n=\alpha'_n+\beta'_n+\gamma'_n=1$. For arbitrarily chosen $x_0\in K$, define a mapping $W: K\to K$ by $Wx=\alpha x_0+\beta T^{n-1}x_0+\gamma T^n[\alpha' x+$

$\beta' S^{n-1}x_0 + \gamma' S^n x]$. Thus for any $x, y \in K$, we have

$$\begin{aligned} \|Wx - Wy\| &= \|\gamma T^n[\alpha' x + \beta' S^{n-1}x_0 + \gamma' S^n x] \\ &\quad -\gamma T^n[\alpha' y + \beta' S^{n-1}x_0 + \gamma' S^n x]\| \\ &\leq \gamma k_n \|\alpha'(x-y) + \gamma'(S^n x - S^n y)\| \\ &\leq \gamma k_n(\alpha'\|x-y\| + \gamma' r_n \|x-y)\|) \\ &= \gamma k_n(\alpha' + \gamma' r_n)\|x-y\|. \end{aligned}$$

If $\gamma k_n(\alpha' + \gamma' r_n) < 1$, then W is a contraction. By Banach contraction mapping principle, W has a unique fixed point. Thus, if $\gamma k_n(\alpha' + \gamma' r_n) < 1$, the implicit iteration processes (1.5) can be employed for the approximation of common fixed points of asymptotically nonexpansive mappings T and S.

Let E be a real normed linear space. The modulus of convexity of E is the function $\delta_E : (0, 2] \to [0, 1]$ defined by

$$\delta_E(\epsilon) = inf\{1 - \|\frac{x+y}{2}\| : \|x\| = \|y\| = 1, \epsilon = \|x-y\|\}.$$

E is called *uniformly convex* if $\delta_E(\epsilon) > 0$ for all $\epsilon \in (0, 2]$. Let K be a nonempty closed subset of a real Banach space E. $T : K \to K$ is said to be *semi-compact* if for any bounded sequence $\{x_n\}$ with $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $\{x_{n_j}\}$ converges strongly to $p \in K$.

Two mappings $T, S : K \to E$ with $F := F(T) \bigcap F(S) = \{x \in K : Tx = Sx = x\} \neq \phi$ are said to *satisfy condition* (A') [2], if there exists a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$, $f(t) > 0$ for all $t > 0$ such that

$\|x - Tx\| \geq f(d(x, F))$ or $\|x - Sx\| \geq f(d(x, F))$,

for all $x \in K$, where $d(x, F) = inf\{\|x - q\| : q \in F\}$.

In what follows, we will state the following useful lemmas:

Lemma 2.1.[6] Let $\{\alpha_n\}$, $\{a_n\}$ and $\{b_n\}$ be sequences of nonnegative real numbers satisfying

$$\alpha_{n+1} \leq (1 + a_n)\alpha_n + b_n, \quad \forall n \geq 1.$$

If $\sum_{n=1}^{\infty} a_n < \infty$ and $\sum_{n=1}^{\infty} b_n < \infty$, then $\lim_{n\to\infty} \alpha_n$ exists in $\mathbf{R}$. If, in addition, $\{\alpha_n\}$ has a subsequence which converges to zero, then $\lim_{n\to\infty} \alpha_n = 0$.

Lemma 2.2.[1] Let E be a real uniformly convex Banach space, K be a nonempty closed convex subset of E and $T : K \to E$ be an asymptotically nonexpansive mapping. Then $I - T$ is demiclosed at zero, that is, for each sequence $\{x_n\}$ in K, if $\{x_n\}$ converges weakly to $q \in K$ and $\{(I - T)x_n\}$ converges strongly to 0, then $(I - T)q = 0$.

Lemma 2.3.[8] Let E be a real uniformly convex Banach space and let a, b be two constants with $0 < a < b < 1$. Suppose that $\{t_n\} \subset [a, b]$ is a real sequence and $\{x_n\}$, $\{y_n\}$ are two sequences in E. Then the conditions

$lim_{n\to\infty}\|t_n x_n + (1-t_n)y_n\| = d$, $\limsup_{n\to\infty} \|x_n\| \le d$, $\limsup_{n\to\infty} \|x_n\| \le d$ imply that $\lim_{n\to\infty} \|x_n - y_n\| = 0$, where $d \ge 0$ is a constant.

3. Main Results

Lemma 3.1 Let E be a real uniformly convex Banach space, K be a nonempty closed convex subset of E. Suppose that $T, S : K \to K$ are two asymptotically nonexpansive mappings with sequences $\{k_n\}$, $\{r_n\} \subset [1, \infty)$ such that $\lim_{n\to\infty} k_n = \lim_{n\to\infty} r_n = 1$ and $\sum_{n=1}^{\infty}(k_n - 1) < \infty$, $\sum_{n=1}^{\infty}(r_n - 1) < \infty$. Let $\{x_n\}$ be generated by (1.5), where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\alpha'_n\}$, $\{\beta'_n\}$, and $\{\gamma'_n\}$ are real sequences in [0,1) satisfying:

(i) $\alpha_n + \beta_n + \gamma_n = \alpha'_n + \beta'_n + \gamma'_n = 1$, $\gamma k_n(\alpha'_n + \gamma'_n r_n) < 1$ for each integer $n \ge 1$;

(ii) $s \le \alpha_n, \beta_n, \gamma_n, \alpha'_n, \beta'_n, \gamma'_n \le 1 - s$, for some $s \in (0, 1)$.

If $F := F(T) \bigcap F(S) \ne \phi$, then

(1) $\lim_{n\to\infty} \|x_n - p\|$ exists for each $p \in F$.

(2) $\lim_{n\to\infty} \|x_n - Tx_n\| = \lim_{n\to\infty} \|x_n - Sx_n\| = 0$.

Proof. (1) Let $p \in F$. Set $k_n = 1 + u_n$, $r_n = 1 + v_n$. Since $\sum_{n=1}^{\infty}(k_n - 1) < \infty$ and $\sum_{n=1}^{\infty}(r_n - 1) < \infty$, so $\sum_{n=1}^{\infty} u_n < \infty$, $\sum_{n=1}^{\infty} v_n < \infty$. Using (1.5), we have

$$\begin{aligned} \|y_n - p\| &\le \alpha'_n\|x_n - p\| + \beta'_n r_{n-1}\|x_{n-1} - p\| + \gamma'_n r_n\|x_n - p\| \\ &= (\alpha'_n + \gamma'_n r_n)\|x_n - p\| + \beta'_n r_{n-1}\|x_{n-1} - p\|, \end{aligned} \tag{3.1}$$

and

$$\begin{aligned} \|x_n - p\| &\le \alpha_n\|x_{n-1} - p\| + \beta_n k_{n-1}\|x_{n-1} - p\| + \gamma_n k_n\|y_n - p\| \\ &= (\alpha_n + \beta_n k_{n-1})\|x_{n-1} - p\| + \gamma_n k_n\|y_n - p\|. \end{aligned} \tag{3.2}$$

Substituting (3.1) into (3.2), we have

$$\begin{aligned} \|x_n - p\| &\le \alpha_n\|x_{n-1} - p\| + \beta_n k_{n-1}\|x_{n-1} - p\| + \\ &\quad +\gamma_n k_n[(\alpha'_n + \gamma'_n r_n)\|x_n - p\| + \beta'_n r_{n-1}\|x_{n-1} - p\|] \\ &= (\alpha_n + \beta_n k_{n-1} + \gamma_n k_n \beta'_n r_{n-1})\|x_{n-1} - p\| + \\ &\quad +\gamma_n k_n(\alpha'_n + \gamma'_n r_n)\|x_n - p\|. \end{aligned} \tag{3.3}$$

which leads to

$$[1 - \gamma_n k_n(\alpha'_n + \gamma'_n r_n)]\|x_n - p\| \le (\alpha_n + \beta_n k_{n-1} + \gamma_n k_n \beta'_n r_{n-1})\|x_{n-1} - p\|.$$

Since $\gamma_n k_n(\alpha'_n + \gamma'_n r_n) < 1$, then $1 - \gamma_n k_n(\alpha'_n + \gamma'_n r_n) > 0$, that is,

$$1 - \gamma_n(1 + u_n)(\alpha'_n + \gamma'_n(1 + v_n)) > 0$$

for all $n \geq 1$. Thus, it implies that

$$\|x_n - q\| \leq \frac{[\alpha_n + \beta_n(1 + u_{n-1}) + \gamma_n\beta'_n(1 + u_n)(1 + v_{n-1})]}{1 - \gamma_n(1 + u_n)(\alpha'_n + \gamma'_n(1 + v_n))}\|x_{n-1} - p\|. \quad (3.4)$$

By using (3.4), we have:

$$\|x_n - p\| \leq$$

$$\leq [1 + \frac{\gamma_n\gamma'_n v_n + \gamma_n u_n + \gamma_n\gamma'_n u_n v_n + \beta_n u_{n-1} + \gamma_n\beta'_n v_{n-1} + \gamma_n\beta'_n u_n v_{n-1}}{1 - \gamma_n(1 - \beta'_n) - \gamma_n\gamma'_n v_n - \gamma_n u_n(1 - \beta'_n) - \gamma_n\gamma'_n u_n v_n)}]\times$$

$$\|x_{n-1} - p\|.$$

On the other hand, since

$$\lim_{n\to\infty} \gamma_n\gamma'_n v_n = \lim_{n\to\infty} \gamma_n u_n(1 - \beta'_n) = \lim_{n\to\infty} \gamma_n\gamma'_n u_n v_n = 0,$$

for given $\frac{\epsilon_0}{3}$, $\frac{\epsilon_1}{3}$, $\frac{\epsilon_2}{3} \in (0, s)$, $\epsilon = max\{\epsilon_0, \epsilon_1, \epsilon_2\}$, there exists positive integer n_0 such that

$$\gamma_n(1 - \beta'_n) + \gamma_n\gamma'_n v_n + \gamma_n u_n(1 - \beta'_n) + \gamma_n\gamma'_n u_n v_n) \leq 1 - s + \epsilon, \quad (3.5)$$

as $n \geq n_0$. From (3.4) and (3.5), we have

$$\begin{aligned}
\|x_n - p\| &\leq (1 + \frac{\gamma_n\gamma'_n}{s - \epsilon}v_n + \frac{\gamma_n}{s - \epsilon}u_n + \frac{\gamma_n\gamma'_n}{s - \epsilon}u_n v_n + \frac{\beta_n}{s - \epsilon}u_{n-1} \\
&\quad + \frac{\gamma_n\beta'_n}{s - \epsilon}v_{n-1} + \frac{\gamma_n\beta'_n}{s - \epsilon}u_n v_{n-1})\|x_{n-1} - p\| \\
&\leq (1 + \frac{1}{s - \epsilon}v_n + \frac{1}{s - \epsilon}u_n + \frac{1}{s - \epsilon}u_n v_n + \frac{1}{s - \epsilon}u_{n-1} \\
&\quad + \frac{1}{s - \epsilon}v_{n-1} + \frac{1}{s - \epsilon}u_n v_{n-1})\|x_{n-1} - p\|. \quad (3.6)
\end{aligned}$$

From $\sum_{n=1}^{\infty} u_n < \infty$ and $\sum_{n=1}^{\infty} v_n < \infty$, we obtain that

$$\sum_{n=2}^{\infty}(\frac{1}{s - \epsilon}v_n + \frac{1}{s - \epsilon}u_n + \frac{1}{s - \epsilon}u_n v_n + \frac{1}{s - \epsilon}u_{n-1} + \frac{1}{s - \epsilon}v_{n-1} + \frac{1}{s - \epsilon}u_n v_{n-1})$$

$$< \infty.$$

Hence, it follows from (3.6) and Lemma 2.1 that $\lim_{n\to\infty} \|x_n - p\|$ exists for each $p \in F$.

(2) From (1), we know that $\lim_{n\to\infty} \|x_n - p\|$ exists for each $p \in F$. We suppose that $\lim_{n\to\infty} \|x_n - p\| = d$, that is,

$$\begin{aligned}
\lim_{n\to\infty} \|x_n - p\| &= \lim_{n\to\infty} \|\alpha_n(x_{n-1} - p) + \beta_n(T^{n-1}x_{n-1} - p) \\
&\quad +\gamma_n(T^n y_n - p)\| \\
&= \lim_{n\to\infty} \|(1-\gamma_n)[\frac{\alpha_n}{1-\gamma_n}(x_{n-1} - p) \\
&\quad +\frac{\beta_n}{1-\gamma_n}(T^{n-1}x_{n-1} - p)] + \gamma_n(T^n y_n - p)\| = d. \qquad (3.7)
\end{aligned}$$

From (3.1) and (3.7), we have

$$\begin{aligned}
\|T^n y_n - p\| &\leq k_n\|y_n - p\| \\
&\leq k_n(\alpha'_n\|x_n - p\| + \beta'_n r_{n-1}\|x_{n-1} - p\| + \gamma'_n r_n\|x_n - p\|) \\
&= k_n[(\alpha'_n + \gamma'_n r_n)\|x_n - p\| + \beta'_n r_{n-1}\|x_{n-1} - p\|] \\
&= k_n[(\alpha'_n + \gamma'_n + \gamma'_n v_n)\|x_n - p\| + \beta'_n(1 + v_{n-1})\|x_{n-1} - p\|] \\
&= k_n[(1 - \beta'_n + \gamma'_n v_n)\|x_n - p\| + (\beta'_n + \beta'_n v_{n-1})\|x_{n-1} - p\|] \\
&= k_n[\|x_n - p\| + \beta'_n(\|x_{n-1} - p\| - \|x_n - p\|) + \gamma'_n v_n\|x_n - p\| + \\
&\quad +\beta'_n v_{n-1}\|x_{n-1} - p\|]. \qquad (3.8)
\end{aligned}$$

Taking lim sup on both sides in the inequality (3.8), we obtain

$$\limsup_{n\to\infty} \|T^n y_n - p\| \leq \limsup_{n\to\infty} \|y_n - p\| \leq d. \qquad (3.9)$$

On the other hand, by using (3.7) we obtain

$$\begin{aligned}
&\limsup_{n\to\infty} \|\tfrac{\alpha_n}{1-\gamma_n}(x_{n-1} - p) + \tfrac{\beta_n}{1-\gamma_n}(T^{n-1}x_{n-1} - p)\| \\
&\leq \limsup_{n\to\infty}(\tfrac{\alpha_n}{1-\gamma_n}\|x_{n-1} - p\| + \tfrac{\beta_n}{1-\gamma_n}k_{n-1}\|x_{n-1} - p)\|) \\
&= \limsup_{n\to\infty}(\tfrac{\alpha_n+\beta_n(1+u_{n-1})}{1-\gamma_n})\|x_{n-1} - p\| \\
&= \limsup_{n\to\infty}(1+\tfrac{\beta_n}{1-\gamma_n}u_{n-1})\|x_{n-1} - p\| = d. \qquad (3.10)
\end{aligned}$$

By using (3.7), (3.9), (3.10) and Lemma 2.3, we obtain that

$$\lim_{n\to\infty} \|\frac{\alpha_n}{1-\gamma_n}x_{n-1} + \frac{\beta_n}{1-\gamma_n}T^{n-1}x_{n-1} - T^n y_n\| = 0.$$

Thus, from (1.5), we have

$$\lim_{n\to\infty} \|x_n - T^n y_n\| = 0. \qquad (3.11)$$

It follows from (3.2) that

$$\|x_n - p\| - (\alpha_n + \beta_n + \beta_n u_{n-1})\|x_{n-1} - p\| \leq \gamma_n k_n\|y_n - p\|,$$

$(\alpha_n+\beta_n)[\|x_n-p\|-\|x_{n-1}-p\|]+\gamma_n\|x_n-p\|-\beta_n u_{n-1}\|x_{n-1}-p\| \leq \gamma_n k_n\|y_n-p\|$
and this implies that

$$\frac{\alpha_n+\beta_n}{\gamma_n}[\|x_n-p\|-\|x_{n-1}-p\|]+\|x_n-p\|-\frac{\beta_n}{\gamma_n}u_{n-1}\|x_{n-1}-p\| \leq k_n\|y_n-p\|. \tag{3.12}$$

Taking lim sup on both sides in the inequality (3.12), we obtain

$$\liminf_{n\to\infty}\|x_n-p\| \leq \liminf_{n\to\infty} k_n\|y_n-p\|$$

and so

$$\liminf_{n\to\infty}\|y_n-p\| \geq d. \tag{3.13}$$

Combining (3.9) and (3.13), we have

$$\lim_{n\to\infty}\|y_n-p\| = d.$$

It implies that

$$\begin{aligned}\lim_{n\to\infty}\|y_n-p\| &= \lim_{n\to\infty}\|\beta_n'(S^{n-1}x_{n-1}-p)+\gamma_n'(S^n x_n-p)\\ &\quad +\alpha_n'(x_n-p)\|\\ &= \lim_{n\to\infty}\|(1-\alpha_n')[\frac{\beta_n'}{1-\alpha_n'}(S^{n-1}x_{n-1}-p)\\ &\quad +\frac{\gamma_n'}{1-\alpha_n'}(S^n x_n-p)]+\alpha_n'(x_n-p)\| = d. \end{aligned} \tag{3.14}$$

We know that $\limsup_{n\to\infty}\|x_n-p\| \leq d$. Put $w_n = max\{v_{n-1}, v_n\}$ for $n \geq 2$. Since $w_n = \frac{v_n+v_{n-1}+|v_n-v_{n-1}|}{2}$ and $\lim_{n\to\infty} v_n = 0$, we have $\lim_{n\to\infty} w_n = 0$. Thus, from (3.14), we have

$$\begin{aligned}&\limsup_{n\to\infty}\|\tfrac{\beta_n'}{1-\alpha_n'}(S^{n-1}x_{n-1}-p)+\tfrac{\gamma_n'}{1-\alpha_n'}(S^n x_n-p)\|\\ &\leq \limsup_{n\to\infty}(\tfrac{\beta_n'}{1-\alpha_n'}r_{n-1}\|x_{n-1}-p\|+\tfrac{\gamma_n'}{1-\alpha_n'}r_n\|x_n-p)\|)\\ &\leq \limsup_{n\to\infty}(\tfrac{\beta_n'(1+w_n)}{1-\alpha_n'}\|x_{n-1}-p\|+\tfrac{\gamma_n'(1+w_n)}{1-\alpha_n'}\|x_n-p)\|)\\ &\leq \limsup_{n\to\infty}[(1+w_n)\tfrac{\beta_n'(\|x_{n-1}-p\|-\|x_n-p\|)+(1-\alpha_n')\|x_n-p\|}{1-\alpha_n'}] = d. \end{aligned} \tag{3.15}$$

By using (3.14), (3.15), Lemma 2.3 and $\limsup_{n\to\infty}\|x_n-p\| \leq d$, we obtain that

$$\lim_{n\to\infty}\|\frac{\beta_n'}{1-\alpha_n'}S^{n-1}x_{n-1}+\frac{\gamma_n'}{1-\alpha_n'}S^n x_n-x_n\| = 0,$$

which means that

$$\lim_{n\to\infty}\|y_n-x_n\| = 0. \tag{3.16}$$

In addition, since

$$\begin{aligned}\lim_{n\to\infty}\|x_n-p\| &= \lim_{n\to\infty}\|\alpha_n(x_{n-1}-p)+\beta_n(T^{n-1}x_{n-1}-p)\\ &\quad+\gamma_n(T^ny_n-p)\|\\ &= \lim_{n\to\infty}\|(1-\beta_n)[\frac{\alpha_n}{1-\beta_n}(x_{n-1}-p)\\ &\quad+\frac{\gamma_n}{1-\beta_n}(T^ny_n-p)]+\beta_n(T^{n-1}x_{n-1}-p)\|=d, \qquad (3.17)\end{aligned}$$

so, we have

$$\limsup_{n\to\infty}\|T^{n-1}x_{n-1}-p\|\le\limsup_{n\to\infty}k_{n-1}\|x_{n-1}-p\|=d, \qquad (3.18)$$

and

$$\begin{aligned}&\limsup_{n\to\infty}\|\tfrac{\alpha_n}{1-\beta_n}(x_{n-1}-p)+\tfrac{\gamma_n}{1-\beta_n}(T^ny_n-p)\|\\ &\quad\le\limsup_{n\to\infty}(\tfrac{\alpha_n}{1-\beta_n}\|x_{n-1}-p\|+\tfrac{\gamma_n}{1-\beta_n}\|T^ny_n-p)\|)\\ &\quad\le\limsup_{n\to\infty}(\tfrac{\alpha_n}{1-\beta_n}\|x_{n-1}-p\|+\tfrac{\gamma_n}{1-\beta_n}k_n\|y_n-p)\|)\\ &\quad=\limsup_{n\to\infty}(\tfrac{\alpha_n\|x_{n-1}-p\|+\gamma_n\|y_n-p\|+\gamma_nu_n\|y_n-p\|}{1-\beta_n})\\ &\quad=\limsup_{n\to\infty}(\tfrac{\gamma_n(\|y_n-p\|-\|x_{n-1}-p\|)+(1-\beta_n)\|x_{n-1}-p\|+\gamma_nu_n\|y_n-p\|}{1-\beta_n})=d. \qquad (3.19)\end{aligned}$$

By the inequalities (3.17), (3.18), (3.19) and using Lemma 2.3, we get

$$\lim_{n\to\infty}\|\frac{\alpha_n}{1-\beta_n}x_{n-1}+\frac{\gamma_n}{1-\beta_n}T^ny_n-T^{n-1}x_{n-1}\|=0,$$

which implies that

$$\lim_{n\to\infty}\|x_n-T^{n-1}x_{n-1}\|=0. \qquad (3.20)$$

By using (3.11), (3.16) and (3.20), we obtain that

$$\lim_{n\to\infty}\|x_n-x_{n-1}\|=0. \qquad (3.21)$$

Using the same method and Lemma 2.3 for the equality (3.14), we have

$$\lim_{n\to\infty}\|y_n-S^nx_n\|=0, \qquad (3.22)$$

and

$$\lim_{n\to\infty}\|y_n-S^{n-1}x_{n-1}\|=0. \qquad (3.23)$$

Therefore, we have

$$\begin{aligned}
\|x_n - Tx_n\| &\leq \|x_n - T^n y_n\| + \|T^n y_n - T^n x_n\| + \|T^n x_n - Tx_n\| \\
&\leq \|x_n - T^n y_n\| + k_n\|y_n - x_n\| + k_1\|T^{n-1}x_n - x_n\| \\
&\leq \|x_n - T^n y_n\| + k_n\|y_n - x_n\| \\
&\quad +k_1(\|T^n x_n - T^{n-1}x_{n-1}\| + \|T^{n-1}x_{n-1} - x_n\|) \\
&\leq \|x_n - T^n y_n\| + k_n\|y_n - x_n\| \\
&\quad +k_1(k_{n-1}\|x_n - x_{n-1}\| + \|T^{n-1}x_{n-1} - x_n\|)
\end{aligned}$$

and it follows from (3.11), (3.16), (3.20) and (3.21) that

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0.$$

Moreover,

$$\begin{aligned}
\|x_n - Sx_n\| &\leq \|x_n - y_n\| + \|y_n - S^n x_n\| + \|S^n x_n - Sx_n\| \\
&\leq \|x_n - y_n\| + \|y_n - S^n x_n\| + r_1\|S^{n-1}x_n - x_n\| \\
&\leq \|x_n - y_n\| + \|y_n - S^n x_n\| \\
&\quad +r_1(\|S^n x_n - S^{n-1}x_{n-1}\| + \|S^{n-1}x_{n-1} - y_n\| + \|y_n - x_n\|) \\
&\leq \|x_n - y_n\| + \|y_n - S^n x_n\| \\
&\quad +r_1(\|x_n - x_{n-1}\| + \|S^{n-1}x_{n-1} - y_n\| + \|y_n - x_n\|),
\end{aligned}$$

and by using (3.16), (3.21), (3.22) and (3.23), we have

$$\lim_{n\to\infty} \|x_n - Sx_n\| = 0.$$

This completes the proof.

Remark 3.2. Lemma 3.1 generalizes Lemma 3.1 of Wang and Zhao [13] to two asymptotically nonexpansive mappings. In addition, if Opial's condition of Theorem 2.1 of [12] is removed, Lemma 3.1 improves Theorem 2.1 of Zhao et al. [12].

Theorem 3.3. Let E be a real uniformly convex Banach space, K be a nonempty closed convex subset of E. Suppose that $T, S : K \to K$ are two asymptotically nonexpansive mappings with sequences $\{k_n\}$, $\{r_n\} \subset [1, \infty)$ such that $\lim_{n\to\infty} k_n = \lim_{n\to\infty} r_n = 1$ and $\sum_{n=1}^{\infty}(k_n - 1) < \infty$, $\sum_{n=1}^{\infty}(r_n - 1) < \infty$. Let $\{x_n\}$ be generated by (1.5) and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\alpha'_n\}$, $\{\beta'_n\}$, and $\{\gamma'_n\}$ be same as in Lemma 3.1. If T and S satisfy condition (A') and $F := F(T) \bigcap F(S) \neq \phi$, then $\{x_n\}$ converges strongly to a common fixed point of T and S.

Proof. From Lemma 3.1, we know that $\lim_{n\to\infty}\|x_n-p\|$ exists for each $p\in F$. Assume $\lim_{n\to\infty}\|x_n-p\|=c$ for some $c\geq 0$. If $c=0$, there is nothing to prove. So, let $c>o$ and it follows from Lemma 3.1 that

$$\begin{aligned}\|x_n-p\| &\leq (1+\frac{\gamma_n\gamma_n'}{s-\epsilon}v_n+\frac{\gamma_n}{s-\epsilon}u_n+\frac{\gamma_n\gamma_n'}{s-\epsilon}u_nv_n+\frac{\beta_n}{s-\epsilon}u_{n-1}+\frac{\gamma_n\beta_n'}{s-\epsilon}v_{n-1}\\ &\quad+\frac{\gamma_n\beta_n'}{s-\epsilon}u_nv_{n-1})\|x_{n-1}-p\|\end{aligned}$$

which leads to

$$\begin{aligned}d(x_n,F) &\leq (1+\frac{\gamma_n\gamma_n'}{s-\epsilon}v_n+\frac{\gamma_n}{s-\epsilon}u_n+\frac{\gamma_n\gamma_n'}{s-\epsilon}u_nv_n+\frac{\beta_n}{s-\epsilon}u_{n-1}\\ &\quad+\frac{\gamma_n\beta_n'}{s-\epsilon}v_{n-1}+\frac{\gamma_n\beta_n'}{s-\epsilon}u_nv_{n-1})d(x_{n-1},F). \qquad (3.24)\end{aligned}$$

Putting

$$\lambda_n=\frac{\gamma_n\gamma_n'}{s-\epsilon}v_n+\frac{\gamma_n}{s-\epsilon}u_n+\frac{\gamma_n\gamma_n'}{s-\epsilon}u_nv_n+\frac{\beta_n}{s-\epsilon}u_{n-1}+\frac{\gamma_n\beta_n'}{s-\epsilon}v_{n-1}+\frac{\gamma_n\beta_n'}{s-\epsilon}u_nv_{n-1}.$$

Since $\sum_{n=1}^{\infty}u_n<\infty$ and $\sum_{n=1}^{\infty}v_n<\infty$, $\sum_{n=2}^{\infty}\lambda_n<\infty$. By using (3.24) and Lemma 2.1 we get $\lim_{n\to\infty}d(x_n,F)$ exists. By Lemma 3.1, we have $\lim_{n\to\infty}\|x_n-Tx_n\|=\lim_{n\to\infty}\|x_n-Sx_n\|=0$. It follows from condition (A') that

$$\lim_{n\to\infty}f(d(x_n,F))\leq\lim_{n\to\infty}\|x_n-Tx_n\|=0,$$

or

$$\lim_{n\to\infty}f(d(x_n,F))\leq\lim_{n\to\infty}\|x_n-Sx_n\|=0.$$

In the both cases, $\lim_{n\to\infty}f(d(x_n,F))=0$. Since $f:[0,\infty)\to[0,\infty)$ is a nondecreasing function satisfying $f(0)=0$, $f(t)>0$ for all $t>0$, we obtain that $\lim_{n\to\infty}d(x_n,F)=0$. Next we show that $\{x_n\}$ is a Cauchy sequence in K. Taking $\sum_{n=2}^{\infty}\lambda_n=M>0$. Since $\lim_{n\to\infty}d(x_n,F)=0$, for any given $\epsilon>0$, there exists a natural number n_0 such that $d(x_n,F)<\frac{\epsilon}{3e^M}$ as $n\geq n_0$. So, we can find $q\in F$ such that $\|x_{n_0}-q\|<\frac{\epsilon}{2e^M}$. For $n\geq n_0$, from (3.6) we have

$$\begin{aligned}\|x_n-q\| &\leq (1+\lambda_n)\|x_{n-1}-q\|\\ &\leq \prod_{i=n_0}^{n}(1+\lambda_i)\|x_{n_0}-q\|\leq e^{\sum_{i=n_0}^{n}\lambda_i}\|x_{n_0}-q\|\leq e^M\|x_{n_0}-q\|.\end{aligned}$$

Therefore, for any $n,m\geq n_0$

$$\|x_n-x_m\|\leq\|x_n-q\|+\|x_m-q\|\leq e^M\|x_{n_0}-q\|+e^M\|x_{n_0}-q\|<\epsilon.$$

This shows that $\{x_n\}$ is a Cauchy sequence and so $\{x_n\}$ is convergent since E is complete. Let $\lim_{n\to\infty} x_n = p$. From Lemma 3.1, we have

$$\begin{aligned}\|p - Tp\| &\leq \|p - x_n\| + \|x_n - Tx_n\| + \|Tx_n - Tp\| \\ &\leq (1 + k_1)\|x_n - p\| + \|x_n - Tx_n\| \to 0, \quad as \quad n \to \infty.\end{aligned}$$

This implies that p is a fixed point of T. Using the same method, we can obtain that p is also a fixed point of S. So $p \in F$. This completes the proof.

Remark 3.4. Theorem 3.3 also extends the result of Wang and Zhao [13] to the case of implicit iteration process for two asymptotically nonexpansive mappings.

Theorem 3.5. Let E be a real uniformly convex Banach space, K be a nonempty closed convex subset of E. Suppose that $T, S : K \to K$ are two asymptotically nonexpansive mappings with sequences $\{k_n\}$, $\{r_n\} \subset [1, \infty)$ such that $\lim_{n\to\infty} k_n = \lim_{n\to\infty} r_n = 1$ and $\sum_{n=1}^{\infty}(k_n - 1) < \infty$, $\sum_{n=1}^{\infty}(r_n - 1) < \infty$. Let $\{x_n\}$ be generated by (1.5) and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\alpha'_n\}$, $\{\beta'_n\}$, and $\{\gamma'_n\}$ be same as in Lemma 3.1. If one of T and S is semi-compact and $F := F(T) \bigcap F(S) \neq \phi$, then $\{x_n\}$ converges strongly to a common fixed point of T and S.

Proof. From Lemma 3.1, we know that $\lim_{n\to\infty} \|x_n - Tx_n\| = \lim_{n\to\infty} \|x_n - Sx_n\| = 0$. Since one of T and S is semi-compact, then there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $\{x_{n_j}\}$ converges strongly to p. It follows from Lemma 2.2 that $p \in F$. Therefore, from Lemma 3.1, $\lim_{n\to\infty} \|x_n - p\|$ exists. Since the subsequence $\{x_{n_j}\}$ converges strongly to p, then $\{x_n\}$ converges strongly to a common fixed point $p \in F$. This completes the proof.

Remark 3.6. Since the class of asymptotically nonexpansive mappings includes the class of nonexpansive mappings, we have that Theorem 3.5 is a generalization of Theorem 3.3 of Zhao and Wang [13] and Theorem 2.2 of Zhao et al. [12].

Remark 3.7. The implicit iteration process (1.5) can be generalized for two finite families asymptotically nonexpansive mappings $\{T_j : j \in J\}$ and $\{S_j : j \in J\}$ (here $J = \{1, 2, \cdots, N\}$).

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