An. Şt. Univ. Ovidius Constanţa Vol. 18(2), 2010, 267–280

REGULARIZATION AND NEW ERROR ESTIMATES FOR A MODIFIED HELMHOLTZ EQUATION

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Abstract

We consider the following Cauchy problem for the Helmholtz equation with Dirichlet boundary conditions at $x = 0$ and $x = \pi$

$$\begin{cases} \Delta u - k^2 u = 0, (x,y) \in (0,\pi) \times (0,1) \\ u(0,y) = u(\pi,y) = 0, y \in (0,1) \\ u_y(x,0) = f(x), (x,y) \in (0,\pi) \times (0,1) \\ u(x,0) = \varphi(x), 0 < x < \pi \end{cases} \quad (1)$$

The problem is shown to be ill-posed, as the solution exhibits unstable dependence on the given data functions. Using a modified regularization method, we regularize the problem and to get some new error estimates. The numerical results show that our methods work effectively. This paper extends the work by T.Wei and H.H.Quin[8].

1 Introduction

Many physical and engineering problems in areas like geophysics and seismology require the solution of a Cauchy problem for the Laplace equation. For example, certain problems related to the search for mineral resources, which involve interpretation of the earth's gravitational and magnetic fields,

Key Words: Elliptic equation, Ill-posed problem, Cauchy problem, Contraction principle
2010 Mathematics Subject Classification: 35K05, 35K99, 47J06, 47H10
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Received: November, 2009
Accepted: January, 2010

are equivalent to the Cauchy problem for the Laplace equation. The continuation of the gravitational potential observed on the surface of the earth in a direction away from the sources of the field is again such a problem.

The Cauchy problem for the Laplace equation and for other elliptic equations is in general ill-posed in the sense that the solution, if it exists, does not depend continuously on the initial data. This is because the Cauchy problem is an initial value problem which represents a transient phenomenon in a time-like variable while elliptic equations describe steady-state processes in physical fields. A small perturbation in the Cauchy data, therefore, affects the solution largely [2, 3, 4]. Due to the severe ill-posedness of the problem, it is impossible to solve Cauchy problem of elliptic equation by using classical numerical methods and it requires special techniques, e.g., regularization strategies. In recent years, the Cauchy problems associated with the modified Helmholtz equation have been studied by using different numerical methods, such as the Landweber method with boundary element method (BEM) [14], the method of fundamental solutions (MFS) [13, 22] and so on.

For a recent paper on modified Helmholtz equation, we refer the reader to [17]. In there, the authors used quasi-reversibility method and truncation method for solving a Cauchy problem of modified Helmhotlz equation in a rectangle domain. They established the error estimates be as in the logarithmic forms (See Theorem 3.1 and Theorem 3.3). The convergence rates is too weak. The main aim of this paper is to present a different regularization method and investigate the error estimate between the regularization solution and the exact one with the Holder form.
The paper is organized as follows. In Section 2, the modified regularization method is introduced; in Section 3 and Section 4, some stability estimate are proved under different priori conditions; in Section 5, some numerical results are reported.

2 Mathematical problem and regularization

We shall consider the following Cauchy problem for the modified Helmholtz equation

$$\begin{cases} \Delta u - k^2 u = 0, (x,y) \in (0,\pi) \times (0,1) \\ u(0,y) = u(\pi,y) = 0, y \in (0,1) \\ u_y(x,0) = f(x), (x,y) \in (0,\pi) \times (0,1) \\ u(x,0) = g(x), 0 < x < \pi \end{cases} \tag{2}$$

where $g(x), f(x)$ are given functions in $L^2(0,\pi)$ and $k > 0$ is a real number. By the method of separation of variables, the solution of problem (2) is as follows. If we denote by

$$t_{nk}(y) = e^{\sqrt{n^2+k^2}y}$$

then:

$$u(x,y) = \sum_{n=1}^{\infty} \left[\left(\frac{t_{nk}(y) + 1/t_{nk}(y)}{2} \right) g_n + \left(\frac{t_{nk}(y) - 1/t_{nk}(y)}{2\sqrt{n^2+k^2}} \right) f_n \right] \sin nx \quad (3)$$

where

$$f(x) = \sum_{n=1}^{\infty} f_n \sin nx, \ g(x) = \sum_{n=1}^{\infty} g_n \sin nx.$$

Physically, g can only be measured, there will be measurement errors, and we would actually have as data some function $g^\epsilon \in L^2(0,\pi)$, for which

$$\|g^\epsilon - g\| \le \epsilon,$$

where the constant $\epsilon > 0$ represents a bound on the measurement error, $\|.\|$denotes the L^2-norm.
The case $f = 0$, the problem (2) becomes

$$\begin{cases} \Delta u - k^2 u = 0, (x,y) \in (0,\pi) \times (0,1) \\ u(0,y) = u(\pi,y) = 0, y \in (0,1) \\ u_y(x,0) = 0, (x,y) \in (0,\pi) \times (0,1) \\ u(x,0) = g(x), 0 < x < \pi. \end{cases} \quad (4)$$

Very recently, in [17], H.H.Quin and T.Wei considered (2) by the quasi-reversibility method. They established the following problem for a fourth-order equation

$$\begin{cases} \Delta u^\epsilon - k^2 u^\epsilon - \beta^2 u^\beta_{xxyy} = 0, (x,y) \in (0,\pi) \times (0,1) \\ u^\epsilon(0,y) = u^\epsilon(\pi,y) = 0, y \in (0,1) \\ u^\epsilon_y(x,0) = 0, (x,y) \in (0,\pi) \times (0,1) \\ u(x,0) = g(x), 0 < x < \pi. \end{cases} \quad (5)$$

The separation of variables leads to the solution of problem (5) as follows

$$u^\epsilon(x,y) = \sum_{n=1}^{\infty} \left(\frac{\exp\{\sqrt{\frac{n^2+k^2}{1+\beta^2 n^2}}y\} + \exp\{-\sqrt{\frac{n^2+k^2}{1+\beta^2 n^2}}y\}}{2} \right) g_n \sin nx. \quad (6)$$

We note the reader that the term $t_{nk}(y)$ in (3) increases rather quickly when n becomes large, so it is the unstability cause. To regularization the problem

(3), we should replace it by the better terms. In [17], the authors replaced $t_{nk}(y)$ and e^{-ny} by two better terms $\exp\{\sqrt{\frac{n^2+k^2}{1+\beta n^2}}y\}$ and $\exp\{-\sqrt{\frac{n^2+k^2}{1+\beta n^2}}y\}$ respectively.
In this paper, we replace $t_{nk}(y)$ by the better terms $\frac{t_{nk}(y)}{1+\alpha t_{nk}(a)}$ for $a > 1$ be a fixed number and modify the exact solution as follows

$$u^\epsilon(x,y) = \\ = \sum_{n=1}^{\infty}\left[\left(\frac{\frac{t_{nk}(y)}{1+\alpha t_{nk}(a)} + \frac{1}{t_{nk}(y)}}{2}\right) g_n + \left(\frac{\frac{t_{nk}(y)}{1+\alpha t_{nk}(a)} - \frac{1}{t_{nk}(y)}}{2\sqrt{n^2+k^2}}\right) f_n\right] \sin nx. \tag{7}$$

Let v^ϵ be the regularizing solution corresponding to the noisy data g^ϵ

$$v^\epsilon(x,y) = \\ \sum_{n=1}^{\infty}\left[\left(\frac{\frac{t_{nk}(y)}{1+\alpha t_{nk}(a)} + \frac{1}{t_{nk}(y)}}{2}\right) g_n^\epsilon + \left(\frac{\frac{t_{nk}(y)}{1+\alpha t_{nk}(a)} - \frac{1}{t_{nk}(y)}}{2\sqrt{n^2+k^2}}\right) f_n\right] \sin nx. \tag{8}$$

where $g_n^\epsilon = \frac{2}{\pi}\int_0^\pi g^\epsilon(x)\sin(nx)dx$.

3 The main results

Theorem 1.
Let $\varphi(x), g(x) \in L^2(0,\pi)$. Then we have

$$\|v^\epsilon(.,y) - u^\epsilon(.,y)\| \le \alpha^{-\frac{y}{a}}\epsilon.$$

Proof. We have

$$v^\epsilon(x,y) - u^\epsilon(x,y) = \sum_{n=1}^{\infty}\left[\left(\frac{\frac{t_{nk}(y)}{1+\alpha t_{nk}(a)} + \frac{1}{t_{nk}(y)}}{2}\right)(g_n^\epsilon - g_n)\right] \sin nx.$$

For $n, x, \alpha, 0 < a < b$, It is not difficult to prove the inequality

$$\frac{e^{na}}{1+\alpha e^{nb}} \le \alpha^{-\frac{a}{b}}. \tag{9}$$

Thus, we have

$$\begin{aligned}\frac{e^{na}}{1+\alpha e^{nb}} &= \frac{e^{na}}{(1+\alpha e^{nb})^{\frac{a}{b}}(1+\alpha e^{nb})^{1-\frac{a}{b}}}\\ &\le \frac{e^{na}}{(1+\alpha e^{nb})^{\frac{a}{b}}}\\ &\le \alpha^{-\frac{a}{b}}.\end{aligned}$$

Using this inequality, we get

$$\begin{aligned}\|v^\epsilon(.,y)-u^\epsilon(.,y)\|^2 &= \frac{\pi}{2}\sum_{n=1}^{\infty}\left[\left(\frac{\frac{t_{nk}(y)}{1+\alpha t_{nk}(a)}+\frac{1}{t_{nk}(y)}}{2}\right)(g_n^\epsilon-g_n)\right]^2\\ &\le \frac{\pi}{2}\sum_{n=1}^{\infty}\left[\left(\frac{e^{ny}}{1+\alpha e^{na}}\right)(g_n^\epsilon-g_n)\right]^2\\ &\le \alpha^{-2\frac{y}{a}}\|g^\epsilon-g\|^2\\ &\le \alpha^{-2\frac{y}{a}}\epsilon^2.\end{aligned}$$

Hence

$$\|v^\epsilon(.,y)-u^\epsilon(.,y)\| \le \alpha^{-\frac{y}{a}}\epsilon.$$

Theorem 2. *Let E, E_3 be positive numbers such that $\|u(.,1)\| \le E$ and*

$$\sum_{n=1}^{\infty} e^{2\sqrt{n^2+k^2}a} f_n^2 < E_3. \tag{10}$$

If we select $\alpha = \epsilon^a$, then one has

$$\|v^\epsilon(.,y)-u(.,y)\| \le \epsilon^{1-y}(\sqrt{2E+\pi\epsilon^{2a-2}E_3}+1), \tag{11}$$

for every $y \in [0,1]$.

Proof. Since (3) and (8) give

$$u(x,y)-u^\epsilon(x,y) =$$
$$\sum_{n=1}^{\infty}\left[\left(\frac{t_{nk}(y)-\frac{t_{nk}(y)}{1+\alpha t_{nk}(a)}}{2}\right)g_n+\left(\frac{t_{nk}(y)-\frac{t_{nk}(y)}{1+\alpha t_{nk}(a)}}{2\sqrt{n^2+k^2}}\right)f_n\right]\sin nx.$$

It follows from (3) that

$$g_n = \frac{2}{e^{\sqrt{n^2+k^2}}+e^{-\sqrt{n^2+k^2}}} < u(x,1), \sin nx > -$$

$$-\frac{e^{\sqrt{n^2+k^2}}-e^{-\sqrt{n^2+k^2}}}{(e^{\sqrt{n^2+k^2}}+e^{-\sqrt{n^2+k^2}})\sqrt{n^2+k^2}}f_n.$$

By directly transform, we have

$$\begin{aligned}
&< u(x,y)-u^{\epsilon}(x,y),\sin nx >= \\
&= \frac{t_{nk}(y)-\frac{t_{nk}(y)}{1+\alpha t_{nk}(a)}}{e^{\sqrt{n^2+k^2}}+e^{-\sqrt{n^2+k^2}}} < u(x,1),\sin nx > + \\
&+\frac{t_{nk}(y)-\frac{t_{nk}(y)}{1+\alpha t_{nk}(y)}}{(e^{\sqrt{n^2+k^2}}+e^{-\sqrt{n^2+k^2}})2\sqrt{n^2+k^2}}(e^{\sqrt{n^2+k^2}}+e^{-\sqrt{n^2+k^2}}-1)g_n \\
&= \frac{\alpha t_{nk}(a+y)}{(1+\alpha t_{nk}(a))(e^{\sqrt{n^2+k^2}}+e^{-\sqrt{n^2+k^2}})} < u(x,1),\sin nx > + \\
&+\frac{\alpha t_{nk}(a+y)(e^{\sqrt{n^2+k^2}}+e^{-\sqrt{n^2+k^2}}-1)}{2\sqrt{n^2+k^2}(1+\alpha t_{nk}(a))(e^{\sqrt{n^2+k^2}}+e^{-\sqrt{n^2+k^2}})}f_n.
\end{aligned}$$

Thus

$$\begin{aligned}
&| < u(x,y)-u^{\epsilon}(x,y),\sin nx > | \le \\
&\frac{\alpha t_{nk}(a+y-1)}{1+\alpha t_{nk}(a)}| < u(x,1),\sin nx > | + \frac{\alpha t_{nk}(y)}{1+\alpha t_{nk}(a)}t_{nk}(a)g_n \\
&\le \alpha^{\frac{1-y}{a}}| < u(x,1),\sin nx > | + \alpha^{1-\frac{y}{a}}t_{nk}(a)f_n.
\end{aligned} \tag{12}$$

Using the inequality $(c+d)^2 \le 2c^2+2d^2$, we obtain

$$\begin{aligned}
\|u(.,y)-u^{\epsilon}(.,y)\|^2 &= \frac{\pi}{2}\sum_{n=1}^{\infty}| < u(x,y)-w^{\epsilon}(x,y),\sin nx > |^2 \\
&\le 2\alpha^{2\frac{1-y}{a}}\frac{\pi}{2}\sum_{n=1}^{\infty}| < u(x,1),\sin nx > |^2 \\
&+ 2\alpha^{2-\frac{2y}{a}}\frac{\pi}{2}\sum_{n=1}^{\infty}e^{2na}f_n^2 \\
&\le 2\alpha^{2\frac{1-y}{a}}\|u(.,1)\|^2+\pi\alpha^{2-\frac{2y}{a}}\sum_{n=1}^{\infty}t_{nk}(a)f_n^2.
\end{aligned}$$

Apply the triangle inequality and we get

$$\begin{aligned}\|u(.,y)-v^\epsilon(.,y)\| &\le \|u(.,y)-u^\epsilon(.,y)\|+\|u^\epsilon(.,y)-v^\epsilon(.,y)\|\\ &\le \sqrt{2\alpha^{2\frac{1-y}{a}}\|u(.,1)\|^2+\pi\alpha^{2-\frac{2y}{a}}\sum_{n=1}^{\infty}t_{nk}(a)f_n^2}+\alpha^{-\frac{y}{a}}\epsilon\\ &\le \sqrt{2\alpha^{2\frac{1-y}{a}}E+\pi\alpha^{2-\frac{2y}{a}}E_3}+\alpha^{-\frac{y}{a}}\epsilon\\ &\le \sqrt{2\alpha^{2-2y}E+\pi\alpha^{2a-2y}E_3}+\epsilon^{1-y}\\ &\le \epsilon^{1-y}(\sqrt{2E+\pi\alpha^{2a-2}E_3}+1).\end{aligned}$$

Remark 1. From (11), as $y\to 1$, the accuracy of regularized solution becomes progressively lower. Moreover, the error in $y=1$ is not given. This is a common thing in the theory of ill-posed problems, if we do not have additional conditions on the smoothness of the solution. To retain the continuous dependence of the solution at $y=1$, we introduce a stronger a priori assumption. We have the following theorem

Theorem 3. *Let* (10) *holds. Suppose that there are positive real numbers* γ, E_4 *such that*

$$\sum_{n=1}^{\infty}e^{2\gamma n}|<u(x,1),\sin nx>|^2<E_4. \tag{13}$$

Let us select $\alpha=\epsilon^{\frac{a}{1+\gamma}}$, $b=\min\{1+\gamma,a\}$, *then one has*

$$\|v^\epsilon(.,y)-u(.,y)\| \le \epsilon^{\frac{b-y}{1+\gamma}}(\sqrt{2E_4+\pi E_3}+1), \tag{14}$$

for every $y\in[0,1]$.

Proof. For the first term on the right-hand side of (12), we have

$$\begin{aligned}&\frac{\alpha t_{nk}(a+y-1)}{1+\alpha t_{nk}(a)}|<u(x,1),\sin nx>|\\ &=\frac{\alpha t_{nk}(a+y-1-\gamma)}{1+\alpha t_{nk}(a)}t_{nk}(\gamma)|<u(x,1),\sin nx>|\\ &\le \alpha\alpha^{\frac{1+\gamma-y-a}{a}}t_{nk}(k)|<u(x,1),\sin nx>|\\ &=\alpha^{\frac{1+\gamma-y}{a}}t_{nk}(\gamma)|<u(x,1),\sin nx>|.\end{aligned}$$

Using the inequality $(c+d)^2 \leq 2c^2 + 2d^2$, we obtain

$$\begin{aligned}
\|u(.,y) - u^\epsilon(.,y)\|^2 &= \frac{\pi}{2}\sum_{n=1}^{\infty} |< u(x,y) - w^\epsilon(x,y), \sin nx >|^2 \\
&\leq 2\alpha^{2\frac{1+\gamma-y}{a}} \frac{\pi}{2}\sum_{n=1}^{\infty} e^{2\gamma n}|< u(x,1), \sin nx >|^2 \\
&+ 2\alpha^{2-\frac{2y}{a}} \frac{\pi}{2}\sum_{n=1}^{\infty} t_{nk}(2a) f_n^2 \\
&\leq 2\alpha^{2\frac{1+\gamma-y}{a}} \sum_{n=1}^{\infty} e^{2\gamma n}|< u(x,1), \sin nx >|^2 + \pi\alpha^{2-\frac{2y}{a}} E_3 \\
&\leq 2\alpha^{2\frac{1+\gamma-y}{a}} E_4 + \pi\alpha^{2-\frac{2y}{a}} E_3.
\end{aligned}$$

Applying the triangle inequality, we get

$$\begin{aligned}
\|u(.,y) - v^\epsilon(.,y)\| &\leq \|u(.,y) - u^\epsilon(.,y)\| + \|u^\epsilon(.,y) - v^\epsilon(.,y)\| \\
&\leq \sqrt{2\alpha^{2\frac{1+\gamma-y}{a}} E_4 + \pi\alpha^{2-\frac{2y}{a}} E_3} + \alpha^{-\frac{y}{a}}\epsilon \\
&\leq \sqrt{2\alpha^{2\frac{1+\gamma-y}{a}} E_4 + \pi\alpha^{2-\frac{2y}{a}} E_3} + \alpha^{-\frac{y}{a}}\epsilon \\
&\leq \sqrt{2\epsilon^{2\frac{1+\gamma-y}{1+k}} E_4 + \pi\epsilon^{\frac{2a-2y}{1+\gamma}} E_3} + \epsilon^{\frac{1+\gamma-y}{1+\gamma}}. \\
&\leq \epsilon^{\frac{b-y}{1+\gamma}}(\sqrt{2E_4 + \pi E_3} + 1).
\end{aligned}$$

Remark 2.
1. We separately consider the case $0 \leq y < 1$ and the case $y = 1$ in order to emphasize the following facts. For the case $0 \leq y < 1$, the a priori bound $\|u(.,1)\| \leq E$ is sufficient. However, for the case $y = 1$, the stronger a priori bound in (13) must be imposed.
2. The error (14) is the order of Holder type for all $y \in [0,1]$. As we know, the convergence rate of ϵ^p, $(0 < p)$ is more quickly than the logarithmic order $\left(\ln(\frac{1}{\epsilon})\right)^{-q}$ $(q > 0)$ when $\epsilon \to 0$. Thus, up to know, Holder order type is optimal order. Note that this error is not investigated in [7,8,9,13,14,16]. This proves the advantages of our method.

4 Numerical result.

In this section, a simple example is devised for verifying the validity of the proposed method. For the reader can make a comparison between this paper with [7] and [17] by using same example with the same parameters, we consider the problem

$$\begin{cases} u_{xx} + u_{yy} = 3u, (x,t) \in (0,\pi) \times (0,1) \\ u(0,y) = u(\pi,y) = 0, yy \in (0,1) \\ u_y(x,0) = 0, (x,y) \in (0,\pi) \times (0,1) \\ u(x,0) = \dfrac{\sin(x)}{4}, 0 < x < \pi. \end{cases} \tag{15}$$

The exact solution to this problem is

$$u(x,y) = \frac{e^{2y} + e^{-2y}}{8} \sin x.$$

Let $y = 1$, we get $u(x,1) = 0.940548922770908 \sin x$.
Let g_m be the measured data

$$g_m(x) = \frac{1}{4}\sin(x) + \frac{1}{m}\sin(mx).$$

So the data error, at $t = 0$, is

$$F(m) = \|g_m - g\| = \sqrt{\int_0^\pi \frac{1}{m^2}\sin^2(mx)dx} = \sqrt{\frac{\pi}{2}}\frac{1}{m} \le \epsilon.$$

The solution of (15) corresponding the g_m is

$$u^m(x,y) = \frac{e^y + e^{-y}}{2}\sin x + \frac{e^{\sqrt{m^2+3}y} + e^{-\sqrt{m^2+3}y}}{2m}\sin mx.$$

The error in $y = 1$ is

$$\begin{aligned} O(n) := \|u^m(.,1) - u(.,1)\| &= \sqrt{\int_0^\pi \frac{(e^{\sqrt{m^2+3}} + e^{-\sqrt{m^2+3}})^2}{4m^2}\sin^2(mx)\,dx} \\ &= \frac{(e^{2\sqrt{m^2+3}} + e^{-2\sqrt{m^2+3}} + 2)}{4m^2}\sqrt{\frac{\pi}{2}}. \end{aligned}$$

Then, we notice that

$$\lim_{m\to\infty} F(m) = \lim_{m\to\infty}\frac{1}{m}\sqrt{\frac{\pi}{2}} = 0, \tag{16}$$

$$\lim_{m\to\infty} O(m) = \lim_{m\to\infty}\frac{(e^{2\sqrt{m^2+3}} + e^{-2\sqrt{m^2+3}} + 2)}{4m^2}\sqrt{\frac{\pi}{2}} = \infty. \tag{17}$$

From the two equalities above, we see that (15) is an ill-posed problem. Hence, the Cauchy problem (15) cannot be solved by using classical numerical methods and it needs regularization techniques.
Let $\epsilon = \sqrt{\frac{\pi}{2}}\frac{1}{m}$. By applying the method in this paper, we have the approximated solution

$$v^{\epsilon}(x,y) = \sum_{n=1}^{\infty}\left[\left(\frac{\frac{e^{\sqrt{n^2+3}y}}{1+\alpha e^{a\sqrt{n^2+3}}} + e^{-\sqrt{n^2+3}y}}{2}\right) < g_m(x), \sin nx > \right]\sin nx. \quad (18)$$

Choose $\alpha = \epsilon^2$ and $a = 2$, let $y = 1$, we have

$$v^{\epsilon}(x,1) = \sum_{n=1}^{\infty}\left[\left(\frac{\frac{e^{\sqrt{n^2+3}}}{1+\epsilon^2 e^{2\sqrt{n^2+3}}} + e^{-\sqrt{n^2+3}}}{2}\right) < g_m(x), \sin nx > \right]\sin nx \quad (19)$$

$$= \frac{\frac{e^2}{1+\epsilon^2 e^2} + e^{-2}}{8}\sin x + \frac{\frac{e^{\sqrt{m^2+3}}}{1+\epsilon^2 e^{2\sqrt{m^2+3}}} + e^{-\sqrt{m^2+3}}}{2m}\sin mx. \quad (20)$$

The error between $v^{\epsilon}(.,1)$ and $u(.,1)$ is as follows

$$\|v^{\epsilon}(.,1) - u(.,1)\|^2 =$$

$$= \frac{\pi}{2}\left[\left(\frac{\frac{e^2}{1+\epsilon^2 e^2} + e^{-2}}{8} - \frac{e^2 + e^{-2}}{8}\right)^2 + \left(\frac{\frac{e^{\sqrt{m^2+3}}}{1+\epsilon^2 e^{2\sqrt{m^2+3}}} + e^{-\sqrt{m^2+3}}}{2m}\right)^2\right]$$

$$= \frac{\pi}{2}\left[\frac{\epsilon^4 e^4}{64(1+\epsilon^2 e^2)^2} + \left(\frac{\frac{e^{\sqrt{m^2+3}}}{1+\epsilon^2 e^{2\sqrt{m^2+3}}} + e^{-\sqrt{m^2+3}}}{2m}\right)^2\right].$$

Table 1 shows the the error between the regularization solution v^{ϵ} and the exact solution u, for three values of ϵ. We have the table numerical test by choose some values as follows ($a_{\epsilon} = \|v_{\epsilon}(.,1) - u(.,1)\|$).

By applying the method in [17], we have the approximated solution

$$w^{\epsilon}(x,y) =$$

$$= \sum_{p=1}^{\infty}\left[\left(\frac{\exp\{\sqrt{\frac{p^2+3}{1+\beta p^2}}y\} + \exp\{-\sqrt{\frac{p^2+3}{1+\beta p^2}}y\}}{2}\right) < g_m(x), \sin px > \right]\sin px.$$

Table 1: The error of the method in this paper.

ϵ	v_ϵ	a_ϵ
$\epsilon_1 = 10^{-2}\sqrt{\frac{\pi}{2}}$	$0.9394781334\sin(x)$ $+1.166691975 \times 10^{-42}\sin(100x)$	0.0013420
$\epsilon_2 = 10^{-5}\sqrt{\frac{\pi}{2}}$	$0.9405489216\sin(x)$ $+1.34122070 \times 10^{-43425}\sin(10^5 x)$	1.343592×10^{-9}
$\epsilon_3 = 10^{-10}\sqrt{\frac{\pi}{2}}$	$0.940548922770\sin(x)$ $+3.977391112813 \times 10^{-434294474}$	$1.3435930 \times 10^{-17}$

Let $y = 1$, we have

$$
\begin{aligned}
&w^\epsilon(x,1) = \\
&= \sum_{p=1}^{\infty}\left[\left(\frac{\exp\{\sqrt{\frac{p^2+3}{1+\beta p^2}}\} + \exp\{-\sqrt{\frac{p^2+3}{1+\beta p^2}}\}}{2}\right) < g_m(x), \sin px > \right]\sin px \\
&= \frac{\exp\{\sqrt{\frac{4}{1+\beta}}\} + \exp\{-\sqrt{\frac{4}{1+\beta}}\}}{8}\sin x + \\
&\frac{\exp\{\sqrt{\frac{m^2+3}{1+\beta m^2}}\} + \exp\{-\sqrt{\frac{m^2+3}{1+\beta m^2}}\}}{2m}\sin mx. \qquad (21)
\end{aligned}
$$

This implies that

$$
\begin{aligned}
&\|w^\epsilon(.,1) - u(.,1)\| = \\
&= \frac{\pi}{2}\left(\frac{\exp\{\sqrt{\frac{4}{1+\beta}}\} + \exp\{-\sqrt{\frac{4}{1+\beta}}\}}{8} - \frac{e^2+e^{-2}}{8}\right)^2 \\
&+ \frac{\pi}{2}\left(\frac{\exp\{\sqrt{\frac{m^2+3}{1+\beta m^2}}\} + \exp\{-\sqrt{\frac{m^2+3}{1+\beta m^2}}\}}{2}\right)^2.
\end{aligned}
$$

We note that if we chooseing $\beta = \epsilon$ and m such that $\epsilon = \sqrt{\frac{\pi}{2}}\frac{1}{m}$ then $\|w^\epsilon(.,1) - u(.,1)\|$ does not converges to zero. Thus, by choose some differrent values, we have the table numerical test as follows

1. $\epsilon = 10^{-2}\sqrt{\frac{\pi}{2}}$ corresponding to $m = 10^{20}$.
2. $\epsilon = 10^{-5}\sqrt{\frac{\pi}{2}}$ corresponding to $m = 10^{20}$.
3. $\epsilon = 10^{-10}\sqrt{\frac{\pi}{2}}$ corresponding to $m = 10^{50}$.

Looking at Tables 1, 2 in order to do a comparison between the three methods,

Table 2: The error of the method in the paper [17]

ϵ	w^ϵ	$\|w^\epsilon - u\|$
$10^{-2}\sqrt{\frac{\pi}{2}}$	$v_1 = 0.929362\sin(x) + 3.7868 \times 10^{-17}\sin(10^{20}x)$	0.0140196
$10^{-5}\sqrt{\frac{\pi}{2}}$	$0.939414\sin(x) + 9.2559 \times 10^{-9}\sin(10^{20}x)$	0.00142200
$10^{-10}\sqrt{\frac{\pi}{2}}$	$0.940435\sin(x) + 3.1049 \times 10^{-12}\sin(10^{50}x)$	0.000142403

we can see the error results in Table 1 are smaller than the errors in Table 2. In the same parameter regularization, the error in Table 1 converges to zero more quickly than in Table 2 . This shows that our approach has a nice regularizing effect and give a better approximation in comparison to many previous results, such as [2, 3, 4, 5, 17, 15, 16].

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