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SOME PROPERTIES OF LOCALLY HOMOGENEOUS GRAPHS

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Abstract

In this paper we determine when the join of two graphs is locally homogeneous. It is shown that the Cartesian product of a locally H_1 graph and a locally H_2 graph is locally $(H_1 \cup H_2)$. All graphs H of order at most 4 for which there are no locally H graphs are determined.

1 Introduction

For undefined concepts, the reader is referred to [2].

Let a graph H be given. A graph G is called *locally* H if for each vertex $v \in V(G)$, the subgraph induced by the set of neighbors of v is isomorphic to H, see [3] and [7]. Given some fixed graph H, it is a natural question to ask for a classification of all connected locally H graphs. For some graphs H, all graphs that are locally H have been determined, see [6] for locally Petersen graphs for example. A connected locally H graph G is *locally recognizable* if, up to isomorphism, G is the unique connected locally H graph, see [5]. The tetrahedron, the octahedron and the icosahedron are locally recognizable, they are the connected locally C_n graphs where $n = 3, 4, 5$, respectively, see [11] and [15]. It is remarked in [11] that the tetrahedron is k-null. The octahedron and the icosahedron are k-divergent, see [4], [13] and [14]. There is an infinite number of locally C_6 graphs, see [9] and [10]. The k-behaviour of locally C_n graphs, for $n \geq 7$ are described in [11]. In section 4, we will characterize the graphs H of order at most 4 for which there are no locally H graphs.

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Locally H graphs are also called locally homogeneous, a *locally homogeneous* graph is a locally H graph for some graph H, see [5] and [15].

If v is a vertex of a graph G and $B \subseteq V(G)$, then $N(v)$ is the set of neighbors of v and $\langle B \rangle$ is the subgraph of G induced by B. For two graphs G_1 and G_2, the union, the join, and the Cartesian product of G_1 and G_2 are denoted by $G_1 \cup G_2$, $G_1 + G_2$, and $G_1 \square G_2$, respectively.

2 When is the join of two graphs locally H?

The aim of this section is to determine precisely for which graphs G_1 and G_2, the join $G_1 + G_2$ is locally homogeneous. First we determine when the join of two locally homogeneous graphs is locally homogeneous.

Theorem 1. *Let G_1 be a locally H_1 graph and G_2 be a locally H_2 graph, for some graphs H_1 and H_2. Then the join $G_1 + G_2$ is locally homogeneous if and only if $H_1 + G_2$ is isomorphic to $G_1 + H_2$. Moreover, if $G_1 + G_2$ is locally H, then H is isomorphic to $H_1 + G_2$.*

Proof. Let $v \in V(G_1)$. Then $N_{G_1+G_2}(v) = N_{G_1}(v) \cup V(G_2)$. Since G_1 is locally H_1, the set $N_{G_1}(v)$ induces in $G_1 + G_2$ a graph isomorphic to H_1. The set $V(G_2)$ induces in $G_1 + G_2$ a graph isomorphic to G_2. But in $G_1 + G_2$, every vertex in $N_{G_1}(v)$ is adjacent to all vertices in $V(G_2)$. Thus $\langle N_{G_1+G_2}(v) \rangle \cong H_1 + G_2$. Similarly, for every vertex $w \in V(G_2)$, we have $\langle N_{G_1+G_2}(v) \rangle \cong H_2 + G_1$. Therefore $G_1 + G_2$ is locally homogeneous if and only if $H_1 + G_2$ is isomorphic to $G_1 + H_2$. Obviously, $G_1 + G_2$ is locally $(H_1 + G_2)$ whenever it is locally homogeneous. $\square$

In particular, if we consider the join of a locally homogeneous graph with itself, we get the following result.

Corollary 2. *If G is a locally H graph, then $G + G$ is locally $(H + G)$.*

If $m > n \geq 1$, then the complete graph $K_m + K_n$ of order $m + n$ is an example of a join of two nonisomorphic locally homogeneous graphs that is locally homogeneous.

Next, we consider the case in which at least one of the two graphs in the join is an edgeless graph. We start by the following lemma.

Lemma 3. *Let G be a graph such that $nK_1 + G$ is locally homogeneous. Then either $G \cong nK_1$ or G is locally homogeneous.*

Proof. Suppose that G is not isomorphic to nK_1. Then G must have positive size, for otherwise $nK_1 + G$ would be not regular. Assume to the contrary that G is not locally homogeneous. Then there exist two vertices x and y of G such that $\langle N_G(x)\rangle \not\cong \langle N_G(y)\rangle$. This implies that $\langle N_{nK_1+G}(x)\rangle \not\cong \langle N_{nK_1+G}(y)\rangle$, which contradicts the assumption that $nK_1 + G$ is locally homogeneous. $\square$

The regular complete r-partite graph $K_{\underbrace{n,\ldots,n}_{r-\text{times}}}$ is denoted by $K_{r(n)}$, see [1]. By $K_{1(n)}$ we will mean the graph nK_1.

Theorem 4. *Let G be any graph. Then the join $nK_1 + G$ is locally homogeneous if and only if $G \cong K_{m(n)}$ for some positive integer m.*

Proof. Suppose that $G \cong K_{m(n)}$ for some positive integer m. Then $nK_1+G \cong K_{(m+1)(n)}$ which is locally $K_{m(n)}$.

For the converse, suppose that $nK_1 + G$ is locally homogeneous. Then, since for each vertex v of nK_1 we have $\langle N_{nK_1+G}(v)\rangle = G$, the graph $nK_1 + G$ is locally G. Let g be a vertex of G. Then $N_{nK_1+G}(g) = V(nK_1) \cup N_G(g)$. But by Lemma 3, either $G \cong nK_1$ or G is locally G_1 where $G_1 = \langle N_G(g)\rangle$. The result follows when $G \cong nK_1 \cong K_{1(n)}$. So, suppose that G is locally G_1. Then $\langle N_{nK_1+G}(g)\rangle \cong nK_1 + G_1$. Thus, since $nK_1 + G$ is locally G, we must have $G \cong nK_1 + G_1$. Now, since G is locally homogeneous, again by Lemma 3, either $G_1 \cong nK_1$ or G_1 is locally G_2 for some graph G_2. If $G_1 \cong nK_1$, then $G \cong nK_1 + nK_1 \cong K_{2(n)}$, hence the result. So, suppose that G_1 is locally G_2 and let $g_1 \in V(G_1)$. Then $\langle N_{nK_1+G_1}(g_1)\rangle \cong nK_1 + G_2$. Thus, since $G \cong nK_1 + G_1$ is locally G_1, we have $G_1 \cong nK_1 + G_2$. Then $G \cong nK_1 + G_1 \cong nK_1 + (nK_1 + G_2)$. Therefore, since G is finite, applying the same argument on G_2 and so on, we must end up with a graph G_i for some $i \geq 0$ (where $G_0 = G$) such that $G_i \cong nK_1$. Then we have

$$\begin{aligned} G &\cong nK_1 + (nK_1 + (\cdots + (nK_1 + nK_1 \underbrace{)\cdots))}_{(i-1)-\text{times}} \\ &\cong K_{(i+1)(n)}. \end{aligned}$$

$\square$

Now we are in a position to determine for which graphs G_1 and G_2, the join $G_1 + G_2$ is locally homogeneous.

Theorem 5. *Let G_1 and G_2 be two graphs. Then the join $G_1 + G_2$ is locally homogeneous if and only if either $G_1 \cong K_{s(n)}$ and $G_2 \cong K_{t(n)}$ for some positive integers s, t and n, or G_1 is locally H_1 and G_2 is locally H_2 for some graphs H_1 and H_2 where $H_1 + G_2 \cong G_1 + H_2$.*

Proof. In view of Theorem 1 and Theorem 4, it would be enough to consider the case in which each of G_1 and G_2 has positive size and at least one of them, say G_1, is not locally homogeneous. But in this case, there exist two vertices x and y in G_1 such that $\langle N_{G_1}(x)\rangle \ncong \langle N_{G_1}(y)\rangle$. Then $\langle N_{G_1+G_2}(x)\rangle \ncong \langle N_{G_1+G_2}(y)\rangle$, which implies that $G_1 + G_2$ is not locally homogeneous. $\square$

3 Cartesian products of locally H graphs

The property of being locally homogeneous is preserved under Cartesian product of graphs, this is shown in the next result.

Theorem 6. *Let G_1 be a locally H_1 graph and G_2 be a locally H_2 graph for some graphs H_1 and H_2. Then the Cartesian product $G_1 \Box G_2$ is locally $(H_1 \cup H_2)$.*

Proof. Let $v = (g_1, g_2)$ be a vertex of $G_1 \Box G_2$. Then $N(v) = A \cup B$ where $A = \{(g_1, y) : y \in V(G_2), yg_2 \in E(G_2)\}$ and $B = \{(x, g_2) : x \in V(G_1), xg_1 \in E(G_1)\}$. The elements of A are the neighbors of v in the copy of G_2 corresponding to the vertex g_1, hence A induces in $G_1 \Box G_2$ a subgraph isomorphic to H_2. Similarly, B induces a subgraph isomorphic to H_1. Obviously, there is no edge in $G_1 \Box G_2$ joining a vertex in A with a vertex in B. Therefore $\langle N(v)\rangle = H_1 \cup H_2$. $\square$

A locally $3K_1$ graph G must have order at least 6, because if v is a vertex of G with $N(v) = \{x_1, x_2, x_3\}$, then $N(x_1) = \{v, y_1, y_2\}$ for some two vertices $y_1, y_2 \notin \{v, x_1, x_2, x_3\}$. Obviously, since G is 3-regular, the order of G must be even. The following result assures the existence of a locally $3K_1$ graph of any even order greater than or equal to 6.

Corollary 7. *For any integer $n \geq 3$, there exists a locally $3K_1$ graph of order $2n$ and diameter $\lfloor \frac{n}{2} \rfloor + 1$.*

Proof. For $n = 3$, the graph $K_{3,3}$ is a locally $3K_1$ graph of order 6 and diameter 2. For $n \geq 4$, the graph $K_2 \Box C_n$ is, by Theorem 6, locally $3K_1$. Clearly $K_2 \Box C_n$ has order $2n$ and diameter $\lfloor \frac{n}{2} \rfloor + 1$. $\square$

Note that the graph $K_2 \Box C_n$ is just the cycle permutation graph $P_\alpha(C_n)$, where α is the identity permutation.

A graph G is a *Hamming graph* if $G = K_{n_1} \Box K_{n_2} \Box \cdots \Box K_{n_r}$, for some $r \geq 1$, where $n_i \geq 2$ for each $i = 1, 2, ..., r$, see [8] and [12]. The following corollary follows by Theorem 6.

Corollary 8. *The Hamming graph $K_{n_1} \Box K_{n_2} \Box \cdots \Box K_{n_r}$ is locally $(K_{n_1-1} \cup K_{n_2-1} \cup \cdots \cup K_{n_r-1})$. In particular, for any integer $r \geq 1$, the hypercube Q_r is locally rK_1.*

4 The graphs H of order at most 4 for which there are no locally H graphs

In this section we characterize the graphs H of order at most 4 for which there are no locally H graphs.

The graph K_2 is locally K_1. The triangle C_3 and the cycle $C_n, n \geq 4$, are locally K_2 and locally $2K_1$, respectively. Thus for any graph H of order less than 3, there exists a locally H graph. According to the following result, 3 is the minimum possible order of a graph H for which there is no locally H graph.

Theorem 9. *For any integer $n \geq 3$, there is no locally $(K_n - e)$ graph, where e is an edge of K_n.*

Proof. Assume to the contrary that there exists a locally $(K_n - e)$ graph G. Let v be a vertex of G with $N(v) = \{x_1, x_2, ..., x_n\}$ where $x_1x_2 \notin E(\langle N(v)\rangle)$. Then $N(x_1) = \{x_3, x_4, ..., x_n\} \cup \{v, y\}$ for some vertex y of G. Clearly y is not adjacent to any of the two vertices v, x_3 because each of them has already its n neighbors. Thus $N(x_1)$ does not induce a subgraph isomorphic to $K_n - e$, a contradiction. $\square$

The 3-cube is locally $3K_1$, the complete graph K_4 is locally K_3, and, by Theorem 6, the graph $K_2 \square K_3$ is locally $(K_1 \cup K_2)$. Therefore, by Theorem 9, we have the following result.

Theorem 10. *Let H be a graph of order 3. Then there is no locally H graph if and only if H is the path P_3.*

Next, we consider locally H graphs where H has order 4.

Lemma 11. *There is no locally $(K_1 + (K_1 \cup K_2))$ graph.*

Proof. Assume to the contrary that there exists a locally $(K_1 + (K_1 \cup K_2))$ graph G. Let v be a vertex of G with $N(v) = \{x_1, x_2, x_3, x_4\}$, where x_1, x_2 are the vertices of degrees $1, 3$ in $\langle N(v)\rangle$, respectively. Clearly, $N(x_2) = \{x_1, v, x_3, x_4\}$. Then $N(x_1) = \{v, x_2, y_1, y_2\}$ for some two vertices $y_1, y_2 \notin \{x_3, x_4\}$. Now, since $y_1, y_2 \notin N(v) \cup N(x_2)$, the neighborhood $N(x_1)$ does not induce a subgraph of G isomorphic to $K_1 + (K_1 \cup K_2)$, a contradiction. $\square$

Lemma 12. *There is no locally $K_{1,3}$ graph.*

Proof. Assume to the contrary that there exists a locally $K_{1,3}$ graph G. Let v be a vertex of G with $N(v) = \{u, x_1, x_2, x_3\}$ where u is the vertex of degree 3 in $\langle N(v)\rangle$. Obviously, $N(u) = \{v, x_1, x_2, x_3\}$. Then $N(x_1) = \{u, v, y_1, y_2\}$ for some two vertices $y_1, y_2 \notin \{x_2, x_3\}$. Since $y_1, y_2 \notin N(u) \cup N(v)$, the set $N(x_1)$ does not induce a subgraph isomorphic to $K_{1,3}$, a contradiction. $\square$

Lemma 13. *There is no locally* $(P_3 \cup K_1)$ *graph.*

Proof. Assume to the contrary that there exists a locally $(P_3 \cup K_1)$ graph G. Let v be a vertex of G with $N(v) = \{x_1, x_2, x_3, y\}$ where $x_1x_2x_3$ is a path in G. Then $N(x_1) = \{v, x_2, y_1, y_2\}$ for some two vertices $y_1, y_2 \notin \{x_3, y\}$. Since v is not adjacent to any of the two vertices y_1, y_2, and $N(x_1)$ induces a subgraph isomorphic to $P_3 \cup K_1$, the vertex x_2 must be adjacent to exactly one of the two vertices y_1 and y_2, say x_2 is adjacent to y_1. Then $\langle N(x_2)\rangle = \langle\{y_1, x_1, v, x_3\}\rangle$ contains the 4-path $y_1x_1vx_3$, and hence does not induce a subgraph isomorphic to $P_3 \cup K_1$, a contradiction. □

Now we can determine for which graph H of order 4 we do not have a locally H graph.

Theorem 14. *Let* H *be a graph of order* 4*. Then there is no locally* H *graph if and only if* H *is isomorphic to one of the four graphs:* $K_1 + (K_1 \cup K_2)$*,* $K_{1,3}$*,* $P_3 \cup K_1$*, and the kite* $K_{1,1,2}$*.*

Proof. There is no locally H graph when H is isomorphic to $K_1 + (K_1 \cup K_2)$, $K_{1,3}$, $P_3 \cup K_1$, or $K_{1,1,2}$, according to Lemma 11, Lemma 12, Lemma 13, and Theorem 9 , respectively. For the converse, it would be enough to find a locally H graph for each of the remaining seven possibilities of the graph H. By Theorem 6, the graphs $K_2\square K_4$, $K_3\square K_3$, $K_3\square K_2\square K_2$, and the hypercube Q_4 are locally $(K_1 \cup K_3)$, locally $2K_2$, locally $(K_2 \cup 2K_1)$, and locally $4K_1$, respectively. The Octahedron is locally C_4. Finally, the complete graph K_5 and the 4-antiprism $C_8(1, 2)$ are locally K_4 and locally P_4, respectively. This completes all possible eleven cases of H. □

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