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UNIQUENESS OF MEROMORPHIC FUNCTIONS SHARING VALUES

He Ping

Abstract

In this paper, we investigate the problem of the uniqueness of meromorphic function sharing values. It is turned out that our results are natural extensions of Q. C. Zhang and G. G. Gundersen.

1 Introduction

We assume that the reader is familiar with the fundamental results in Nevanlinna's value distribution theory of meromorphic functions of single complex variable in the open complex plane. In this paper, a transcendental meromorphic function is meromorphic in the whole complex plane. We say that f and g share the value a CM (counting multiplicities) if f and g have the same a-points with the same multiplicity and we say that f and g share the value a IM (ignoring multiplicities) if we do not consider the multiplicities.

In 1929, R. Nevanlinna proved that for two nonconstant meromorphic functions f and g in the complex plane, if they share five distinct values IM, then $f \equiv g$; if they share four distinct values CM, then f is a Möbius transformation of g. After his very deep work, many results on uniqueness of meromorphic functions concerning shared values in the complex plane have been obtained (see [3]). We will use the standard notations of the Nevanlinna's theory such

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as $T(r,f)$,$N(r,f)$ and $m(r,f)$. For references, please see [1, 2, 3]. We say that $\bar{E}(a,f)$ is the set of those a-points of $f(z)$, ignoring multiplicity.

J. H. Zheng (see [5]) took into account of the uniqueness dealing with five shared values in some angular domains of $\mathbb{C}$. It is also interesting how to extend some important uniqueness results in the whole complex plane to an angular domain. In this paper, we study the uniqueness of meromorphic functions on the angular domain. So we also introduce its fundamental notations (see [1]). We denote

$$A(r,\Delta_\delta,f)=\frac{\omega}{\pi}\int_1^r(\frac{1}{t^\omega}-\frac{t^\omega}{r^{2\omega}})\{\log^+|f(te^{i(\theta_0-\delta)})|+\log^+|f(te^{i(\theta_0+\delta)})|\}\frac{dt}{t},$$

$$B(r,\Delta_\delta,f)=\frac{2\omega}{\pi r^\omega}\int_{\theta_0-\delta}^{\theta_0+\delta}\log^+|f(re^{i\varphi})|sin\omega(\varphi-\theta_0+\delta)d\varphi,$$

$$C(r,\Delta_\delta,f=\infty)=2C(r,\Delta_\delta,f=\infty)=$$

$$=2\sum_{\substack{1<\rho_n\le r\\|\psi_n-\theta_0|\le\delta}}(\frac{1}{\rho_n^\omega}-\frac{\rho_n^\omega}{r^{2\omega}})\sin\omega(\psi_n-\theta_0+\delta),$$

$$C(r,\Delta_\delta,f=a)=C(r,\Delta_\delta,\frac{1}{f-a}),a\in\mathbb{C},$$

$$S_N(r,\Delta_\delta,f)=A(r,\Delta_\delta,f)+B(r,\Delta_\delta,f)+C(r,\Delta_\delta,f=\infty),$$

where $\omega=\frac{\pi}{2\delta}$ and $\rho_n e^{i\psi_n}(n=1,2,\cdots)$ are the poles of f in the angular domain Δ_δ and each pole of multiplicity m appears m times.

2 Notations and main results

Definition 1. Let$f(z)$ be a non-constant meromorphic function, $a\in\mathbb{C}\bigcup\{\infty\}$, we say that $\overline{E}(a,\Delta_\delta,f)$ is the set of those a-points of $f(z)$ in $\Delta_\delta=\{z||argz-\theta_0|\le\delta\}(0<\delta<\pi)$, ignoring multiplicity.

Definition 2. Let $f(z)$ and $g(z)$ be two nonconstant meromorphic functions, $a\in\mathbb{C}\bigcup\{\infty\}$, we say that $f(z)$and $g(z)$ share the value a IM in $\Delta_\delta=\{z||argz-\theta_0|\le\delta\}(0<\delta<\pi)$ if $f-a$ and $g-a$ have the same zeros in Δ_δ (ignoring multiplicities); they share the value a CM in Δ_δ, if $f-a$ and $g-a$ have the same zeros with the same multiplicities in Δ_δ (counting multiplicities).

C. C. Yang (see [3]) and Q. C. Zhang (see [4]) proved the following well-known theorems.

Theorem A [3]. Let $f(z)$ and $g(z)$ be nonconstant meromorphic functions, $a_j(j=1,2,3,4,5)$ be differentiable complex numbers. If $\overline{E}(a_j,f)\subseteq$

$\overline{E}(a_j, g)(j = 1, 2, 3, 4, 5)$ and

$$\frac{\liminf\limits_{r\to\infty} \sum\limits_{j=1}^{5} \overline{N}(r, \frac{1}{f-a_j})}{\liminf\limits_{r\to\infty} \sum\limits_{j=1}^{5} \overline{N}(r, \frac{1}{g-a_j})} > \frac{1}{2},$$

then $f(z) \equiv g(z)$.

Theorem B [4]. Let $f(z)$ and $g(z)$ be nonconstant meromorphic functions, $a_j(j = 1, 2, 3, 4, 5)$ be different complex numbers, and let $\Delta_\delta = \{z||argz-\theta_0| \leq \delta\}(0 < \delta < \pi)$ be an angular domain satisfying

$$\limsup_{\varepsilon\to o^+} \limsup_{r\to+\infty} \frac{logT(r, \Delta_{\delta-\varepsilon}, f)}{logr} > \omega, \tag{1}$$

where $\omega = \frac{\pi}{2\delta}, T(r, \Delta_{\delta-\varepsilon}, f)$ denotes the Ahlfors characteristic function of f in $\Delta_{\delta-\varepsilon}$. If $f(z)$ and $g(z)$ share $a_j(j = 1, 2, 3, 4, 5)$ IM in $\Delta_{\delta-\varepsilon}$, then $f(z) \equiv g(z)$.

In this paper,we improve the Theorem B and obtain

Theorem 1. Let $f(z)$ and $g(z)$ be two nonconstant meromorphic functions, $a_j(j = 1, 2, 3, 4, 5)$ be different complex numbers. If $\overline{E}(a_j, \Delta_\delta, f) \subseteq \overline{E}(a_j, \Delta_\delta, g), (j = 1, 2, 3, 4, 5)$ and

$$\frac{\liminf\limits_{r\to\infty} \sum\limits_{j=1}^{5} \overline{C}(r, \Delta_\delta, \frac{1}{f-a_j})}{\liminf\limits_{r\to\infty} \sum\limits_{j=1}^{5} \overline{C}(r, \Delta_\delta, \frac{1}{g-a_j})} > \frac{1}{2}, \tag{2}$$

then$f(z) \equiv g(z)$.

G. G. Gundersen (see [2])and Q. C. Zhang (see [4]) obtained the following theorems.

Theorem C [2]. Let $f(z)$ and $g(z)$ be two distinct transcendental meromorphic functions, $a_j \in \hat{\mathbb{C}}(j = 1, 2, 3, 4)$ be different complex numbers. If $f(z)$ and $g(z)$ share a_1, a_2, a_3 CM, and share a_4 IM, then $f(z) \equiv T(g)$, where T is a linear fractional transformation.

Theorem D [4]. Let $f(z)$ and $g(z)$ be two distinct transcendental meromorphic functions, $a_j \in \hat{\mathbb{C}}(j = 1, 2, 3, 4)$ be different complex numbers, and let $\Delta_\delta = \{z||argz - \theta_0| \leq \delta\}(0 < \delta < \pi)$, be an angular domain satisfying (1). If $f(z)$ and $g(z)$ share $a_j(j = 1, 2, 3, 4)$ CM in $\Delta_{\delta-\varepsilon}$, then $f(z)$ is a linear fractional transformation of $g(z)$.

We consider $3CM + 1IM = 4CM$ in one angular domain and get

Theorem 2. Let $f(z)$ and $g(z)$ be two nonconstant meromorphic functions, $a_j \in \hat{\mathbb{C}}(j = 1, 2, 3, 4)$ be different complex numbers. Let $T(r) = max\{T(r, f), T(r, g)\}$,$\rho(r)$ be the precise order of $T(r)$ satisfying

$$\limsup_{r\to 0^+} \limsup_{r\to\infty} \frac{T(r, \Delta_{\delta-\varepsilon}, f)}{\rho(r)\log r} > 0.$$

If $f(z)$ and $g(z)$ share a_1, a_2, a_3 CM in Δ_δ, which share a_4 IM in Δ_δ, then $f(z) \equiv T(g)$, where T is a linear fractional transformation.

3 Main Lemmas

Lemma 1 [1]. Let $f(z)$ be a meromorphic function on an angular domain $\Delta_\delta = \{z||argz - \theta_0| \le \delta\}(0 < \delta < \pi)$. Then for any $a \in \mathbb{C}$, $S_N(r, \Delta_\delta, \frac{1}{f-a}) = S_N(r, \Delta_\delta, f) + O(1)$, and for any $(q \ge 3)$ distinct values $a_j \in \hat{\mathbb{C}}(1, 2, \cdots, q)$,

$$(q-2)S_N(r, \Delta_\delta, f) \le \sum_{i=1}^{q} \overline{C}(r, \Delta_\delta, f = a_j) + R(r, \Delta_\delta, f),$$

$$R(r, \Delta_\delta, f) = A(r, \Delta_\delta, \frac{f'}{f}) + B(r, \Delta_\delta, \frac{f'}{f}) + \sum_{i=1}^{q}\{A(r, \Delta_\delta, \frac{f'}{f-a_j})$$
$$+B(r, \Delta_\delta, \frac{f'}{f-a_j})\} + O(1),$$

where $\overline{C}(r, \Delta_\delta, f = a_j)$ is the corresponding reduced case of $C(r, \Delta_\delta, f = a_j)$, in this case each multiple zero of $f - a_j$ appears only once (ignoring multiplicities).

Lemma 2 [1]. Let $f(z)$ be a meromorphic function in $\mathbb{C}$, $\Delta_\delta = \{z||argz - \theta_0| \le \delta\}(0 < \delta < \pi)$ be an angular domain, $\omega = \frac{\pi}{2\delta}$, then

$$A(r, \Delta_\delta, \frac{f'}{f}) \le K\{(\frac{R}{r})^\omega \int_1^R \frac{\log^+ T(t, f)}{t^{\omega+1}}dt + \log^+ \frac{r}{R-r} + \log\frac{R}{r} + 1\},$$

$$B(r, \Delta_\delta, \frac{f'}{f}) \le \frac{4\omega}{r^\omega} m(r, \frac{f'}{f}),$$

where $1 < r < R < +\infty, K$ is a nonzero constant.

Lemma 3 [4]. Let $f(z)$ be a meromorphic function in $\mathbb{C}$, $\Delta_\delta = \{z||argz - \theta_0| \le \delta\}(0 < \delta < \pi)$ be an angular domain, then

$$R(r, \Delta_\delta, \frac{f'}{f}) = \begin{cases} O(1), \\ O(log(U(r))). \end{cases}$$

where $R(r, \Delta_\delta, f)$ is defined as in (3), $U(r) = r^{\rho(r)}$, and $\rho(r)$ is the precise order of $T(r, f)$ when $f(z)$ is of infinite order.

4 Proof of main results

Theorem 1. Let $f(z)$ and $g(z)$ be two nonconstant meromorphic functions, $a_j (j = 1, 2, 3, 4, 5)$ be different complex numbers. If $\overline{E}(a_j, \Delta_\delta, f) \subseteq \overline{E}(a_j, \Delta_\delta, g), (j = 1, 2, 3, 4, 5)$ and

$$\frac{\liminf\limits_{r\to\infty} \sum\limits_{j=1}^{5} \overline{C}(r, \Delta_\delta, \frac{1}{f-a_j})}{\liminf\limits_{r\to\infty} \sum\limits_{j=1}^{5} \overline{C}(r, \Delta_\delta, \frac{1}{g-a_j})} > \frac{1}{2}. \tag{3}$$

then $f(z) \equiv g(z)$.

Proof. On the contrary, we assume that $f \neq g$. Without loss of generality, let $a_j (j = 1, 2, 3, 4, 5)$ be finite complex numbers, from lemma 1, we have

$$3S_N(r, \Delta_{\delta-\varepsilon}, f) \leq \sum_{j=1}^{5} \overline{C}(r, \Delta_{\delta-\varepsilon}, f = a_j) + R(r, \Delta_{\delta-\varepsilon}, f), \tag{4}$$

and

$$3S_N(r, \Delta_{\delta-\varepsilon}, g) \leq \sum_{j=1}^{5} \overline{C}(r, \Delta_{\delta-\varepsilon}, g = a_j) + R(r, \Delta_{\delta-\varepsilon}, g). \tag{5}$$

Thus

$$\sum_{j=1}^{5} \overline{C}(r, \Delta_{\delta-\varepsilon}, f = a_j) = O(S_N(r, \Delta_{\delta-\varepsilon}, f)), (r \notin E), \tag{6}$$

and

$$\sum_{j=1}^{5} \overline{C}(r, \Delta_{\delta-\varepsilon}, g = a_j) = O(S_N(r, \Delta_{\delta-\varepsilon}, g)), (r \notin E).$$

We suppose $f(z) \neq g(z)$, according to $\overline{E}(a_j, \Delta_\delta, f) \subseteq \overline{E}(a_j, \Delta_\delta, g), (j = 1, 2, 3, 4, 5)$, then

$$\begin{aligned}\sum_{j=1}^{5} \overline{C}(r, \Delta_{\delta-\varepsilon}, f = a_j) &\leq \overline{C}(r, \Delta_{\delta-\varepsilon}, \frac{1}{f-g}) \\ &\leq S_N(r, \Delta_{\delta-\varepsilon}, f) + S_N(r, \Delta_{\delta-\varepsilon}, g) + O(1).\end{aligned}$$

From (5) and (6), we obtain

$$\sum_{j=1}^{5} \overline{C}(r, \Delta_{\delta-\varepsilon}, f = a_j) \le (\frac{1}{3} + O(1)) \sum_{j=1}^{5} \overline{C}(r, \Delta_{\delta-\varepsilon}, f = a_j) +$$

$$+(\frac{1}{3} + O(1)) \sum_{j=1}^{5} \overline{C}(r, \Delta_{\delta-\varepsilon}, g = a_j),$$

$$(\frac{2}{3} + O(1)) \sum_{j=1}^{5} \overline{C}(r, \Delta_{\delta-\varepsilon}, f = a_j) \le (\frac{1}{3} + O(1)) \sum_{j=1}^{5} \overline{C}(r, \Delta_{\delta-\varepsilon}, g = a_j), \qquad (7)$$

from (8), we have

$$\frac{\liminf\limits_{r\to\infty} \sum\limits_{j=1}^{5} \overline{C}(r, \Delta_\delta, \frac{1}{f-a_j})}{\liminf\limits_{r\to\infty} \sum\limits_{j=1}^{5} \overline{C}(r, \Delta_\delta, \frac{1}{g-a_j})} \le \frac{1}{2},$$

which is a contradiction to our hypothesis (2). The proof of Theorem 1 is complete.

Theorem 2. Let $f(z)$ and $g(z)$ be two nonconstant meromorphic functions, $a_j \in \hat{\mathbb{C}}(j = 1, 2, 3, 4)$ be different complex numbers. Let $T(r) = max\{T(r, f), T(r, g)\}$, $\rho(r)$ be the precise order of $T(r)$ satisfying

$$\limsup_{r\to 0^+} \limsup_{r\to\infty} \frac{T(r, \Delta_{\delta-\varepsilon}, f)}{\rho(r) \log r} > 0.$$

If $f(z)$and $g(z)$ share a_1, a_2, a_3 CM in Δ_δ, which share a_4 IM in Δ_δ, then

$$f(z) \equiv T(g),$$

where T is a linear fractional transformation.

Proof. On the contrary, we assume that f is not a linear fractional transformation of $g(z)$. Without loss of generality, let $a_1 = \infty$ and

$$F = \frac{f'(f - a_4)}{(f - a_2)(f - a_3)} - \frac{g'(g - a_4)}{(g - a_2)(g - a_3)}. \qquad (8)$$

Suppose $F(z) \not\equiv 0$,

$$A(r, \Delta_{\delta-\varepsilon}, F) + B(r, \Delta_{\delta-\varepsilon}, F) \le$$

$$\leq \sum_{j=2}^{4}\{A(r,\Delta_{\delta-\varepsilon},\frac{f'}{f-a_j})+B(r,\Delta_{\delta-\varepsilon},\frac{f'}{f-a_j})\}+$$

$$+\sum_{j=2}^{4}\{A(r,\Delta_{\delta-\varepsilon},\frac{g'}{g-a_j})+B(r,\Delta_{\delta-\varepsilon},\frac{g'}{g-a_j})\}+O(1).$$

We have that $f(z)$ and $g(z)$ share a_1, a_2, a_3 CM in Δ_δ, so $F(z)$ has no poles, and $C(r,\Delta_{\delta-\varepsilon},F)=0$. Hence

$$S_N(r,\Delta_{\delta-\varepsilon},F)\leq C(r,\Delta_{\delta-\varepsilon},F)+R(r,\Delta_{\delta-\varepsilon},F)\leq R(r,\Delta_{\delta-\varepsilon},F).$$

Noting that share $f(z)$ and $g(z)$ share a_4 IM in Δ_δ, and from (9), we get

$$\begin{aligned}\overline{C}(r,\Delta_{\delta-\varepsilon},\frac{1}{f-a_4}) &= \overline{C}(r,\Delta_{\delta-\varepsilon},\frac{1}{g-a_4})\\ &\leq C(r,\Delta_{\delta-\varepsilon},\frac{1}{F})\leq S_N(r,\Delta_{\delta-\varepsilon},\frac{1}{F})\\ &\leq S_N(r,\Delta_{\delta-\varepsilon},F)+O(1)\\ &\leq R(r,\Delta_{\delta-\varepsilon},F).\end{aligned}$$

Without loss of generality, we suppose

$$\overline{C}(r,\Delta_{\delta-\varepsilon},\frac{1}{f-a_2})\neq R(r,\Delta_{\delta-\varepsilon},F). \tag{9}$$

Let

$$G=\frac{f'(f-a_2)}{(f-a_3)(f-a_4)}-\frac{g'(g-a_2)}{(g-a_3)(g-a_4)}.$$

If $G(z)\equiv 0$, then a_4 is a CM sharing value of $f(z)$ and $g(z)$ in Δ_δ, hence the Theorem 2 holds.

Suppose that $G(z)\not\equiv 0$,

$$A(r,\Delta_{\delta-\varepsilon},G)+B(r,\Delta_{\delta-\varepsilon},G)\leq R(r).$$

Since $f(z)$and $g(z)$ share a_1, a_2, a_3 CM in Δ_δ, which share a_4 IM in Δ_δ, therefore

$$C(r,\Delta_{\delta-\varepsilon},G)\leq \overline{C}(r,\Delta_{\delta-\varepsilon},\frac{1}{f-a_4})\leq R(r).$$

Hence

$$S_N(r,\Delta_{\delta-\varepsilon},G)\leq R(r).$$

Moreover

$$\overline{C}(r,\Delta_{\delta-\varepsilon},\frac{1}{f-a_2})\leq C(r,\Delta_{\delta-\varepsilon},\frac{1}{G})\leq S_N(r,\Delta_{\delta-\varepsilon},G)+O(1)\leq R(r),$$

which contradicts that (10). The proof is complete.

Remark. Nevanlinna theory in an angular domain plays a key role in this paper. Theorem 1 and Theorem 2 extend Theorem A and Theorem C in the whole complex plane to an angular domain. If (2) in the Theorem 1 instead of (1), we can obtain Theorem B. When we count the multiplicities of sharing values, we get Theorem D from Theorem 2.

References

[1] A. A. Gol'dberg, I. V. Ostrovskii, The Distribution of Values of Meromorphic Function, *Nauka*, Moscow, 1970.

[2] G. G. Gundersen, Meromorphic Functions that share three or four vaues, *London Math. Soc.*, 20(1979), 457-466.

[3] C. C. Yang, H.-X. Yi, Uniqueness Theory of Meromorphic Functions, *Science Press*, Beijing (2003).

[4] Q. C. Zhang, Meromorphic functions sharing values in an angular domain, *Math. Anal. Appl.*, 349 (2009), 100-112.

[5] J.H. Zheng, On uniqueness of meromorphic functions with shared values in some angular domains, *Canad. Math. Bull.*, 47(2004),152-160.

Department of Mathematics
Honghe University, Mengzi, 661100, China,
Email: hepingky@163.com