An. Şt. Univ. Ovidius Constanţa Vol. 18(2), 2010, 189–200

A VERSION OF THE GABRIEL-POPESCU THEOREM

Constantin Năstăsescu, Costel Chiteş

Abstract

The aim of this article is to prove another version of the celebrated Gabriel-Popescu theorem and to present some applications.

Introduction

Let $\mathcal{C}$ be a Grothendieck category with the generator U. Denote by A the ring $\mathrm{End}_{\mathcal{C}}(U)$ of endomorphisms of U and by S the functor $\mathrm{Hom}_{\mathcal{C}}(U,-)$ from the category $\mathcal{C}$ to the category $\mathrm{Mod}-A$ of right A-modules. Let $T : \mathrm{Mod}-A \longrightarrow \mathcal{C}$ be a left adjoint of S (such a T always exists according to Gabriel). Denote by $\Phi : T \circ S \longrightarrow 1_{\mathcal{C}}$ and by $\Psi : 1_{\mathrm{Mod}-A} \longrightarrow S \circ T$ the functorial morphisms associated to the adjoint functors S and T.

The Gabriel-Popescu Theorem. *With the above notations the following assertions hold:*

1. *Φ is a functorial isomorphism.*
2. *The functor S is fully faithful.*
3. *The functor T is exact.*

Key Words: Gabriel-Popescu theorem, Grothendieck category, enough idempotents, enough units.
2010 Mathematics Subject Classification: 18E15, 16D90.
Received: November, 2009
Accepted: September, 2010

Let $\mathrm{Ker}(T) = \{M \in \mathrm{Mod}-A \,|\, T(M) = 0\}$. This is a localizing subcategory of $\mathrm{Mod} - A$ (i.e. it is closed under subobjects, quotient objects, extensions and arbitrary direct sums). The Gabriel-Popescu Theorem says that $\mathcal{C}$ is equivalent to the quotient category of $\mathrm{Mod} - A$ by the localizing subcategory $\mathrm{Ker}(T)$.

We know at least three proofs of this theorem.

The first is the original proof of P. Gabriel and N. Popescu which appears in [4]. More detailed presentations can be found in [13] and [14].

The second one is a very short proof given by M. Takeuchi [15]. The reader can find detailed presentations of this proof in [10] and [11].

The third proof belongs to B. Mitchell [9]. It is based on a result of Grothendick from [6] saying that a Grothendick category has enough injective objects.

In this paper, using the Gabriel-Popescu Theorem, we give a short proof of the fact that any Grothendieck category $\mathcal{C}$ is a quotient category of $\mathrm{Funct}(\mathcal{U}, \mathrm{Ab})$, where $\mathcal{U}$ is a family of generators for $\mathcal{C}$. This result appears in [5] with quite a long proof that follows closely the initial proof of the Gabriel-Popescu Theorem.

1 Preliminaries

In this section we review some preliminary results. For more details, the reader is referred to [16] and [1].

Let R be a ring (not necessarily with unit). We say that:

(1) *R has enough units* if for all finite non-empty subset X of R there exists an idempotent $e \in R$ such that $X \subseteq eRe$.

(2) *R has enough right idempotents* if there exists an orthogonal family $\{e_i\}_{i\in I}$ of idempotents (i.e. $e_ie_j = \delta_{ij}e_i$, for all $i, j \in I$) such that $R = \bigoplus_{i\in I} Re_i$.

(3) *R has enough idempotents* if there exists an orthogonal family $\{e_i\}_{i\in I}$ of idempotents such that $R = \bigoplus_{i\in I} Re_i = \bigoplus_{i\in I} e_iR$.

A ring with enough right idempotents has enough units (see [1]). As a consequence, any ring with enough idempotents is a ring with enough units.

Let R be a ring with enough units. We define the category $\mathrm{MOD} - R$ as follows: the objects are the right R-modules with the property $MR = M$ and the morphisms are the right R-module morphisms. We list some properties of $\mathrm{MOD} - R$:

(i) If R has a unity, $\mathrm{MOD} - R$ is the category of unitary right R-modules.

(ii) If R is a ring with enough units, $\mathrm{MOD} - R$ is a localizing subcategory of $\mathrm{Mod} - R$ (see [1]).

In fact, $\mathrm{MOD} - R$ is similar to the category of unitary modules over a unitary ring (there are some differences: for example, if $(M_i)_{i\in I}$ is a family of objects in $\mathrm{MOD} - R$, the direct product is obtained by multiplying the cartesian product $\prod_{i\in I} M_i$ by R; so the direct product of the given family in $\mathrm{MOD} - R$ is $(\prod_{i\in I} M_i)R$).

2 A Version of the Gabriel-Popescu Theorem

Let $\mathcal{A}$ be a Grothendick category and $\mathcal{U} = \{U_i\}_{i\in I}$ be a family of generators for $\mathcal{A}$. We construct a ring $R_{\mathcal{U}}$ with enough idempotents as follows. As an additive group, $R_{\mathcal{U}} = \bigoplus_{i,j\in I} \mathrm{Hom}_{\mathcal{A}}(U_i, U_j)$; the multiplication is defined by the rule: if $f \in \mathrm{Hom}_{\mathcal{A}}(U_i, U_j)$ and $g \in \mathrm{Hom}_{\mathcal{A}}(U_k, U_l)$, then $fg = f \circ g$ if $i = l$ and 0 otherwise. If we put $e_i = 1_{U_i}$, $i \in I$, then $\{e_i\}_{i\in I}$ is a family of orthogonal idempotents. One can easily see that

$$R_{\mathcal{U}} = \bigoplus_{i\in I} R_{\mathcal{U}} e_i = \bigoplus_{i\in I} e_i R_{\mathcal{U}},$$

hence $R_{\mathcal{U}}$ is a ring with enough idempotents.

We denote by $\mathrm{MOD} - R_{\mathcal{U}}$ the category of $R_{\mathcal{U}}$-right modules M which satisfy $MR_{\mathcal{U}} = M$. In this category, $(e_i R_{\mathcal{U}})_{i\in I}$ is a family of finitely generated projective generators. Moreover,

$$\mathrm{Hom}_{R_{\mathcal{U}}}(e_i R_{\mathcal{U}}, e_j R_{\mathcal{U}}) \cong e_j R_{\mathcal{U}} e_i \cong \mathrm{Hom}_{\mathcal{A}}(U_i, U_j).$$

In particular, $\mathrm{End}_R(e_i R_{\mathcal{U}}) \cong e_i R_{\mathcal{U}} e_i \cong \mathrm{End}_{\mathcal{A}}(U_i)$. Also, if $U = \bigoplus_{i\in I} U_i$, then it is clear that $R_{\mathcal{U}}$ is a non-unital subring of $\mathrm{End}_{\mathcal{A}}(U)$.

The family $\mathcal{U}$ can be viewed as a small subcategory of $\mathcal{A}$ with objects U_i, $i \in I$. Consider the category $\mathrm{Funct}(\mathcal{U}, \mathrm{Ab})$ which is equivalent to $\mathrm{MOD} - R_{\mathcal{U}}$ (see [3]).

In this section we give a short proof of the following result which, as noted in the introduction, may be found, for example, in [5]:

Theorem 2.1. *Let $\mathcal{A}$ be a Grothendick category and $\mathcal{U} = \{U_i\}_{i\in I}$ a family of generators for $\mathcal{A}$. Consider the canonical functor $F : \mathcal{A} \longrightarrow \mathrm{MOD} - R_{\mathcal{U}}$ with $F(M) = \bigoplus_{i\in I} \mathrm{Hom}_{\mathcal{A}}(U_i, M)$, for $M \in \mathcal{A}$. Then F has a left adjoint exact functor G and $G \circ F \cong 1_{\mathcal{A}}$. In particular, G induces an equivalence between $\mathcal{A}$ and the quotient category $\mathrm{MOD} - R_{\mathcal{U}}/\mathrm{Ker}(G)$, where $\mathrm{Ker}(G) = \{M \in \mathrm{MOD} - R_{\mathcal{U}} \mid G(M) = 0\}$ is a localizing subcategory of $\mathrm{MOD} - R_{\mathcal{U}}$.*

We note that for $\mathcal{U} = \{U\}$, we obtain the classical Gabriel-Popescu Theorem. We have $F(U_i) = e_i R_{\mathcal{U}}$ and $G(e_i R_{\mathcal{U}}) = U_i$.

Proof. Our proof is made up of two steps.

Step 1. We construct a ring R with enough right idempotents associated to the family of generators $\mathcal{U}$. Let $A = \mathrm{End}_{\mathcal{A}}(U)$, where $U = \bigoplus_{i\in I} U_i$. Following [8], we can consider the family of orthogonal idempotent elements $\{\eta_i\}_{i\in I}$ of A, where $\eta_i : U \longrightarrow U$, $\eta_i = \epsilon_i \circ \pi_i$, with ϵ_i and π_i being the canonical injections and projections. For any $X \in \mathcal{A}$ and $f \in \mathrm{Hom}_{\mathcal{A}}(U, X)$, consider the set

$$\overline{\mathrm{Hom}}_{\mathcal{A}}(U, X) = \{f \in \mathrm{Hom}_{\mathcal{A}}(U, X) \,|\, \mathrm{Supp}(f) < \infty\},$$

where $\mathrm{Supp}(f) = \{i \in I \,|\, f \circ \eta_i \neq 0\}$. We denote by $R = \overline{\mathrm{End}}_{\mathcal{A}}(U)$ the set $\overline{\mathrm{Hom}}_{\mathcal{A}}(U, U)$, which is a non-unital subring of the endomorphism ring $A = \mathrm{End}_{\mathcal{A}}(U)$. In fact, R is a left idempotent ideal of A and $R = \bigoplus_{i\in I} R\eta_i$ is a ring with enough right idempotents [8]. Hence $\overline{\mathrm{Hom}}_{\mathcal{A}}(U, X) = \mathrm{Hom}_{\mathcal{A}}(U, X)R$. Moreover, if every U_i is a small object (i.e. the covariant functor $\mathrm{Hom}_{\mathcal{A}}(X, -)$ commutes with direct sums), then $R = \bigoplus_{i\in I} R\eta_i = \bigoplus_{i\in I} \eta_i R$, so R is a ring with enough idempotents (see [8]).

Consider the category $\mathrm{MOD} - R$ (the right R-modules M with $MR = M$). From [1, Proposition 1.1] we have the following sequence of categories and adjoint functors:

$$\begin{tikzcd} \mathcal{A} \arrow[r, "S"] & \mathrm{Mod} - A \arrow[l, bend left, "T"] \arrow[r, "t"] & \mathrm{MOD} - R \arrow[l, bend left, "-\otimes_R A"] \end{tikzcd},$$

where S and T are the functors from the Gabriel-Popescu theorem and t is defined by $t(N) = NR$, for all $N \in \mathrm{Mod} - A$, and it is a right exact adjoint functor of $- \otimes_R A$ such that $t \circ (- \otimes_R A) \simeq 1_{\mathrm{Mod}-R}$.

For any object $X \in \mathcal{A}$, we define the functor $S' = t \circ S$ by $S'(X) = \mathrm{Hom}_{\mathcal{A}}(U, X)R = \overline{\mathrm{Hom}}_{\mathcal{A}}(U, X)$.

For the second step, we need some preliminary results.

Proposition 2.2. *Let $\mathcal{A}$ be a Grothendieck category, $\{U_i\}_{i\in I}$ be a family of generators of $\mathcal{A}$ and S, T, t, S' be the above functors. Then the following assertions hold:*

1. *$T' = T \circ (- \otimes_C A)$ is a left adjoint functor of $S' = t \circ S$.*

2. *T' is an exact functor.*

3. *$T' \circ S' \simeq 1_A$.*

Moreover, S' induces an equivalence between the category $\mathcal{A}$ and the quotient category of $\mathrm{MOD} - R$ *corresponding to the torsion class* $\mathrm{Ker}(T')$*.*

A proof of this result, consisting of making a series of simple constructions and applying the Gabriel-Popescu Theorem, can be found in [1]. An alternate proof, which follows closely the initial proof of the Gabriel-Popescu Theorem from [4], is given in [8].

For the convenience of the reader we sketch the proofs of the parts in the above proposition which are important in the sequel.

(1) This follows immediately from the Gabriel-Popescu Theorem and the fact that the composition of adjoint functors gives adjoint functors (see [7]).

(2) Let $0 \longrightarrow M' \xrightarrow{u} M \xrightarrow{v} M'' \longrightarrow 0$ be an exact sequence in $\mathrm{MOD}-R$. By applying the functor $- \otimes_R A$, we obtain the exact sequence

$$M' \otimes_R A \xrightarrow{u\otimes A} M \otimes_R A \xrightarrow{v \otimes A} M'' \otimes_R A \longrightarrow 0.$$

If we put $X = \mathrm{Ker}(u \otimes A)$, then we have $t(X) = t(\mathrm{Ker}(u \otimes A)) \simeq \mathrm{Ker}(u) = 0$. Therefore, $XR = 0$ or $X(RA) = 0$. For $RA \in \mathrm{Ker}(T)$, we have $X \in \mathrm{Ker}(T)$, so $T(X) = 0$. Hence, T' is an exact functor.

(3) Let X be an object of $\mathcal{A}$. Consider the canonical morphism of A-modules:

$$\gamma : S(X)R \otimes_R A \longrightarrow S(X),\ \gamma(x \otimes a) = xa,$$

for any $x \in S(X)R$ and $a \in A$. If we apply the functor t to the exact sequence

$$0 \longrightarrow \mathrm{Ker}(\gamma) \longrightarrow S(X)R \otimes_R A \xrightarrow{\gamma} S(X) \longrightarrow \mathrm{Coker}(\gamma) \longrightarrow 0,$$

then we obtain $t(\mathrm{Ker}(\gamma)) = t(\mathrm{Coker}(\gamma)) = 0$. It follows that $\mathrm{Ker}(\gamma)R = \mathrm{Coker}(\gamma)R = 0$, and $\mathrm{Ker}(\gamma)(RA) = \mathrm{Coker}(\gamma)(RA) = 0$, so $\mathrm{Ker}(\gamma)$, $\mathrm{Coker}(\gamma) \in \mathrm{Ker}(T)$. In conclusion, $T(\gamma)$ is an exact isomorphism for T, hence $T'S'(X) \simeq T(t(S(X)) \otimes_R A) \simeq T(S(X)) \simeq X$.

We need two more lemmas.

Lemma 2.3. *Let $R = \bigoplus_{i \in I} R\eta_i$ be a ring with enough right idempotents. The following assertions hold:*

1. *ηR is a finitely generated projective R-module, for every idempotent $\eta \in R$.*

2. *The family $\{\eta_i R\}_{i \in I}$ is a system of small projective generators of* $\mathrm{Mod}-R$*.*

Proof. (1) Let $\eta \in R$ be an arbitrary idempotent. We will prove first that ηR is R-projective. Consider the following diagram in $\mathrm{MOD} - R$:

$$\begin{tikzcd} & \eta R \arrow[dl, "h"'] \arrow[d, "f"] & \\ M \arrow[r, "g"] & M' \arrow[r] & 0 \end{tikzcd} .$$

If $f(\eta) = m'$, then $f(\eta) = g(m)$, for a given $m \in M$. For every $r \in R$, $h : \eta R \longrightarrow M$, $\eta r \mapsto m\eta r$, is a morphism of right R-modules and $g \circ h = f$, so the diagram is commutative. Hence, ηR is R-projective.

Let $\eta R = \bigcup_{i\in I} M_i$, where $\{M_i\}_{i\in I}$ is filtering family of subobjects of ηR. Because $\eta \in \eta R$, there exists an $i \in I$ such that $\eta \in M_i$. Therefore, $\eta R \subseteq M_i R = M_i$, so ηR is finitely generated in $\mathrm{MOD} - R$.

(2) We will prove that $\{\eta_i R\}_{i\in I}$ is a family of generators for $\mathrm{MOD} - R$. Since $\eta_i R$ is generated by one element, it is a small object in $\mathrm{MOD} - R$ for any $i \in I$.

Consider $M \in \mathrm{MOD} - R$, $M' \subseteq M$ with $M' \neq M$, and $x \in M \setminus M'$. Then $MR = M$ and $x = \sum_{i\in I} m_i \eta_i^x$. From the orthogonality relation $x\eta_j^x = m_j \eta_j^x$ we obtain that there exists an $i \in I$ such that $m_i \eta_i^x \notin M'$. So there exists $m \in M$ with $m\eta_i \notin M'$ for some $i \in I$. Therefore, we have $\mathrm{Im}(f) \nsubseteq M'$, where $f : \eta_i R \longrightarrow M$, $\eta_i r \mapsto m\eta_i r$, since $f(\eta_i) = m \cdot \eta_i \notin M'$. Hence $\{\eta_i R\}_{i\in I}$ is a family of generators for $\mathrm{MOD} - R$. □

Lemma 2.4. *If $\mathcal{A}$ is a category with a family of small projective generators then it is equivalent to a category of modules over a ring with enough idempotents.*

Proof. The proof follows from Proposition 2.2. □

Now we continue the proof of the main theorem.

Step 2. If R is the ring constructed in Step 1, then $\mathcal{U}' = \{\eta_i R\}_{i\in I}$ is a family of finitely generated projective generators for the category $\mathrm{MOD} - R$. We have the ring isomorphism $R_{\mathcal{U}'} \cong R_{\mathcal{U}}$.

Using Lemma 2.4, we conclude that $\mathrm{MOD} - R$ is equivalent to $\mathrm{MOD} - R_{\mathcal{U}}$. The equivalence is given by the scalar restriction functor $i_* : \mathrm{MOD} - R \longrightarrow \mathrm{MOD} - R_{\mathcal{U}}$ with the functor $- \otimes_{R_{\mathcal{U}}} R$ as its inverse, where $i : R_{\mathcal{U}} \longrightarrow R$ is the inclusion morphism. According to Proposition 2.2, we have the following sequence of categories and adjoint functors:

$$\begin{tikzcd} \mathcal{A} \arrow[r, "S"] & \mathrm{Mod} - A \arrow[r, "t"] \arrow[l, bend left, "T"] & \mathrm{MOD} - R \arrow[r, "i_*"] \arrow[l, bend left, "-\otimes_R A"] & \mathrm{MOD} - R_{\mathcal{U}} \arrow[l, bend left, "-\otimes_{R_{\mathcal{U}}}"] \end{tikzcd} .$$

We put $F = i_* \circ t \circ S : \mathcal{A} \longrightarrow \mathrm{MOD} - R_{\mathcal{U}}$ and $G = T \circ (- \otimes_R A) \circ (-\otimes_{R_{\mathcal{U}}}) :$ $\mathrm{MOD} - R_{\mathcal{U}} \longrightarrow \mathcal{A}$. We have $F(X) = \bigoplus_{i\in I} \mathrm{Hom}_{\mathcal{A}}(U_i, X)$ for any $X \in \mathcal{A}$. From the above lemmas, we conclude that G is a left adjoint for F. Moreover, G is an exact functor and $G \circ F \cong 1_{\mathcal{A}}$, so the proof ends. □

As an immediate corollary of Theorem 2.1, we obtain the following result from [5]:

Corollary 2.5. *Let $\mathcal{A}$ be a Grothendick category and $\mathcal{U} = \{U_i\}_{i\in I}$ a family of generators for $\mathcal{A}$. Then $\mathcal{A}$ is a quotient category of* $\mathrm{Funct}(\mathcal{U}, Ab)$.

Proof. Since we know from a result of Gabriel (in [3]) that $\mathrm{Funct}(\mathcal{U}, Ab)$ is equivalent to $\mathrm{MOD} - R_{\mathcal{U}}$, the conclusion follows now if we apply Theorem 2.1. □

We recall some facts about semiartinian categories and the Loewy series. Details can be found, for example, in [11].

Let $\mathcal{A}$ be a category. An object $M \in \mathcal{A}$ is called *semiartinian* if for every subobject M' of M with $M' \neq M$, M/M' contains a simple object.

If M is an object of $\mathcal{A}$, we construct by transfinite induction, an ascending chain of subobjects of M as follows. Let α be an arbitrary ordinal.

If $\alpha = 0$, we set $L_0(M) = 0$.

If $\alpha = 1$, we set $L_1(M) = s_0(M)$, where s_0 is the the *socle* of M, i.e. the sum of all the simple subobjects of M.

If $\alpha = \beta + 1$, then $L_\alpha(M)$ is the subobject of M given by

$$L_\alpha(M)/L_\beta(M) = s_0(M/L_\beta(M)).$$

If α is a limit ordinal (i.e. it has no predecessor), we set

$$L_\alpha(M) = \bigcup_{\beta<\alpha} L_\beta(M).$$

In this way, we obtain an ascending chain of subobjects of M

$$0 = L_0(M) \subseteq L_1(M) \subseteq \ldots \subseteq L_\alpha(M) \subseteq L_{\alpha+1}(M) \subseteq \ldots$$

indexed by the set of ordinal numbers. This chain is called the *Loewy series associated to M*. The semisimple objects $L_{\alpha+1}(M)/L_\alpha(M)$ are called the *factors* of the Loewy series. The smallest ordinal number that satisfies $L_\alpha(M) = L_{\alpha+1}(M) = \ldots$ is called the *Loewy length* of the series and it is denoted by $\lambda(M)$.

It is clear that for two ordinal numbers $\xi < \eta \leq \lambda(M)$ we have that $L_\xi(M) \neq L_\eta(M)$. If there is an ordinal ξ with $L_\xi(M) = M$, then we say that M is *of defined Loewy length.*

Some well-known properties of semiartinian categories and of the Loewy series are listed below:

1. If M este the direct sum of the family of subobjects $(M_i)_{i\in I}$, then

$$L_\alpha(M) = \bigoplus_{i\in I} L_\alpha(M_i),$$

for any ordinal number α.

2. If N is a subobject of M, the following assertions hold:
   (a) $\lambda(N) \leq \lambda(M)$.
   (b) If M is of defined Loewy length, then $\lambda(M/N) \leq \lambda(M)$.
   (c) $\lambda(M) \leq \lambda(N) + \lambda(M/N)$.

3. If M is semiartinian object, then $s_0(M)$ is an essential subobject of M.

4. Let N be a subobject of M. Then M is semiartinian if and only if N and M/N are semiartinian.

5. A direct sum of semiartinian objects is a semiartinian object.

6. An object M is semiartinian if and only if M is of defined Loewy length.

7. If $0 \to M' \to M \to M'' \to 0$ is an exact sequence of objects, then the sequence $0 \to s_0(M') \to s_0(M) \to s_0(M'') \to 0$ is also exact.

If $\mathcal{A}$ is a semiartinian Grothendick category (i.e. any object is semiartinian) with the family of generators $\mathcal{U} = \{U_i\}_{i\in I}$, let $\lambda(\mathcal{A}) = \sup\{\lambda(M) \mid M \in \mathcal{A}\} = \sup\{\lambda(U_i) \mid i \in I\}$. Then $\lambda(\mathcal{A})$ is an ordinal number and it always exists.

Corollary 2.6. *Let $\mathcal{A}$ be a semiartinian Grothendick category and $\mathcal{U} = \{U_i\}_{i\in I}$ a family of generators of $\mathcal{A}$ such that $\lambda(\mathcal{A}) < \infty$ and the U_i's are injective objects. Then $\mathcal{A}$ is a semiperfect category (i.e. any simple object is contained in a projective one). In particular, $\mathcal{A}$ has enough projective objects.*

Proof. Following the steps of the proof of [1, Theorem 4.1] and applying Theorem 2.1, we obtain the desired conclusion. $\square$

Let C be a coalgebra over a field k. We denote by ${}^C\mathcal{M}$ the category of left comodules over the coalgebra C. For details on coalgebras and comodules, the reader is referred to [2]. If $M \in^C \mathcal{M}$, we consider the Loewy series of M:

$$0 = M_0 \subseteq M_1 \subseteq \ldots \subseteq M_n \subseteq \ldots \subseteq M.$$

We have that $\bigcup_{i\geq 0}^{\infty} M_i = M$ (see [12]), so the Loewy length has the property $\lambda(M) \leq \omega$, where ω is the first transfinite ordinal. If $\lambda(M) = n < \infty$, we say that M is a comodule of finite length. If $M = C$ (as a left C-comodule), the series $0 = C_0 \subseteq C_1 \subseteq \ldots \subseteq C_n \subseteq \ldots \subseteq C$ is called the *coradical filtration* of the coalgebra C.

Corollary 2.7. *Let C be a coalgebra with a finite coradical filtration. If C is a generator of ${}^C\mathcal{M}$, then C is a quasi-co-Frobenius coalgebra.*

Proof. C can be written as the sum of a family of injective indecomposable objects $\{U_i\}_{i\in I}$ in ${}^C\mathcal{M}$, because C is a coalgebra that has the coradical filtration of finite length $n \geq 1$. From $\lambda(C) = n$, we obtain that $\lambda(U_i) \leq n$, for all $i \in I$, hence $\mathrm{End}_{{}^C\mathcal{M}}(U_i)$ is a semiprimary ring, for all $i \in I$ (i.e. it is a local ring with the Jacobson radical of nilpotence degree smaller or equal to n). Since the conditions of the Corollary 2.6 hold in our case, we deduce that the category ${}^C\mathcal{M}$ is semiperfect, so from [12, Theorem 3.1] we obtain that C is quasi-co-Frobenius. $\square$

3 Acknowledgment

The work of C. Năstăsescu was supported by Grant ID_1005, contract no. 434/01.10.2007 of C.N.C.S.I.S.

References

[1] Castaño-Iglesias, F., Enache, P., Năstăsescu, C. and Torrecillas, B., *Gabriel-Popescu type theorems and applications*, Bull. Sci. Math., **128**(4) (2004), 323–332.

[2] Dăscălescu, S., Năstăsescu, C. and Raianu, Ş, *Hopf Algebras. An Introduction*, Pure and Applied Mathematics Volume, **235**, Marcel Dekker, New York-Basel, 2001.

[3] Gabriel, P. *Des catégories abéliennes*, Bull. Soc. Math. France, **90** (1962), 323–448.

[4] Gabriel, P. and Popescu, N., *Caractérisation des catégories abéliennes avec générateurs et limites inductives exactes*, C. R. Acad. Sci. Paris, **258** (1964), 4188–4190.

[5] Garkusha, G., *Grothendieck categories*, St. Petersburg Math. J., **13**(2) (2002), 149–200.

[6] Grothendieck, G., *Sur quelques points d'algèbre homologique*, Tôhoku Math. J., **9** (1957), 119–221.

[7] Mac Lane, S., *Categories for the Working Mathematician*, Graduate Texts in Mathematics, Volume **5**, Springer-Verlag, New York-Berlin-Heidelberg, 1971.

[8] Menini, C., *Gabriel-Popescu type theorems and graded modules*, in Van Oystaeyen, F. and Le Bruyn, L. (eds.), Proceedings of the N.A.T.O. Advanced Research Workshop on *Perpectives in Ring Theory* (1987, Antwerp, Belgium), Kluwer Academic Publishers, Dordrecht, 1988, pp. 239–251.

[9] Mitchell, B., *A quick proof of the Gabriel-Popescu theorem*, J. Pure Appl. Algebra, **20** (1981), 313–315.

[10] Năstăsescu, C., *Teorie della torsione*, Quaderni dei Gruppi di Ricerca del Consiglio Nazionale delle Ricerche, Istituto Matematico dell' Università di Ferrara, 1974.

[11] Năstăsescu, C., *Inele. Module. Categorii*, Ed. Academiei, Bucureşti, 1976.

[12] Năstăsescu, C., Torrecillas, B. and Van Oystaeyen, F., *When is a coalgebra a generator?*, Algebr. Represent. Theory, **11** (2008), 179–190.

[13] Popescu, N., *Abelian Categories with Applications to Rings and Modules*, L.M.S. Monograph No. 3, Academic Press, London, 1973.

[14] Stenström, B., *Rings of Quotients*, Springer-Verlag, Berlin-Heidelberg-New York, 1975.

[15] Takeuchi, M., *A simple proof of Gabriel and Popesco's theorem*, J. Algebra, **18** (1971), 112–113.

[16] Wisbauer, R., *Foundations of Module and Ring Theory*, Gordon and Breach Science Publishers, Reading, 1991.

University of Bucharest,
Faculty of Mathematics and Informatics,
e-mail: constantin_nastasescu@yahoo.com

"Tudor Vianu" High School,
Str. Arhitect Ion Mincu 10, Bucharest 1, Romania
e-mail: costelchites@yahoo.com