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SOME PROPERTIES OF AUTOMORPHISM GROUPS OF PAVING MATROIDS

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Abstract

This paper deals with the relation between the automorphism groups of some paving matroids and $\mathbb{Z}_3$, where $\mathbb{Z}_3$ is the additive group of modulo 3 over $\mathbb{Z}$. It concludes that for paving matroids under most cases, $\mathbb{Z}_3$ is not isomorphic to the automorphism groups of these paving matroids. Even in the exceptional cases, we reasonably conjecture that $\mathbb{Z}_3$ is not isomorphic to the automorphism groups of the corresponding paving matroids. Actually, the result here is relative to the Welsh's open problem that for any group G, there is a paving matroid with automorphism group isomorphic to G.

1 Introduction and Preliminaries

Welsh indicates [5,p.40] that paving matroids are essentially a class of relatively well-behaving matroids. Additionally, J.Oxley points out [4,p.26] that paving matroids is an important class of matroids. In fact, there are many unsolved open problems relative with paving matroids such as the open problem respectively in [5,p.41], [5,p.331] and so on. This paper is relative to the open problem in [5,p.331]. The problem is that for any group H, whether there is a paving matroid with automorphism group isomorphic to H. Actually, if we take $H = \mathbb{Z}_3$, i.e. the additive group of modulo 3, then under most cases except the unsolved completely special cases, we get that the automorphism group of a paving matroid is not isomorphic to H. Even for the unsolved special cases, by the discussion in this paper, we conjecture that for any of paving matroids belonged to these unsolved special cases, its automorphism

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group is not isomorphic to H.

It starts by reviewing some knowledge what we need in the sequel. We assume that E is a finite set. In this paper, all informations relative to matroids are referred to [4,5] and that relative to permutations and groups are found in [1].

Definition 1 [4,p.26&5] Let M be a matroid. Then M is *uniform* if and only if it has no circuits of size less than $\rho(M)+1$. M is *paving* if it has no circuits of size less than $\rho(M)$.

Lemma 1 (1)[5,p.9&4] A collection $\mathcal{C}$ of subsets of E is the set of circuits of a matroid on E if and only if conditions (c1) and (c2) are satisfied
(c1) If $X \neq Y \in \mathcal{C}$, then $X \nsubseteq Y$;
(c2) If $C_1, C_2 \in \mathcal{C}, C_1 \neq C_2$ and $z \in C_1 \cap C_2$, then there exists $C_3 \in \mathcal{C}$ satisfying $C_3 \subseteq (C_1 \cup C_2) \setminus z$.

(2)[3] Let M be a matroid on E. A permutation $\pi : E \to E$ is an automorphism of M if πX is a circuit in M if and only if X is a circuit in M.

(3)[1,p.26] If S_n is denoted the symmetric group on n letters, then $|S_n| = n!$.

(4)[2,p.439] Every restriction of a paving matroid is paving.

Remark 1 We denote the automorphism group of a matroid M by $Aut(M)$. Based on (1) in Lemma 1, a matroid M on E with $\mathcal{C}(M)$ as its collection of circuits can be in notation $(E, \mathcal{C}(M))$. In addition, if a group H_1 is isomorphic to a group H_2, then it is denoted as $H_1 \cong H_2$. Otherwise, we write $H_1 \ncong H_2$.

2 Properties relative to matroids

This section presents some properties of a matroid in preparation for Section 3.

Lemma 2 Let $M = (E, \mathcal{C}(M) = \{C_1, C_2, \ldots, C_k\})$ be a matroid, $n = |\bigcup_{j=1}^{k} C_j|$ and $E \setminus \bigcup_{j=1}^{k} C_j = \{x_1, x_2, \ldots, x_m\}$. Then the following properties are true.

(1) $|Aut(M)| \geq m!$.

(2) $M' = (\bigcup_{j=1}^{k} C_j, \mathcal{C}(M))$ is a matroid and $|Aut(M)| \geq |Aut(M')| \times m!$.

(3) If $k = 1$, then $|Aut(M)| = |C_1|! \times m!$.

(4) If there is $C_i \in \mathcal{C}(M)$ satisfying $C_i \cap C_j = \emptyset, (j \neq i; j = 1, 2, \ldots, k)$.

Then $|Aut(M)| \geq |C_i|! \times m!$. Besides, $M_{C_i} = ((\bigcup_{j=1}^{k} C_j) \setminus C_i, \mathcal{C}(M) \setminus C_i)$ is still a matroid, and further, if M is paving and $m = 0$, then M_{C_i} is paving.

(5) If $\rho(M) = 0$, then $Aut(M) \cong S_{n+m}$.

(6) Let $C_{i1}, C_{i2} \in \mathcal{C}(M)$ and $m = 0$. Then $N = (C_{i1} \cup C_{i2}, \{C_j : C_j \in \mathcal{C}(M)$ and $C_j \subseteq C_{i1} \cup C_{i2}\})$ is a matroid. If M is paving, and so is N.

(7) If $m = 0$ and $k \geq 2$. Then M satisfies $C_1 \cap C_2 \cap \ldots \cap C_k = \emptyset$.

Proof Routine verification from the related definitions and Lemma 1.

3 Main results

Let $H = \mathbb{Z}_3$, i.e the additive group of $\mathbb{Z}$ modulo 3. Evidently, $|H| = 3$. Let $M = (E, \mathcal{C}(M) = \{C_1, \ldots, C_k\})$ be a paving matroid, $n = |\bigcup_{j=1}^{k} C_j|$ and $m = |E \setminus \bigcup_{j=1}^{k} C_j|$. In this section, we consider that under what conditions, M will satisfy $Aut(M) \cong H$.

First, we may state that $Aut(M) \ncong H$ holds if one of the following ($\alpha 1$), ($\beta 1$), ($\gamma 1$) happens.

($\alpha 1$) If M is uniform. Definition 1 informs $|Aut(M)| = C_{n+m}^{\rho(M)+1} \times (\rho(M)+1)! \neq 3$.

($\beta 1$) If $\rho(M) = 0$. (5) in Lemma 2 shows $Aut(M) \cong S_{n+m}$. Lemma 1 proves $|S_{n+m}| = (n+m)!$. No matter the values of n and m, it has $|Aut(M)| \neq 3$.

($\gamma 1$) If $|E \setminus \bigcup_{j=1}^{k} C_j| = m \geq 2$. Let $M' = (\bigcup_{j=1}^{k} C_j, \mathcal{C}(M))$. (2) in Lemma 2 demonstrates $|Aut(M)| \geq 2! \times |Aut(M')|$. In addition, in view of $\mathcal{C}(M') = \mathcal{C}(M)$ and (2) in Lemma 1, we may describe that

if $|Aut(M')| = 1$ holds, it leads to $|Aut(M)| = |Aut(M')| = 1 \times m!$, further, $|Aut(M)| \neq 3$;

if $|Aut(M')| \geq 2$ holds, it causes $|Aut(M)| \geq 2! \times 2 = 4$, and so $|Aut(M)| \neq 3$.

Second, (2) in Lemma 1 expresses that if $E \setminus \bigcup_{j=1}^{k} C_j = \{x\}$ and $\rho(M) \geq 1$, then both $Aut(M) \cong Aut(M')$ and $\pi(x) = x$ for any $\pi \in Aut(M)$ are correct, where $M' = (\bigcup_{j=1}^{k} C_j, \mathcal{C}(M))$.

According to the above two points, in what follows, we only consider the non-uniform paving matroid $M = (E, \mathcal{C}(M))$ with $\rho(M) \geq 1$, where $\mathcal{C}(M) = \{C_j : j = 1, 2, \ldots, k\}$ and E satisfies $|E \setminus \bigcup_{j=1}^{k} C_j| = m = 0$. We will divide different cases into discussion.

The following Lemma 3 is to consider the case of $\rho(M) = 1$.

Lemma 3 Let $M = (\bigcup_{j=1}^{k} C_j, \mathcal{C}(M) = \{C_1, C_2, \ldots, C_k\})$ be a non-uniform paving matorid and $\rho(M) = 1$. Then $Aut(M) \ncong H$ and

if $k \geq 2$, then $|Aut(M)| \geq 2$; if $k \geq 3$, then $|Aut(M)| \geq 4$.

Proof Assume $k = 1$. Lemma 2 shows $|Aut(M)| = |C_1|!$, and so $Aut(M) \ncong H$.

Since M is paving, one has $1 = \rho(M) \leq |C_j| \leq \rho(M)+1 = 2, (j = 1, \ldots, k)$.

Let $k \geq 2$.

If $|C_i| = \rho(M) = 1$. This causes $C_i = \{a_i\}, (i = 1, \ldots, k)$. Then $\pi : a_i \mapsto a_j$ $(i = 1, \ldots, k; j = 1, \ldots, k)$ satisfies $\pi(C_i) \in \mathcal{C}(M)$ $(i = 1, \ldots, k)$, and so $\pi \in Aut(M)$. Thus $|Aut(M)| = |\bigcup_{i=1}^{k} C_i|! = k!$. So if $k = 2$, then $|Aut(M)| = 2$; if $k \geq 3$, then $|Aut(M)| \geq 6 \geq 4$. These follow $Aut(M) \ncong H$.

Suppose that there is C_i satisfying $|C_i| = \rho(M) + 1 = 2$. No matter to suppose $|C_k| = 2$. Distinguishing four steps to fulfil the proof.

Step 1. Assume $k = 2$.

It is no harm to suppose $C_1 = \{a_1, \ldots, a_t\}$ and $C_2 = \{a_{s-1}, a_s\}$ where $1 \leq t \leq 2$. In virtue of Lemma 1, $C_i \nsubseteq C_j$ is correct $(i \neq j; i, j = 1, 2)$. We assert $C_1 \cap C_2 = \emptyset$. Otherwise, $a_s \in C_1 \cap C_2$ and Lemma 1 lead to $C_3 \subseteq C_1 \cup C_2 \setminus a_s$ and $C_3 \in \mathcal{C}(M) \setminus \{C_1, C_2\}$, this is a contradiction to $k = |\mathcal{C}(M)| = 2$. Thus, we have $|Aut(M)| \geq |C_1|! \times |C_2|! \geq t! \times 2! \geq 2$.

Obviously, $t = 1$ follows $|Aut(M)| = 2$; $t = 2$ follows $|Aut(M)| \geq 4$. So $Aut(M) \ncong H$.

Step 2. Assume $k = 3$.

The following (2.1) and (2.2) will carry out the proof of this step.

(2.1) Let $C_1 = \{a_1\}$ and $C_3 = \{a_2, a_3\}$. Divided two cases (α) and (β) for discussion.

(α) If $|C_2| = 1$, i.e. $C_2 = \{a_4\}$. Then by Lemma 1, $a_i \neq a_j, (i \neq j; i, j = 1, 2, 3, 4)$. Define

$\pi_{01} : a_i \mapsto a_i$ $(i = 1, 2, 3, 4)$; $\pi_{11} : a_1 \mapsto a_4, a_4 \mapsto a_1, a_i \mapsto a_i$ $(i = 2, 3)$; $\pi_{21} : a_i \mapsto a_i$ $(i = 1, 4), a_2 \mapsto a_3, a_3 \mapsto a_2$; $\pi_{31} : a_1 \mapsto a_4, a_4 \mapsto a_1, a_2 \mapsto a_3, a_3 \mapsto a_2$.

Then $Aut(M) \supseteq \{\pi_{01}, \pi_{11}, \pi_{21}, \pi_{31}\}$, so $Aut(M) \ncong H$ and $|Aut(M)| \geq 4$.

(β) If $|C_2| = 2$.

By Lemma 1, $C_2 \cap C_3 \neq \emptyset$ yields out $C_2 = \{a_2, a_5\}$. However, $C_2 \cup C_3 \setminus a_2 = \{a_3, a_5\} \nsupseteq C_j, (j \in \{1, 2, 3\})$ will bring about a contradiction to Lemma 1. Moreover, $C_2 \cap C_3 = \emptyset$, i.e. $C_2 = \{a_4, a_5\}$, and in addition, $a_i \neq a_j, (i \neq j; i, j = 1, 2, 3, 4, 5)$. Define

$\pi_{02} : a_i \mapsto a_i$ $(i = 1, 2, 3, 4, 5)$; $\pi_{12} : a_2 \mapsto a_3, a_3 \mapsto a_2, a_i \mapsto a_i$ $(i = 1, 4, 5)$;

$\pi_{22} : a_i \mapsto a_i (i = 1, 2, 3), a_4 \mapsto a_5, a_5 \mapsto a_4$;

$\pi_{32} : a_1 \mapsto a_1, a_2 \mapsto a_3, a_3 \mapsto a_2, a_4 \mapsto a_5, a_5 \mapsto a_4$.

Then $\pi_{j2} \in Aut(M)$ $(j = 0, 1, 2, 3)$. So $Aut(M) \ncong H$ and $|Aut(M)| \geq 4$.

(2.2) Let $|C_j| = \rho(M) + 1 = 2$ $(j = 1, \ldots, k)$. Then M satisfies one of the following statuses

(i) $C_1 = \{a_1, a_2\}, C_2 = \{a_1, a_3\}, C_3 = \{a_2, a_3\}$ $(a_i \neq a_j, i \neq j; i, j = 1, 2, 3)$.

(ii) $C_1 = \{a_1, a_2\}, C_2 = \{a_3, a_4\}, C_3 = \{a_5, a_6\}$ $(a_i \neq a_j, i \neq j; i, j = 1, 2, 3, 4, 5, 6)$.

We may easily obtain that if (i) happens, then $|Aut(M)| \geq 6 \geq 4$; if (ii) happens, then $|Aut(M)| \geq 8 \geq 4$. Hence, no matter which happens between (i) and (ii), it follows $Aut(M) \ncong H$.

Step 3. Let $|C_j| = 2$ $(j = 1, \ldots, k)$ and $3 < k$.

(3.1) Assume $C_1 \cap C_j = \emptyset$ $(j = 2, \ldots, k)$. We will carry out the proof using the induction method. In light of Lemma 2, it has $|Aut(M)| \geq |C_1|! \times |Aut(N)| = 2|Aut(N)|$, where $N = (\bigcup_{j=1}^{k} C_j \setminus C_1, \mathcal{C}(M) \setminus C_1)$. Recalling Step 2, $k - 1 \geq 3$ and the supposition of the inductive, we may indicate $|Aut(N)| \geq 4$, and so $|Aut(M)| \geq |Aut(N)| \geq 4$, and hence $Aut(M) \ncong H$.

(3.2) Assume for any $C_i \in \mathcal{C}(M)$, there is $C_j \in \mathcal{C}(M) \setminus C_i$ satisfying $C_i \cap C_j \neq \emptyset$.

Using (6) in Lemma 2, $N_{ij} = (C_i \cup C_j, \{C_p : C_p \subseteq C_i \cup C_j, C_p \in \mathcal{C}(M)\}) = (C_i \cup C_j, C_i = \{a_{i1}, a_{i2}\}, C_j = \{a_{i1}, a_{j2}\}, \{a_{i2}, a_{j2}\})$ is a paving matroid. In addition $N_{ij} \neq M$ is effective because of $k > 3$. Additionally, $|\mathcal{C}(N_{ij})| = 3$ and Step 2 together produce $|Aut(N_{ij})| \geq 4$.

First of all, we prove that if for any paving matroid $N = (\bigcup_{p=1}^{t} C_{i_p}, \{C_{i_p} : C_{i_p} \in \mathcal{C}(M), p = 1, \ldots, t\}) \neq M$, there is $C_h \in \mathcal{C}(M) \setminus \mathcal{C}(N)$ satisfying $C_h \cap \bigcup_{p=1}^{t} C_{i_p} \neq \emptyset$, then we assert that M is uniform.

Combining $\rho(M) = 1$ and $|C_j| = 2$ $(j = 1, \ldots, k)$ with the property of M as a non-uniform together, it brings about the existence of $C = \{1, 2\} \subseteq \bigcup_{j=1}^{k} C_j$ and $C \notin \mathcal{C}(M)$. Herein, there is $C_1, C_2 \in \mathcal{C}(M)$ satisfying $1 \in C_1 = \{1, q\}$, $2 \in C_2$ and $C_1 \cap C_2 \neq \emptyset$. In view of $|C_2| = 2$, it follows $C_2 = \{2, 3\}$. If $\bigcup_{j=1}^{k} C_j = \{1, 2, 3\}$, then it follows $|\mathcal{C}(M)| \leq 3$. This is a contradiction to $k > 3$. That is to say, there is at least $4 \in \bigcup_{j=1}^{k} C_j \setminus \{1, 2, 3\}$. Let $\{2, 3, 4\} \subseteq \bigcup_{j=1}^{k} C_j$ and $\{2, 3\} \in \mathcal{C}(M)$.

We notice that $\{2, 3\} \in \mathcal{C}(M)$ and the supposition above for N taken together leads to $\{3,4\} \in \mathcal{C}(M)$, and further $\{2,4\} \in \mathcal{C}(M)$. Hence, $N_{23} = (\{2,3,4\}, \{\{2,3\}, \{2,4\}, \{3,4\}\})$ is a paving matroid and $N_{23} \neq M$. This causes $C_5 = \{4,5\} \in \mathcal{C}(M) \setminus \mathcal{C}(N)$ and $C_5 = \{4,5\} \cap \{2,3,4\} \neq \emptyset$. Thus, $N_{2345} = \{\{2,3,4,5\}, \{\{i,j\} : i \neq j, i,j = 2,3,4,5\}) \neq M$ is a uniform matroid with $\rho(N_{2345}) = 1$ and $1 \notin \{2,3,4,5\}$. By the supposition, induction and $k < \infty$, we may express that there is a uniform matroid $N \neq M$ satisfying $\mathcal{C}(N_{2345}) \subseteq \mathcal{C}(N) \subseteq \mathcal{C}(M)$, and $\mathcal{C}(M) \ni C_1 = \{1,q\}$ and $C_1 \cap (\bigcup_{C_p \in \mathcal{C}(N)} C_p) \neq \emptyset$.

If $C_1 \cap (\bigcup_{C_p \in \mathcal{C}(N)} C_p) = 1$. That is $\{1,t\} \in \mathcal{C}(N)$. In light of $2 \in \bigcup_{C_p \in \mathcal{C}(N)} C_p$ and the uniform property of n, it follows $\{2,t\} \in \mathcal{C}(N)$. According to Lemma 1, it assures $\{1,t\} \cup \{2,t\} \setminus t = \{1,2\} \in \mathcal{C}(M)$, a contradiction.

If $C_1 \cap \bigcup_{C_p \in \mathcal{C}(N)} C_p = \{q\}$. Therefore, $\{s,q\} \in \mathcal{C}(N)$ and $\{1,q\} \in \mathcal{C}(M)$ follows $\{1,s\} \in \mathcal{C}(M)$. Since $\{2,s\} \in \mathcal{C}(N)$, it gets $\{1,s\} \cup \{2,s\} \setminus s = \{1,2\} \in \mathcal{C}(M)$, a contradiction.

But the uniform of the assertion is a contradiction to the non-uniform property of M.

Second, if for any $C_h \in \mathcal{C}(M) \setminus (C_i \cup C_j), C_h \cap (C_i \cup C_j) = \emptyset$ holds. Let $\pi_{ij} \in Aut(N_{ij})$. We define $\pi : x \mapsto \pi_{ij}(x)$ for $x \in C_i \cup C_j$, $\quad x \mapsto x$ for $\bigcup_{t=1}^{k} C_t \setminus (C_i \cup C_j)$. We may easily have $\pi \in Aut(M)$. Further, it follows $|Aut(M)| \geq |Aut(N_{ij})| \geq 4$.

If there exists a paving matroid $N = (\bigcup_{p=1}^{t} C_{i_p}, \{C_{i_p} \in \mathcal{C}(M), p = 1, \ldots, t\}) \neq M$, $(t > 2)$. Then for any $C_h \in \mathcal{C}(M) \setminus \mathcal{C}(N), C_h \cap \bigcup_{p=1}^{t} C_{i_p} = \emptyset$ holds. Herein, there is $\mathcal{C}(M) \setminus \mathcal{C}(N) \neq \emptyset$. Additionally, it evidently obtains $|Aut(M)| \geq |Aut(N)|$. By induction and $t > 2$, it has $|Aut(N)| \geq 4$. So it follows $|Aut(M)| \geq |Aut(N)| \geq 4$ and $Aut(M) \ncong H$.

Step 4. Suppose there are $C_i, C_j \in \mathcal{C}(M)$ satisfying $|C_j| = 1, (j = 1, \ldots, t, 1 \leq t < k)$ and $|C_i| = 2, (i = t+1, \ldots, k)$. Recalling Lemma 2, we bring about $|Aut(M) \geq t! \times |Aut(M_t)|$ where $M_t = (\bigcup_{i=t+1}^{k} C_i, \{C_{t+1}, \ldots, C_k\})$ is a paving matroid by Lemma 2.

If $t = 1$. Then by $k-1 \geq 3$, Step 2 and Step 3, we may indicate $|Aut(M_t)| \geq 4$. Furthermore, $|Aut(M)| \geq 4$ is right. So $Aut(M) \ncong H$ holds.

If $t \geq 2$. Then $k - 2 \geq 2$ and Step 1 together ensure $|Aut(M_t)| \geq 2$. Therefore, it leads to $|Aut(M)| \geq 2 \times 2 = 4$, and so $Aut(M) \ncong H$.

In the following, we will handle with $\rho(M) \geq 2$.

Lemma 4 Let $M = (\bigcup_{j=1}^{k} C_j, \mathcal{C}(M) = \{C_1, \ldots, C_k\})$ be a non-uniform paving matroid with $r = \rho(M) \geq 2$.

(1) Assume $k = 1$. Then $Aut(M) \not\cong H$ is right.

(2) Assume $k \geq 2$. Then there are the following expressions.

(i) If there is $C_i \in \mathcal{C}(M)$ satisfying $C_i \cap C_j = \emptyset, (j \neq i;\ j = 1, 2, \ldots, k)$, then $|Aut(M)| \geq 4$ and $Aut(M) \not\cong H$.

(ii) Suppose for any $C_i \in \mathcal{C}(M)$, there is $C_{j_i} \in \mathcal{C}(M) \setminus C_i$ satisfying $C_i \cap C_{j_i} \neq \emptyset$. If there is $C_{i_1}, C_{i_2} \in \mathcal{C}(M)$ $(i_1 \neq i_2)$ such that $C_{i_1} \cap C_{i_2} \neq \emptyset, C_{i_3}, \ldots, C_{i_p} \subseteq C_{i_1} \cup C_{i_2}$, and $C_{i_t} \cap (C_{i_1} \cup C_{i_2}) = \emptyset, (t = p+1, \ldots, k)$, where $C_{i_j} \in \mathcal{C}(M)$ $(j = 1, 2, \ldots, p, p+1, \ldots, k)$ and $0 \neq p < k$ and $k - p \geq 1$. Let $M_1 = (C_{i_1} \cup C_{i_2}, \{C_{i_1}, \ldots, C_{i_p}\})$ and $M_2 = (\bigcup_{t=p+1}^{k} C_{i_t}, \{C_{i_t} : t = p+1, \ldots, k\})$. Then, we have the following statements.

State 1. If one of M_1 and M_2 are uniform, then $|Aut(M)| \geq 4$ and $Aut(M) \not\cong H$.

State 2. If both of M_1 and M_2 are non-uniform paving, and in addition, for some $h \in \{1, 2\}$, M_h satisfies

(a1) there exist $C_t, C_s \in \mathcal{C}(M)$ satisfying $C_t \cap C_s \neq \emptyset$ and $N_{ts} = (C_t \cup C_s, \{C_{ts} \in \mathcal{C}(M_h) : C_{ts} \subseteq C_t \cup C_s\}) \neq M_h$;

(a2) for any $C_j \in \mathcal{C}(M_h) \setminus \mathcal{C}(N_{ts})$, it has $C_j \cap (C_t \cup C_s) = \emptyset$,
where $1 \leq |\mathcal{C}(N_{ts})| < p$.

Then $|Aut(M)| \geq 4$ and $Aut(M) \not\cong H$.

Proof (1) Assume $k = 1$. By Lemma 2, it follows $|Aut(M)| = |C_1|!$, and so $Aut(M) \not\cong H$.

(2) (i) According to Lemma 2, $M' = (\bigcup_{j\neq i, j=1}^{k} C_j, \{C_j : j \neq i, j = 1, 2, \ldots, k\})$ is a paving matroid. Evidently, $|Aut(M)| \geq |C_i|! \times |Aut(M')|$ is correct. In light of $r = \rho(M) \leq |C_t| \leq \rho(M) + 1, (t = 1, \ldots, k)$, we may carry out $|Aut(M)| \geq r! \times |Aut(M')|$. Hence, if $r \geq 3$, then $|Aut(M)| \geq r! \geq 4$. So $Aut(M) \not\cong H$ is true.

Next we prove that if $r = 2$, then $|Aut(M)| \geq 4$ and $Aut(M) \not\cong H$.

Assume $k = 2$. $C_1 \cap C_2 = \emptyset$ holds, and in addition, $M' = (C_2, C_2)$ holds. Furthermore, it yields out $|Aut(M)| \geq |C_1|! \times |C_2|! \geq r^2 \geq 4$, and so $Aut(M) \not\cong H$.

Assume $k > 2$.

If M' is uniform. $(\alpha 1)$ informs us $|Aut(M')| \geq 2$, and so $|Aut(M)| \geq 4$. Thus $Aut(M) \not\cong H$.

If M' is non-uniform and there is $C_{i_0} \cap (\bigcup_{j\neq i, j\neq i_0, j=1}^{k} C_j) = \emptyset$. By the induction supposition, we obtain $|Aut(M')| \geq 4$, and so $|Aut(M)| \geq 2 \times 4 = 8$. Therefore, it causes $Aut(M) \ncong H$.

If M' is non-uniform paving, and in addition, for any $C_s \in \mathcal{C}(M')$, there is $C_t \in \mathcal{C}(M')$ fitting $C_s \cap C_t \neq \emptyset$. Then we may easily indicate that by induction on $|\mathcal{C}(M')|$, it assures that M' is the following status:

Status: Posit $N_j = (\bigcup_{q=1}^{m_j} C_{j_q}, \{C_{j_q} \in \mathcal{C}(M') : q = 1, \ldots, m_j\}), (j = 1, 2)$. We may carry out $\mathcal{C}(M') = \mathcal{C}(N_1) \cup \mathcal{C}(N_2)$; N_j is a uniform with $|\mathcal{C}(N_j)| > 1, (j = 1, 2)$; $C_{1_x} \cap C_{2_y} = \emptyset$ for any $C_{1_x} \in \mathcal{C}(N_1)$ and $C_{2_y} \in \mathcal{C}(N_2)$.

Evidently, for this status, $|Aut(M')| \geq 4$ is true. Moreover, $|Aut(M)| \geq 4$ is real, and so $Aut(M) \ncong H$.

(ii) By Lemma 2, both of $M_1 = (C_{i_1} \cup C_{i_2}, \{C_{i_1}, C_{i_2}, \ldots, C_{i_p}\})$ and $M_2 = (\bigcup_{t=p+1}^{k} C_{i_t}, \{C_{i_t} : t = p+1, \ldots, k\})$ are paving matroids. In view of the given, we may easily receive that $|Aut(M)| \geq |Aut(M_1)| \times |Aut(M_2)|$ and $\rho(M) \leq |C_j| \leq \rho(M) + 1, (j = 1, \ldots, k)$.

If $k = 2$, $\mathcal{C}(M_1) \neq \emptyset$ and $\mathcal{C}(M_2) \neq \emptyset$. Then, the need result is followed from (i).

If $k = 2$, $\mathcal{C}(M_1) \neq \emptyset$ and $\mathcal{C}(M_2) = \emptyset$. Then, it follows $k - p \ngeq 1$, a contradiction.

In one word, if $k = 2$, it will have $|Aut(M)| \geq 4$ and $Aut(M) \ncong H$.

By induction on k, we will prove $|Aut(M)| \geq 4$ and $Aut(M) \ncong H$.

According to the given, we know $\mathcal{C}(M_j) \neq \emptyset\, (j = 1, 2)$ and $|\mathcal{C}(M_1)| = p \geq 1$, $\mathcal{C}(M_2) = k - p \geq 1$.

State 1. Assume both of M_1 and M_2 are uniform. By Lemma 2, one gets $|Aut(M_1)| \geq |(C_{i_1} \cup C_{i_2})|! \geq 3!$ and $|Aut(M_2)| \geq |(\bigcup_{t=p+1}^{k} C_{i_t})|! \geq 1$. Hence, we get the need result.

Assume M_1 is uniform and M_2 is non-uniform. This assumption and Lemma 2 together cause $|Aut(M_1)| \geq 6$. Additionally, it causes $|Aut(M_2)| \geq 1$. Thus the need consequent is followed.

Assume M_2 is uniform and M_1 is non-uniform. If $k - p = 1$. Then (i) brings about $|Aut(M)| \geq 4$ and $Aut(M) \ncong H$. If $k - p > 1$. Then Lemma 2 yields out $|Aut(M_2)| \geq 4$. Hence, it easily produces $|Aut(M)| \geq 4$, and so, $Aut(M) \ncong H$ is provided.

State 2. Assume both M_1 and M_2 are non-uniform paving. According to (i) or the inductive supposition and the property of M_h, we have $|Aut(M_h)| \geq 4$, and so $|Aut(M)| \geq 4 \times 1 = 4$, further, $Aut(M) \ncong H$.

Lemma 5 Let $M = (\bigcup_{j=1}^{k} C_j, \mathcal{C}(M) = \{C_j : j = 1, \ldots, k\})$ and $k \geq 2$ be a

non-uniform paving matroid with $\rho(M) = r \geq 2$. If M satisfies the following (1) and (2)
(1) for any $C_i \in \mathcal{C}(M)$, there is $C_j \in \mathcal{C}(M) \setminus C_i$ satisfying $C_i \cap C_j \neq \emptyset$;
(2) for any $C_{i_1}, C_{i_2} \in \mathcal{C}(M)$, if $C_{i_1} \cap C_{i_2} \neq \emptyset$, then $N = (\bigcup_{t=1}^{q} C_{i_t} = C_{i_1} \cup C_{i_2}, \mathcal{C}(N) = \{C_{i_t} : C_{i_t} \subseteq C_{i_1} \cup C_{i_2}, C_{i_t} \in \mathcal{C}(M), t = 1, 2, \ldots, q\}) = M$.

Then $3 \leq |\mathcal{C}(M)| \leq 4$.

Proof Since M is non-uniform and $C_1 \cap C_2 = \{1, \ldots, t\} \neq \emptyset$. We will suppose $C_1 = \{1, \ldots, t, a_{1(t+1)}, \ldots, a_{1r_1}\}$ and $C_2 = \{1, \ldots, t, a_{2(t+1)}, \ldots, a_{2r_2}\}$, where $r_1, r_2 \in \{r, r+1\}$.

By the given condition and $C_j \cap C_3 \neq \emptyset$ $(j = 1, 2)$, we present $C_1 \cup C_2 \setminus 1 \supseteq C_3 \in \mathcal{C}(M)$ and $N = (\bigcup_{j=1}^{p} C_{1_j}, \{C_{1_j} : C_{1_j} \subseteq C_1 \cup C_3, C_{1_1} = C_1, C_{1_2} = C_3, C_{1_j} \in \mathcal{C}(M)\}) = (\bigcup_{j=1}^{p} C_{2_j}, \{C_{2_j} : C_{2_j} \subseteq C_1 \cup C_2, C_{2_1} = C_1, C_{2_2} = C_2, C_{2_j} \in \mathcal{C}(M)\}) = M$. This compels $C_1 \cup C_2 = C_1 \cup C_3$, and hence $\{a_{2(t+1)}, \ldots, a_{2r_2}\} \subseteq C_3$. Furthermore, $C_2 \cup C_3 = C_1 \cup C_2$ follows $\{a_{1(t+1)}, \ldots, a_{1r_1}\} \subseteq C_3$. Namely, $\{a_{1(t+1)}, \ldots, a_{1r_1}, a_{2(t+1)}, \ldots, a_{2r_2}\} \subseteq C_3$.

By Lemma 1, $C_3 \cup C_1 \setminus a_{1(t+1)} \supseteq C_i \in \mathcal{C}(M)$ for some C_i, and so $a_{1(t+1)} \notin C_i$. If $C_i \neq C_2$, then $C_i = C_4$, and in addition, $C_4 \cap C_2 \neq \emptyset$. Therefore, it follows $C_4 \cup C_2 \neq C_1 \cup C_2$, a contradiction with the property of M. That is to say, $C_i = C_2$. Similarly to $C_3 \cup C_1 \setminus a_{1j}$ $(j = t+2, \ldots, r)$ and $C_3 \cup C_2 \setminus a_{2s}$, $(s = t+1, \ldots, r_2)$.

Additionally, if $j \in C_3$ for some $j \in \{1, \ldots, t\}$, it follows $C_1 \cup C_3 \setminus j \supseteq C_\alpha \in \mathcal{C}(M)$, but $j \in C_1, C_2, C_3$, and so $C_\alpha \notin \{C_1, C_2, C_3\}$. No matter to denote $C_\alpha = C_4$. By Lemma 1, $C_4 \nsubseteq C_1, C_3$. Combining the close result above and $C_4 \subseteq C_1 \cup C_3$, we may indicate $C_4 \cap C_1 \neq \emptyset$ and $C_4 \cap C_3 \neq \emptyset$. This follows $a_{2p} \in C_4$ for some $p \in \{t+1, \ldots, r_2\}$. So it causes $C_4 \cap C_2 \neq \emptyset$. Thus, it presents $C_2 \cup C_4 = C_1 \cup C_2$. This compels $\{a_{1(t+1)}, \ldots, a_{1r_1}\} \subseteq C_4$. Since $C_1 \cup C_4 = C_1 \cup C_2$ compels $\{a_{2(t+1)}, \ldots, a_{2r_2}\} \subseteq C_4$, one has $\{a_{1(t+1)}, \ldots, a_{1r_1}, a_{2(t+1)}, \ldots, a_{2r_2}\} \subseteq C_4$. No harm to suppose $\{1, \ldots, s\} \subseteq C_3$ $(s \leq t)$. In view of $C_3 \cup C_4 = C_1 \cup C_2$, we may earn $\{s+1, \ldots, t\} \subseteq C_4$. In addition, $|C_3| \leq r+1$ and $C_1 \cap C_2 \neq \emptyset$ together assure $s < t$.

Suppose $C_3 \cap C_4 \cap \{1, \ldots, t\} \neq \emptyset$, i.e. there is $\beta \in \{1, \ldots, t\}$ satisfying $\beta \in C_3 \cap C_4$. Then $C_3 \cup C_4 \setminus \beta \supseteq C_\gamma \in \mathcal{C}(M)$. But we know $C_\gamma \notin \{C_1, C_2, C_3, C_4\}$. No harm to denote C_γ to be C_5. Obviously, $C_5 \cap C_3 \neq \emptyset$ and $C_5 \cap C_4 \neq \emptyset$. Let $\{1, \ldots, t\} \supseteq \{\beta_1, \ldots, \beta_q\} \subseteq C_5$.

If $C_5 \cap \{a_{2(t+1)}, \ldots, a_{2r_2}\} = \emptyset$, then $C_5 \subseteq C_1$, a contradiction.

Similarly, $C_5 \cap \{a_{1(t+1)}, \ldots, a_{1r_1}\} \neq \emptyset$.

Therefore, by the supposition of M, we may obtain $C_5 \cup C_2 = C_5 \cup C_1 = C_1 \cup C_2$, and so $\{a_{1(t+1)}, \ldots, a_{1r_1}, a_{2(t+1)}, \ldots, a_{2r_2}\} \subseteq C_5$. Moreover, using this augmentation repeated, we may state that $N = (C_3 \cup C_4, \mathcal{C} = \{C_j : C_j \subseteq C_3 \cup C_4, C_j \in \mathcal{C}(M)\})$ is a paving matroid with $\mathcal{C}(N) = \mathcal{C}$ and $\{a_{1(t+1)}, \dots, a_{1r_1}, a_{2(t+1)}, \dots, a_{2r_2}\} \subseteq C_j \in \mathcal{C}$, and in addition, $N \neq M$, a contradiction to the supposition of M. Namely, $C_3 = \{1, \dots, s, a_{1(t+1)}, \dots, a_{1r_1}, a_{2(t+1)}, \dots, a_{2r_2}\}$ and $C_4 = \{s+1, \dots, t, a_{1(t+1)}, \dots, a_{1r_1}, a_{2(t+1)}, \dots, a_{2r_2}\}$. Thus $\mathcal{C}(M) = \{C_1, C_2, C_3, C_4\}$, and hence $3 \leq |\mathcal{C}(M)| \leq 4$.

Assume $s = 0$. Then, one has $|\mathcal{C}(M)| = 3$ and $C_3 = \{a_{1(t+1)}, \dots, a_{1r_1}, a_{2(t+1)}, \dots, a_{2r_2}\}$, and in addition, no C_4 exists. That is to say, if $|\mathcal{C}(M)| = 4$, it must have $1 \leq s$ and $1 \leq t - s$.

Based on Lemma 5, we may demonstrate the following Lemma 6.

Lemma 6 Let M be defined as that in Lemma 5. Then

(I) Assume $|\mathcal{C}(M)| = 3$. Then there are the following results.

(1) If $|C_1| = r, |C_2| = r+1, C_1 \cap C_2 \neq \emptyset$ and $|C_1 \cap C_2| = r - 1$. Then $Aut(M) \ncong H$.

(2) If $|C_1| = |C_2| = r, C_1 \cap C_2 \neq \emptyset$ and $|C_1 \cap C_2| = r - 1$. Then $Aut(M) \ncong H$.

(3) Suppose $|C_1| = r$ and for $C_2 \in \mathcal{C}(M), C_1 \cap C_2 = \{1, \dots, t\} \neq \emptyset$. If $t < r-1$, then $Aut(M) \ncong H$.

(4) If $|C_1| = r+1 = |C_2|$ and $C_1 \cap C_2 = \{1, \dots, t\} \neq \emptyset$. Then $Aut(M) \ncong H$.

(II) Assume $|\mathcal{C}(M)| = 4$. Then, we have $Aut(M) \ncong H$.

Proof It is only to testify the truth of every case in (I) and (II) respectively. Because all these checks are not difficult, we omit them here.

Assume M is defined as Lemma 5. If $C_1 \cap C_2 = \emptyset$. Then it assures $C_3 \cap C_1 \neq \emptyset$ and $C_3 \cap C_2 \neq \emptyset$, additionally, $C_1 \cup C_3 = C_2 \cup C_3$. Hence, it is no harm to suppose that $C_1 \cap C_2 \neq \emptyset$ if M is defined as in Lemma 5. This result together with Lemma 6 proves the following Theorem 1.

Theorem 1 If M is defined as that in Lemma 5. Then $Aut(M) \ncong H$.

Summing up, we have the following Theorem 2.

Theorem 2 Let $M = (\bigcup_{j=1}^{k} C_j, \mathcal{C}(M) = \{C_1, \dots, C_k\})$ be a non-uniform paving matroid with $\rho(M) \geq 2$.

(1) If $k = 1$. Then $Aut(M) \ncong H$.

(2) Assume $k \geq 2$. Then there are the following consequences.

(i) If there is $C_i \in \mathcal{C}(M)$ satisfying $C_i \cap C_j = \emptyset, (j \neq i;\ j = 1, 2, \dots, k)$, then $|Aut(M)| \geq 4$ and $Aut(M) \ncong H$.

(ii) Suppose for any $C_i \in \mathcal{C}(M)$, there is $C_{j_i} \in \mathcal{C}(M) \setminus C_i$ satisfying $C_i \cap C_{j_i} \neq \emptyset$. If there is $C_{i_1}, C_{i_2} \in \mathcal{C}(M)$ $(i_1 \neq i_2)$ such that $C_{i_1} \cap C_{i_2} \neq \emptyset, C_{i_3}, \dots, C_{i_p} \subseteq C_{i_1} \cup C_{i_2}$, and $C_{i_t} \cap (C_{i_1} \cup C_{i_2}) = \emptyset, (t = p+1, \dots, k)$, where $C_{i_j} \in \mathcal{C}(M)$ $(j = 1, 2, \dots, p, p+1, \dots, k)$ and $0 \neq p < k$ and $k - p \geq 1$. Let

$M_1 = (C_{i_1} \cup C_{i_2}, \{C_{i_1}, \dots, C_{i_p}\})$ and $M_2 = (\bigcup_{t=p+1}^{k} C_{i_t}, \{C_{i_t} : t = p+1, \dots, k\})$. We have the following statements.

State 1. If one of M_1 and M_2 are uniform, then $|Aut(M)| \geq 4$ and $Aut(M) \ncong H$.

State 2. If both of M_1 and M_2 are non-uniform, and in addition, for some $h \in \{1, 2\}$, M_h satisfies

(a1) there is $C_t, C_s \in \mathcal{C}(M)$ satisfying $C_t \cap C_s \neq \emptyset$ and $N_{ts} = (C_t \cup C_s, \{C_{ts} \in \mathcal{C}(M_1) : C_{ts} \subseteq C_t \cup C_s\}) \neq M_h$, where $1 \leq |\mathcal{C}(N_{ts})| < p$.

(a2) for any $C_j \in \mathcal{C}(M_h) \setminus \mathcal{C}(N_{ts})$, it has $C_j \cap (C_t \cup C_s) = \emptyset$.

Then $|Aut(M)| \geq 4$ and $Aut(M) \ncong H$.

State 3. If both M_1 and M_2 are non-uniform paving and one of M_1 and M_2, no matter to assume M_1, satisfies that for any $C_t \in \mathcal{C}(M_1)$, it exists $C_s \in \mathcal{C}(M_1)$ satisfying $C_t \cap C_s \neq \emptyset$, but $N_{ts} = (C_t \cup C_s, \{C_{ts} \in \mathcal{C}(M_j) : C_{ts} \subseteq C_t \cup C_s\}) = M_1$.

Remark 2 Up till now, for paving matroids, there exists another circumstance left to be dealt with. That is, $M = (C_1 \cup C_2, \mathcal{C}(M) = \{C_j : C_j \subseteq C_1 \cup C_2, j = 1, \dots, k\})$ is a non-uniform paving matroid with $\rho(M) = r \geq 2$ and owns the following properties:

(α) $C_1 \cap C_2 \neq \emptyset$;

(β) for any $C_p \in \mathcal{C}(M)$, there is $C_q \in \mathcal{C}(M) \setminus C_p$ satisfying $C_p \cap C_q \neq \emptyset$;

(γ) for any $C_t, C_s \in \mathcal{C}(M)$ and $C_t \cap C_s \neq \emptyset, (t \neq s)$, if $N = (C_t \cup C_s, \{C_j : C_j \subseteq C_t \cup C_s, C_j \in \mathcal{C}(M)\}) \neq M$, then there is $C_p \in \mathcal{C}(M) \setminus \mathcal{C}(N) \neq \emptyset$ satisfying $C_p \cap (C_t \cup C_s) \neq \emptyset$.

This circumstance will be considered in what follows.

Theorem 3 Let M be defined as that in Remark 2. Then

(1) Let $|C_1| = |C_2| = \rho(M) = r$. If $|C_1 \cap C_2| = r - 1$, then $Aut(M) \ncong H$.

(2) Let $|C_1| = \rho(M) = r$. If $|C_1 \cap C_2| = r - 1$ and $|C_j| = r + 1$ for $C_j \in \mathcal{C}(M) \setminus C_1, j = 2, \dots, k$. Then $Aut(M) \ncong H$.

(3) Let $|C_1| = |C_2| = \rho(M) + 1 = r + 1$. If $|C_1 \cap C_2| = r$, then $Aut(M) \ncong H$.

Proof (1) Let $C_j = \{a_1, a_2, \dots, a_{r-1}, a_{jr}\}, (j = 1, 2)$. Then by Lemma 1, it causes $C_1 \cup C_2 \setminus a_1 = \{a_2, \dots, a_{r-1}, a_{1r}, a_{2r}\} \supseteq C_{31}$. Since $r \leq |C_{31}| \leq r + 1$ and $|\{a_2, \dots, a_{r-1}, a_{1r}, a_{2r}\}| = r$, it follows $C_{31} = \{a_2, \dots, a_{r-1}, a_{1r}, a_{2r}\}$. Similarly, $C_1 \cup C_2 \setminus a_j = C_{3j}$ $(j = 2, \dots, r - 1)$. We may easily testify $C_{3i} \cup C_{3j} \setminus a_{tr} \supseteq C_t, (t = 1, 2; i = 1, \dots, r - 1; j \neq i, j = 1, \dots, r - 1)$. It assures $C_1 \cup C_{3i} \setminus a_j = C_{3j}, (i \neq j; i, j = 1, \dots, r - 1)$. That is to say, it should have $\mathcal{C}(M) = \{C_1, C_2, C_{3j} = \{a_1, \dots, a_{j-1}, a_{j+1}, \dots, a_{r-1}, a_{1r}, a_{2r}\}, j = 1, \dots, r - 1\}$. We define

$\pi_1 : a_1 \mapsto a_{i_1}, a_2 \mapsto a_{i_2}, \dots, a_{r-1} \mapsto a_{i_{r-1}}, a_{1r} \mapsto a_{1r}, a_{2r} \mapsto a_{2r};$

$\pi_2 : a_1 \mapsto a_{i_1}, a_2 \mapsto a_{i_2}, \ldots, a_{r-1} \mapsto a_{i_{r-1}}, a_{1r} \mapsto a_{2r}, a_{2r} \mapsto a_{1r}$,
where $\{i_1, i_2, \ldots, i_{r-1}\} = \{1, 2, \ldots, r-1\}$.

It obviously follows $\pi_1, \pi_2 \in Aut(M)$, and further, $|Aut(M)| \geq (r-1)! \times 2!$.

Assume $r > 2$. Then it has $|Aut(M)| \geq 4$, and hence $Aut(M) \ncong H$.

Assume $r = 2$. Then we obtain $C_1 = \{a_1, a_{12}\}, C_2 = \{a_1, a_{22}\}$ and $C_1 \cup C_2 \setminus a_1 = C_3 = \{a_{12}, a_{22}\}$. But this does not satisfy that M is defined as that in Remark 2, a contradiction.

(2) Let $C_1 = \{a_1, \ldots, a_{r-1}, a_{1r}\}$ and $C_2 = \{a_1, \ldots, a_{r-1}, a_{2r}, a_{2(r+1)}\}$. Since $C_1 \cup C_2 \setminus a_j = \{a_1, \ldots, a_{j-1}, a_{j+1}, \ldots, a_{r-1}, a_{1r}, a_{2r}, a_{2(r+1)}\} = C_{3j}, (j = 1, 2, \ldots, r-1)$. We testify $C_1 \cup C_{3j} \setminus a_{1r} = C_2; C_2 \cup C_{3j} \setminus a_{2s} \supseteq C_1, C_{3i} \cup C_{3j} \setminus a_{2s} \supseteq C_1, (s = r, r+1; j = 1, \ldots, r-1); C_{3p} \cup C_{3q} \setminus a_j = C_{3j}, (a_j \in C_{3p}, C_{3q}; p \neq q; j = 1, \ldots, r-1; p, q = 1, \ldots, r-1)$. Hence, it causes $\mathcal{C}(M) = \{C_1, C_2, C_{3j}, j = 1, 2, \ldots, r-1\}$. We define

$\pi_{11} : a_j \mapsto a_{i_j} \quad (j = 1, 2, \ldots, r-1), a_{1r} \mapsto a_{1r}, a_{2r} \mapsto a_{2r}, a_{2(r+1)} \mapsto a_{2(r+1)}$;

$\pi_{12} : a_j \mapsto a_{i_j} \quad (j = 1, 2, \ldots, r-1), a_{1r} \mapsto a_{1r}, a_{2r} \mapsto a_{2(r+1)}, a_{2(r+1)} \mapsto a_{2r}$,
where $\{i_j : j = 1, 2, \ldots, r-1\} = \{1, 2, \ldots, r-1\}$.

So $|Aut(M)| \geq (r-1)! \times 2$.

Assume $r > 3$. Then it yields out $|Aut(M)| \geq 4$, and so $Aut(M) \ncong H$.

Assume $r = 2$. Then it yields out $C_1 = \{a_1, a_{12}\}, C_2 = \{a_1, a_{22}, a_{23}\}$ and $C_3 = C_1 \cup C_2 \setminus a_1 = \{a_{12}, a_{22}, a_{23}\}, C_1 \cup C_3 \setminus a_{12} = \{a_1, a_{22}, a_{23}\} = C_2, C_2 \cup C_3 \setminus a_{22} = \{a_1, a_{12}, a_{23}\} \supseteq C_1, C_2 \cup C_3 \setminus a_{23} = \{a_1, a_{12}, a_{22}\} \supseteq C_1$. Thus, we may obtain $\mathcal{C}(M) = \{C_1, C_2, C_3\}$. However, $C_1 \cup C_3 = C_1 \cup C_2 = C_2 \cup C_3$ follows that M is not defined as that in Remark 2, a contradiction to the given supposition.

Assume $r = 3$. Then it causes $C_1 = \{1, 2, a_{13}\}$ and $C_2 = \{1, 2, a_{23}, a_{24}\}$. Therefore, it proves $C_3 = \{2, a_{13}, a_{23}, a_{24}\}, C_4 = \{1, a_{13}, a_{23}, a_{24}\} \in \mathcal{C}(M)$. This is just one of case in Lemma 5, a contradiction to M defined as that in Remark 2.

(3) Similarly to the discussion in (1), it follows the need consequences.

Recalling back all the discussion from Lemma 3 to the beyond, we may state that for a paving matroid M, there are the following cases and only the following cases not be solved for considering $Aut(M) \cong H$ or $Aut(M) \ncong H$. Actually, we may indicate that M should be defined as that in Remark 2.

Case 1. $|C_1| = r, |C_1 \cap C_2| < r-1$ and there exists $C_j \in \mathcal{C}(M) \setminus C_1$ satisfying $|C_j| = r$.

Case 2. $|C_1| = r+1 = |C_2|$ and $|C_1 \cap C_2| < r$.

We will use some Examples to handle these cases partly.

Suppose M is defined as that in Remark 2 and $\rho(M) = 2$.

Let $C_1 = \{1, 2\}$. If $|C_2| = 2$. Then, we may understand that $C_2 = \{1, 3\}, C_3 = \{2, 3\}$, and in addition, $(C_1 \cup C_2, \{C_1, C_2, C_3\})$ is a paving matroid. In fact, $C_1 \cup C_2 = C_2 \cup C_3 = C_1 \cup C_3$ are true, a contradiction to the supposition. Thus, it assures $|C_2| = 3$. However, since $C_1 \cap C_2 \neq \emptyset$ and Lemma 1 together ask $|C_1 \cap C_2| = 1$, and so $C_2 = \{1, 3, 4\}$. Additionally, there are $C_1 \cup C_2 \setminus 1 = \{2, 3, 4\} \supseteq C_3$. Assume $C_3 = \{2, 3\}$ (or $\{2, 4\}$). Then $C_1 \cup C_3 \setminus 2 = \{1, 3\} \subset C_2$ (or $C_1 \cup C_3 \setminus 2 = \{1, 4\} \subset C_2$). This leads to a contradiction to Lemma 1. Thus, there is $C_3 = \{2, 3, 4\}$. Furthermore, $(C_1 \cup C_2, \{C_1, C_2, C_3\})$ is a paving matroid, but this is a contradiction with the supposition.

That is to say, $|C_1| = 3$. Similarly, $|C_2| = 3$.

Example 1 Let $C_1 = \{1, 2, 3\}$ and $C_2 = \{1, 4, 5\}$. M is defined as that in Remark 2 with $\rho(M) = 2$. Assume $C_j \in \mathcal{C}(M) \setminus \{C_1, C_2\}, |C_j| = \rho(M) = 2$ and $C_1 \cup C_2 \setminus 1 \supseteq C_3$. Since M is non-uniform, it assures $\rho(M) = 2$.

If any $C_j \in \mathcal{C}(M)$ satisfies $|C_j| = 3$, then we may state that M is uniform. This is a contradiction.

Let $|C_3| = 2$. Then $C_3 = \{2, 4\}$, in addition, $C_1 \cup C_3 \setminus 2 = \{1, 3, 4\} \supseteq C_4$. But $C_4 = \{3, 4\}$ will follow a contradiction to Lemma 1 because $C_3 \cup C_4 \setminus 4 = \{2, 3\} \subseteq C_1$. Thus, we may express that $C_4 = \{1, 3, 4\}$ and $N = (C_1 \cup C_3, \{C_1, C_3, C_4\})$ is a non-uniform matroid.

$C_2 \cup C_4 \setminus 4 = \{1, 2, 5\}$. Similarly to the above, if $C_p \subseteq \{1, 2, 5\}$ and $|C_p| = 2$, then it follows a contradiction. Thus, it causes $C_5 = \{1, 2, 5\}$. Therefore, it provides $C_1 \cup C_5 \setminus 1 = \{2, 3, 5\} \supseteq C_6$. Divided the following (1)-(3) to discuss.

(1) If $C_6 = \{2, 5\}$, then $C_3 \cup C_6 \setminus 2 = \{4, 5\} \subseteq C_2$. This causes a contradiction to Lemma 1.

(2) If $C_6 = \{3, 5\}$, then $C_1 \cup C_6 \setminus = \{1, 2, 5\} = C_5$. We can prove that $(C_1 \cup C_2, \{C_j : j = 1, 2, \ldots, 6\})$ is a non-uniform paving matroid defined as that in Remark 2. Define

$\pi_0 : x \mapsto x, x \in C_1 \cup C_2$; $\pi_1 : 2 \mapsto 4, 4 \mapsto 2, x \mapsto x, x \in \{1, 3, 5\}$;

$\pi_2 : 3 \mapsto 5, 5 \mapsto 3, x \mapsto x, x \in \{1, 2, 4\}$; $\pi_3 : 2 \mapsto 4, 4 \mapsto 2, 3 \mapsto 5, 5 \mapsto 3, 1 \mapsto 1$.

Then, we may easily find out $\pi_j \in Aut(M), (j = 0, 1, 2, 3)$. So $|Aut(M)| \geq 4$ holds. Hence $Aut(M) \ncong H$ is followed.

(3) If $C_6 = \{2, 3, 5\}$. We prove that M, i.e. $(C_1 \cup C_2, \{C_1, C_2, C_3 = \{2, 4\}, C_4 = \{1, 3, 4\}, C_5 = \{1, 2, 5\}, C_6 = \{2, 3, 5\}, C_7 = \{1, 3, 5\}, C_8 = \{3, 4, 5\}\})$, is one of the non-uniform paving matroid defined as that in Remark 2. As the discussion in Theorem 3, there is $Aut(M) \ncong H$.

Let M' be defined as in Remark 2 with $\rho(M') = 2$. Then it is not difficult to demonstrate that M' is isomorphic to one of matroids appeared in Example

1 and Theorem 3. Namely, up to isomorphism, if M is defined as that in Remark 2 and $\rho(M)=2$, then $Aut(M)\ncong H$.

Next we consider with $\rho(M)=3$.

Example 2 Let $C_1=\{1,2,3\}$ and $C_2=\{1,4,5,6\}$. $M=(C_1\cup C_2,\{C_j: j=1,\ldots,10\})$ where $C_3=\{2,3,4\},C_4=\{1,3,4\},C_5=\{1,2,4\},C_6=\{3,4,5,6\},C_7=\{1,3,5,6\},C_8=\{2,4,5,6\},C_9=\{1,2,5,6\},C_{10}=\{2,3,5,6\}$. It obviously demonstrates that M is a non-uniform paving matroid. Additionally, we may easily search out $N=(C_1\cup C_3,\mathcal{C}(N)=\{C_1,C_3,C_4,C_5\})$ and $C_6\cap(C_1\cup C_3)\neq\emptyset$. Define

$\pi_0: x\mapsto x$ for $x\in C_1\cup C_2$; $\pi_1: 1\mapsto 2, 2\mapsto 1, x\mapsto x$ for $x\in\{3,4,5,6\}$;

$\pi_2: 5\mapsto 6, 6\mapsto 5, x\mapsto x$ for $x\in\{1,2,3,4\}$; $\pi_3: 1\mapsto 4, 4\mapsto 1, x\mapsto x$ for $x\in\{2,3,5,6\}$;

Then evidently, there are $\pi_j\in Aut(M),(j=0,1,\ldots,3)$, and so $Aut(M)\ncong H$ and $|Aut(M)|\geq 4$.

Let $C_1=\{1,2,3\},C_2=\{1,4,5\}$, and M be a paving matroid with $\rho(M)=3$ defined on $C_1\cup C_2$. We prove that if M is presented as that in Remark 2 with $\rho(M)=3$ and $1\leq|C_1\cap C_2|<2$, then C_1 (or C_2) satisfies $|C_1|=4$ (or $|C_2|=4$). Thus, similar to Theorem 3 and Example 2, assuming M to be defined on $C_1\cup C_2$ with $\rho(M)=3$ and given as Remark 2 and $|C_1|=3,|C_2|=4$. We earn $Aut(M)\ncong H$ up to isomorphism.

Let M be a paving matroid defined on $C_1\cup C_2, C_1=\{1,2,3,4\},C_2=\{1,2,3,5\}$ with $\rho(M)=3$. Then up to isomorphism, M is $(C_1\cup C_2,\mathcal{C}(M)=\{C_1,C_2,C_3=\{3,4,5\},C_4=\{1,2,4,5\}\})$. We may find out that M is shown as in Remark 2. Thus, if M is defined as that in Remark 2 on $C_1\cup C_2$ with $\rho(M)=3$, then there is $|C_1\cap C_2|\leq 2$. Assume $|C_1\cap C_2|=2$. Then we get $C_1=\{1,2,3,4\}$ and $C_2=\{1,2,5,6\}$.

Example 3 Let $C_1=\{1,2,3,4\},C_2=\{1,2,5,6\}$, $C_3=\{2,3,5\},C_4=\{1,3,4,5\}$, $C_5=\{1,2,4,5\},C_6=\{1,3,5,6\},C_7=\{1,2,3,6\},C_8=\{2,3,4,5\}$ and $C_9=\{1,2,3,5\}$. Then $N=(C_1\cup C_3,\{C_1,C_3,C_4,C_5\})$ is a non-uniform matroid and $M=(C_1\cup C_2,\{C_j: j=1,2,\ldots,9\})$ is defined as that in Remark 2 on $C_1\cup C_2$ with $\rho(M)=3$ according to $N\neq M$ and $C_2\cap(C_1\cup C_3)\neq\emptyset$. Define

$\pi_0: x\mapsto x, x\in C_1\cup C_2$; $\pi_1: 2\mapsto 3, 3\mapsto 2, x\mapsto x, x\in\{1,4,5,6\}$;

$\pi_2: 2\mapsto 5, 5\mapsto 2, x\mapsto x, x\in\{1,3,4,6\}$; $\pi_3: 3\mapsto 5, 5\mapsto 3, x\mapsto x, x\in\{1,2,4,6\}$.

It is easy to see $\pi_j\in Aut(M)$ $(j=0,1,2,3)$, and so $Aut(M)\ncong H$.

Combining Theorem 3, Example 2 and Example 3 with the above discussion, we may state that if $M = (C_1 \cup C_2, \mathcal{C}(M) = \{C_j : j = 1, \ldots, k\})$ is defined as that in Remark 2 with $\rho(M) = 3$, then up to isomorphism, $Aut(M) \ncong H$ holds.

We partially answer to the Welsh's problem. But based on the discussion in this paper, we conjecture that none of paving matroids M satisfies $Aut(M) \cong \mathbb{Z}_3$.

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