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A NOTE ON Θ-CLOSED SETS AND INVERSE LIMITS

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Abstract

For every Hausdorff space X the space X_Θ is introduced. If X is H-closed, then X_Θ is a quasy-compact T_1-space.

If $f : X \to Y$ is a mapping, then there exists the mapping $f_\Theta : X_\Theta \to Y_\Theta$. We say that a mapping $f : X \to Y$ is Θ-closed if f_Θ is a closed mapping. If X and Y are H-closed and if $f : X \to Y$ is a HJ-mapping, then f_Θ is Θ-closed.

Let $\mathbf{X} = \{X_a, p_{ab}, A\}$ be an inverse system of H-closed spaces X_a and Θ-closed bonding mappings f_{ab}. If X_a are non-empty spaces, then $X = \lim \mathbf{X}$ is non-empty. If the bonding mappings p_{ab} are HJ, then $X = \lim \mathbf{X}$ is non-empty and H-closed

1 Introduction

Troughout this paper a space X always denotes a topological space. A mapping $f : X \to Y$ means a continuous map (function).

The convention and elementary results on inverse limits of topological spaces are those given in [4].

An open subset $U \subset X$ is said to be *regularly open* if $U = \operatorname{Int} \operatorname{Cl} U$. Similarly, a closed subset $F \subset X$ is said to be regularly closed if $F = \operatorname{Cl} \operatorname{Int} F$.

Definition 1.1. *[11]. A mapping $f : X \to Y$ is said to be skeletal (HJ) if for each open (regularly open) subset $U \subset X$ we have* $\operatorname{Int} f^{-1}(\operatorname{Cl} U) \subset \operatorname{Cl} f^{-1}(U)$.

The composition of (continuous) skeletal maps is skeletal [11, p. 22].

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Proposition 1. *[11, p. 22].A mapping $f : X \to Y$ is HJ if and only if the counterimage of the boundary of each regularly open set is nowhere dense.*

An HJ mapping is called in [13, p. 236] a **c**-mapping (see also [3]).

Let X be a Hausdorff space. A map $p : Y \xrightarrow{onto} X$ is said to be *irreducible [11, p. 26]* if for each regularly closed subset A of Y

$$A \neq Y \quad \text{implies} \quad \operatorname{Cl} p(A) \neq X.$$

A mapping $f : X \to Y$ is said to be *semi-open* provided $\operatorname{Int} f(U) \neq \emptyset$ for each non-empty open $U \subset X$. From Proposition 1 it follows the following result (see [11, 1.1, p. 27], [13, p. 236]).

Lemma 1.1. *Each semi-open, each open and each closed irreducible mapping is HJ.*

2 The spaces X_Θ and the mappings f_Θ

The notion of H-closed spaces was introduced by Aleksandrov and Urysohn [1].

A Hausdorff space X is *H-closed* [1] if it is closed in any Hausdorff space in which it is embedded.

The following two characterizations are given in [1].

Proposition 2. *[1, Theorem 1]. A Hausdorff space X is H-closed if and only if every family $\{U_\mu : U_\mu$ is open in X, $\mu \in \Omega\}$ with the finite intersection property has the property $\cap\{\operatorname{Cl} U_\mu : \mu \in \Omega\} \neq \emptyset$.*

Proposition 3. *[1, Theorem 2]. A Hausdorff space X is H-closed if for each open cover $\{U_\mu : \mu \in M\}$ of X there exists a finite subfamily $\{U_{\mu_1}, ..., U_{\mu_k}\}$ such that $\{\operatorname{Cl} U_{\mu_1}, ..., \operatorname{Cl} U_{\mu_k}\}$ is a cover of X.*

The Θ-closed sets were introduced by Veličko [14].

Definition 2.1. *A point $x \in X$ is in the Θ-closure of a set $A \subset X$, $x \in |A|_\Theta$, if $\operatorname{Cl} V \cap A \neq \emptyset$ for any open set V containing x. A subset $A \subset X$ is Θ-closed if $A = |A|_\Theta$. A subset $B \subset X$ is Θ-open if $X \setminus B$ is Θ-closed.*

Lemma 2.1. *[10]. A set $A \subset X$ is Θ-closed set if and only if $A = \cap\{\operatorname{Cl} V_\lambda : V_\lambda$ is open in $X, A \subset V_\lambda\}$, where $\mathcal{V} = \{V_\lambda : \lambda \in \Lambda\}$ is a maximal family of open subsets containing A.*

Theorem 2.2. *[6, Theorem 2]. In any topological space:*

(a) *the empty set and the whole space are Θ-closed,*

(b) *arbitrary intersection and finite unions of* Θ*-closed sets are* Θ*-closed,*

(c) $\mathrm{Cl}\, K \subset |K|_\Theta$ *for each subset* K*,*

(d) *a* Θ*-closed subset is closed.*

From (a) and (b) on gets the following result.

Lemma 2.3. *If* X *is a Hausdorff space, then for each* $Y \subset X$ *there exists a minimal* Θ*-closed subset* $Z \subset X$ *such that* $Y \subset Z$*.*

Proof. The collection Φ of all Θ-closed subsets W of X which contains Y is non-empty since $X \in \Phi$. By (b) of Theorem 2.2 we infer that $Z = \cap\{W : W \in \Phi\}$ is a minimal Θ-closed subset $Z \subset X$ containing Y. ∎

From Theorem 2.2 it follows that the family of all Θ-open subsets of (X, t) is a new topology t_Θ on X.

Definition 2.2. *Let* (X, t) *be a topological space. The* Θ*-space of* X *is the space* (X, t_Θ)*. In the sequel we shall use denotations* X *and* X_Θ*.*

Lemma 2.4. *If* X *is a Hausdorff space, then* X_Θ *is* T_1*-space.*

Proof. Let x be any point of X. For every another point $y \in X$, $y \neq x$, there exists a pair of open disjont set U, V such such that $x \in U$ and $y \in V$. It follows that $U \cap \mathrm{Cl}\, V = \emptyset$. We conclude that $\{x\}$ is Θ-closed and, consequently, X_Θ is T_1-space. ∎

Lemma 2.5. *The identity mapping* $id_\Theta : X \to X_\Theta$ *is continuous.*

Theorem 2.6. *If* X *is H-closed, then every family* $\{A_\mu : \mu \in \Omega\ \}$ *of* Θ*-closed subsets of* X *with the finite intersection property has a non-empty intersection* $\cap\{A_\mu :, \mu \in \Omega\ \}$*.*

Proof. A Hausdorff space X is H-closed [6] iff for every family $\{A_\mu : A_\mu \subset X, \mu \in \Omega\ \}$ with the finite intersection property there exists a point $x \in X$ such that $\mathrm{Cl}\, V \cap A \neq \emptyset$ for every open set V containing x and every A_μ. The point x is called Θ-*accumalation point.* From this characterization it follows Lemma. ∎

We say that a space X is an *Urysohn space* ([7], [9]) if for every pair $x, y, x \neq y$, of points of X there exist open sets V and W about x and y such that $\mathrm{Cl}\, V \cap \mathrm{Cl}\, W = \emptyset$.

A Hausdorf space is *nearly-compact* [8] if every open cover $\{U_\mu : \mu \in M\}$ has a finite subcollection $\{U_{\mu_1}, ..., U_{\mu_n}\}$ such that $\mathrm{Int}\,\mathrm{Cl}\, U_{\mu_1} \cup ... \cup \mathrm{Int}\,\mathrm{Cl}\, U_{\mu_n} = X$. Every nearly-compact space is H-closed.

Lemma 2.7. *[8]. A space X is nearly-compact if and only if it is H-closed and Urysohn.*

Lemma 2.8. *If X is H-closed and Urysohn, then X_Θ is a Hausdorff space.*

Theorem 2.9. *If X is an H-closed space, then X_Θ is a quasi-compact T_1-space.*

Proof. Let $\{F_\mu : \mu \in M\}$ be a family of closed sets in X_Θ with the finite intersection property. By virtue of Definition 2.2 it follows that $F_\mu = \cap\{F_{\mu,a} : a \in A, F_{\mu,a}$ is Θ-closed in $X\}$. Lemma 2.6 implies that there exists a $x \in X$ with the property $x \in \cap\{F_{\mu,a} : \mu \in M, a \in A\}$. Clearly $x \in \cap\{F_\mu : \mu \in M\}$. ∎

Problem 1. *Is it true that X is H-closed if X_Θ is a quasi-compact T_1-space?*

From Lemma 2.8 and Theorem 2.9 on gets the following result.

Theorem 2.10. *If X is nearly-compact, then X_Θ is a quasi-compact Hausdorff space.*

Definition 2.3. *Let $f : (X, \tau) \to (Y, \sigma)$ be a mapping. We define a mapping $f_\Theta : X_\Theta \to Y_\Theta$ by $f_\Theta(x) = f(x)$ for every $x \in X$, i.e., the following diagram*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow id & & \downarrow id \\ X_\Theta & \xrightarrow{f_\Theta} & Y_\Theta \end{array} \tag{2.1}$$

commutes.

Lemma 2.11. *The mapping $f_\Theta : X_\Theta \to Y_\Theta$ is continuous.*

Proof. Let us prove that $f_\Theta^{-1}(F)$ is closed in X_Θ if F is closed in Y_Θ. It suffices to prove that $f^{-1}(F)$ is Θ-closed in X if F is Θ-closed in Y. If $x \in X \setminus f^{-1}(F)$, then $f(x) \notin F$. There exists an open set U such that $f(x) \in U$ and $\mathrm{Cl}\, U \cap F = \emptyset$ since F is Θ-closed in Y. The open set $f^{-1}(U)$ contains x and $\mathrm{Cl}\, f^{-1}(U) \cap f^{-1}(F) = \emptyset$ since $f^{-1}(\mathrm{Cl}\, U) \cap f^{-1}(F) = \emptyset$. Hence, if $x \in X \setminus f^{-1}(F)$, then $x \in X \setminus \left| f^{-1}(F) \right|_\Theta$, and, consequently, $f^{-1}(F)$ is Θ-closed in X. ∎

Definition 2.4. *A mapping $f : X \to Y$ is said to be Θ-closed if $f(F)$ is Θ-closed for each Θ-closed subset $F \subset X$.*

Lemma 2.12. *Let $f : X \to Y$ be a continuous mapping. The following conditions are equivalent:*

(a) *f is Θ-closed,*

(b) *for every $B \subset Y$ and each Θ-open set $U \supseteq f^{-1}(B)$ there exists a Θ-open set $V \supseteq B$ such that $f^{-1}(V) \subset U$.*

(c) *f_Θ is a closed mapping.*

Proof. The proof is similar to the proof of the corresponding theorem for closed mappings [4, p. 52]. ∎

From 2.10 and 2.12 we obtain the following result.

Theorem 2.13. *If X and Y are nearly-compact spaces, then every continuous mapping $f : X \to Y$ is Θ-closed.*

Theorems 2.10 and 2.12 imply the following result.

Theorem 2.14. *If $f : X \to Y$ is a continuous mappinga between H-closed e.d. spaces X and Y, then f is Θ-closed.*

Now we prove the following important theorem.

Theorem 2.15. *If X and Y are H-closed, then every HJ-mapping $f : X \to Y$ is Θ-closed.*

Proof. Let A be a Θ-closed subset of X. By Definition 2.4 it suffices to prove that $f(A)$ is Θ-closed in Y.

Claim 1. By Lemma 2.1 we infer that

$$A = \cap\{\operatorname{Cl} V_\lambda : V_\lambda \text{ is open in } X,\ A \subset V_\lambda\}, \tag{2.2}$$

where $\mathcal{V} = \{V_\lambda : \lambda \in \Lambda\}$ is a maximal family of open subsets containing A.

Claim 2. *There exists a family $U = \{U_\mu : \mu \in M\}$ of all open subsets $U_\mu \subset Y$ such that there exists $V_\lambda \in \mathcal{V}$ with the property $f(V_\lambda) \subset U_\mu$. Clearly, $f(A) \subset U_\mu$ for each $U_\mu \in U$.* For each $a \in A$ there is a V_a such that $V_a \subset U_\mu$ for fixed $\mu \in M$. Let $V_\lambda = \cup\{V_a : a \in A\}$. It is clear that $f(V_\lambda) \subset U_\mu$.

Claim 3. *We prove that*

$$f(A) = \cap\{\operatorname{Cl} U_\mu : U_\mu \in \mathcal{U}\} \tag{2.3}$$

We prove only $f(A) \supset \cap\{\operatorname{Cl} U_\mu : U_\mu \in \mathcal{U}\}$ since $f(A) \subset \cap\{\operatorname{Cl} U_\mu : U_\mu \in \mathcal{U}\}$. Suppose that $y \in \cap\{\operatorname{Cl} U_\mu : U_\mu \in \mathcal{U}\}$. For every open $W \ni y$ we have $\operatorname{Cl} W \cap f(V_\lambda) \neq \emptyset$ since $\operatorname{Cl} W \cap f(V_\lambda) = \emptyset$ implies $Y \setminus \operatorname{Cl} W \supset f(V_\lambda)$, $Y \setminus \operatorname{Cl} W \in \mathcal{U}$ and $y \in \operatorname{Cl}(Y \setminus \operatorname{Cl} W)$. Now, the set $W^* = \operatorname{Int} \operatorname{Cl} W$ is regularly open and, by virtue of Definition 1.1, we have

$$\operatorname{Int} f^{-1}(\operatorname{Cl} W^*) \subset \operatorname{Cl} f^{-1}(W^*). \tag{2.4}$$

From (2.4) and $f^{-1}(\operatorname{Cl} W^*) \cap V_\lambda \neq \emptyset$ it follows $f^{-1}(W^*) \cap V_\lambda \neq \emptyset$ for each $V_\lambda \in \mathcal{V}$. The family $\mathcal{V}^* = \{V_\lambda^* : V_\lambda^* = f^{-1}(W^*) \cap V_\lambda\}$ has the finite intersection property. From the H-closedness of X it follows that there exists a point $x \in \cap\{\operatorname{Cl} V_\lambda^* : V_\lambda^* \in \mathcal{V}^*\}$. It is easily to prove that $x \in A$ and $f(x) \in \cap\{\operatorname{Cl} W : W$ is open set containing $y\}$. This means that $y = f(x)$ since Y is a Hausdorff space. Hence, $f(A) \supset \cap\{\operatorname{Cl} U_\mu : U_\mu \in \mathcal{U}\}$. The proof of (2.3) is completed. ∎

Corollary 2.16. *Let $f : X \to Y$ be a mapping between H-closed spaces. If f is open (semi-open, irreducible), then f is Θ-closed.*

Proof. By virtue of Lemma 1.1 these mapping are HJ. Apply Theorem 2.15. ∎

Example. *There exists a Θ-closed mapping which is not an HJ-mapping.* Let $X = [0, 1]$ with the following topology. The neighbourhoods of every point $x \neq 0$ are the same as those in the usual topology, but the the neighbourhoods of $x = 0$ are the sets of the form $[0, \varepsilon) \setminus D$, where $D = \{0, \frac{1}{2}, ..., \frac{1}{n}, ...\}$, $0 < \varepsilon < 1$. The space X is H-closed and Urysohn, i.e., X is nearly-compact (see Theorem 2.7). Let us define $f : X \to X = Y$ by

$$f(x) = \begin{cases} x & \text{if} \quad x < 0.6, \\ 0.6 & \text{if} \quad 0.6 \leq x < 0.8, \\ 2x - 1 & \text{if} \quad 0.8 \leq x \leq 1. \end{cases}$$

The mapping $f : X \to X$ is continuous. Moreover, f is Θ-closed since X and Y are nearly-compact. Let us prove that f is not an HJ-mapping. Let $V = (0, 0.6]$ be regularly open subset of Y. Now $\operatorname{Bd} V = \{0.6\}$ and $f^{-1}(\operatorname{Bd} V) = [0.6, 1]$. It is clear that $f^{-1}(\operatorname{Bd} V)$ contains an open set since $(0.6, 1) \subset [0.6, 1]$. By Proposition 1 f is not HJ.

Lemma 2.17. *Let $f : X \to Y$ be a surjective mapping. If F is Θ-closed in Y, then $f^{-1}(F)$ is Θ-closed in X.*

Proof. Let us prove that $X \setminus f^{-1}(F)$ is Θ-open. If x is a point of $X \setminus f^{-1}(F)$, then $f(x) \in Y \setminus F$. There exists an open set U such that $f(x) \in U$ and $\operatorname{Cl} U \cap F = \emptyset$ since F is Θ-closed. Now $x \in f^{-1}(U)$ and $\operatorname{Cl} f^{-1}(U) \cap F = \emptyset$. We infer that $X \setminus f^{-1}(F)$ is Θ-open. Hence, $f^{-1}(F)$ is Θ-closed. ∎

Let (X, t) be a topological space and $A \subset X$. If for every open t-open cover $\{U_i : i \in I\}$ of A, there exists a finite subset I_0 of I such that $A \subset \cup\{\operatorname{Cl} U_i : i \in I_0\}$, then A is said to be an *H-set [16]*.

Theorem 2.18. *[16, Theorem 3.3]. Every H-set in (X, t) is compact in (X, t_Θ).*

Theorem 2.19. *[16, Corollary 3.4]. If (X, t_Θ) is Hausdorff, then H-set in (X, t) is Θ-closed.*

Theorem 2.20. *A Θ-closed subset of an H-closed space is an H-set.*

Proof. See [2] and [14]. ∎

3 Inverse system $\mathbf{X}_\Theta$

For every inverse system $\mathbf{X} = \{X_a, p_{ab}, A\}$ we shall introduce inverse system $\mathbf{X}_\Theta$. Namely, for every space X_a there exists the space $(X_a)_\Theta$ which is defined in Definition 2.2. Moreover, for every mapping $p_{ab} : X_b \to X_a$ there exists the mapping $(p_{ab})_\Theta$ (see Definition 2.3 and Lemma 2.11). Transitivity condition

$$(p_{ab})_\Theta (p_{bc})_\Theta = (p_{ac})_\Theta$$

it follows from the commutativity of the diagram 2.1. This means that we have the following result.

Proposition 4. *For every inverse system $\mathbf{X} = \{X_a, p_{ab}, A\}$ there exists the inverse system $\mathbf{X}_\Theta = \{(X_a)_\Theta, (p_{ab})_\Theta, A\}$ such that commutes the following diagram*

$$\begin{array}{ccccccccc} X_a & \overset{p_{ab}}{\longleftarrow} & X_b & \overset{p_{bc}}{\longleftarrow} & X_c & \dots & \lim \mathbf{X} \\ \downarrow i_a & & \downarrow i_b & & \downarrow i_c & & \downarrow i \\ (X_a)_\Theta & \overset{(p^*_{ab})_\Theta}{\longleftarrow} & (X_b)_\Theta & \overset{(p_{bc})_\Theta}{\longleftarrow} & (X_c)_\Theta & \dots & \lim \mathbf{X}_\Theta \end{array}$$

where i and each i_a is the identity for every $a \in A$.

Proposition 5. *Let $\mathbf{X} = \{X_a, p_{ab}, A\}$ be an inverse system. There exists a mapping $p_\Theta : (\lim \mathbf{X})_\Theta \to \lim \mathbf{X}_\Theta$ such that $i = p_\Theta i_\Theta$, where $i_\Theta : \lim \mathbf{X} \to (\lim \mathbf{X})_\Theta$ is the identity.*

Proof. By Definition 2.1 for each $a \in A$ there is $(p_a)_\Theta : (\lim \mathbf{X})_\Theta \to (X_a)_\Theta$. This mapping is continuous (Lemma 2.11). The collection $\{(p_a)_\Theta : a \in A\}$ induces a continuous mapping $p_\Theta : (\lim \mathbf{X})_\Theta \to \lim \mathbf{X}_\Theta$. Hence we have the following diagram.

$$\begin{array}{ccc} \lim \mathbf{X} & \overset{id}{\to} & \lim \mathbf{X} \\ \downarrow i & & \downarrow i_\Theta \\ \lim \mathbf{X}_\Theta & \overset{p_\Theta}{\leftarrow} & (\lim \mathbf{X})_\Theta \end{array}$$

∎

In the sequel we shall use the following results.

Theorem 3.1. *[12, Theorem 3, p. 206]. Let* $\mathbf{X} = \{X_a, p_{ab}, A\}$ *be an inverse system of quasi-compact non-empty* T_0 *spaces and closed bonding mapping* p_{ab}*. Then* $\lim \mathbf{X}$ *is non-empty.*

Theorem 3.2. *[12, Theorem 5, p. 208].Let* $\mathbf{X} = \{X_a, p_{ab}, A\}$ *be an inverse system of quasi-compact* T_0 *spaces and closed bonding mapping* p_{ab}*. Then* $\lim \mathbf{X}$ *is quasi-compact.*

We shall prove the following result.

Lemma 3.3. *Let* $\mathbf{X} = \{X_a, p_{ab}, A\}$ *be an inverse system of quasi-compact non-empty* T_0 *spaces and closed surjective bonding mapping* p_{ab}*. Then the projections* $p_a : \lim \mathbf{X} \to \mathbf{X}_a, a \in A$*, are surjective and closed.*

Proof. Let us prove that the projections p_a are surjective. For each $x_a \in X_a$ the sets $Y_b = p_{ab}^{-1}(x_a)$ are non-empty closed sets. This means that the system $\mathbf{Y} = \{Y_b, p_{bc}|Y_c, a \leq b \leq c\}$ satisfies Theorem 3.1 and has a non-empty limit. For every $y \in Y$ we have $p_a(y) = x_a$. Hence, p_a is surjective. Let us prove that p_a is closed. It suffices to prove that for every $x_a \in X_a$ and every neighbourhood U of $p_a^{-1}(x_a)$ in $\lim \mathbf{X}$ there exists an open set U_a containing x_a such that $p_a^{-1}(U_a) \subset U$. For every $x \in p_a^{-1}(x_a)$ there is a basic open set $p_{a(x)}^{-1}(U_{a(x)})$ such that $x \in p_{a(x)}^{-1}(U_{a(x)}) \subset U$. From the quasi-compactness of $p_a^{-1}(x_a)$ it folovs that there exists a finite set $\{x_1, ..., x_n\}$ of the points of $p_a^{-1}(x_a)$ such that $\{p_{a(x_1)}^{-1}(U_{a(x_1)}), ..., p_{a(x_n)}^{-1}(U_{a(x_n)})\}$ is an open cover of $p_a^{-1}(x_a)$. Let $b \geq a(x), a(x_1), ..., a(x_n)$ and let $U_b = \cup\{p_{a(x_1)b}^{-1}(U_{a(x_1)}), ..., p_{a(x_n)b}^{-1}(U_{a(x_n)})\}$. It folows that $p_b^{-1}(U_b) \subset U$ and $p_{ab}^{-1}(x_a) \subset U_b$. From the closedness of p_{ab} it follows that there is an open set U_a containing x_a such that $p_{ab}^{-1}(U_a) \subset U_b$. Finally, $p_a^{-1}(U_a) \subset U$. The proof is complete. ∎

Theorem 3.4. *Let* $\mathbf{X} = \{X_a, p_{ab}, A\}$ *be an inverse system of quasi-compact non-empty* T_0 *spaces and closed surjective bonding mapping* p_{ab}*. Then the limit* $\lim \mathbf{X}$ *is connected if and only if each* X_a *is connected.*

Proof. If $\lim \mathbf{X}$ is connected, then each X_a is connected since, by Theorem 3.3, the projections $p_a : \lim \mathbf{X} \to \mathbf{X}_a$ are surjective mappings. Let us prove the converse. Suppose that X is not connected. There exists a pair of clopen sets U, V such that $U \cup V = X$. Now, $p_a(U), p_a(V)$ is a pair of closed sets since p_a is closed. Moreover, $X_a = p_a(U) \cup p_a(V)$. Now, $Y_a = p_a(U) \cap p_a(V)$ is non-empty since X_a is connected. Moreover, Y_a is closed and each $p_a^{-1}(Y_a)$ is closed. The collection $\{p_a^{-1}(Y_a) : a \in A\}$ has the finite intersection property. By quasi-compactnes of $\lim \mathbf{X}$ (Theorem 3.3) $Y = \cap\{p_a^{-1}(Y_a) : a \in A\}$ is non-empty. It is clear that $Y \subset U$ and $Y \subset V$. This is imposible since U and V are disjoint closed sets. ∎

The following is the main result of this paper.

Theorem 3.5. . *Let* $\mathbf{X} = \{X_a, p_{ab}, A\}$ *be an inverse system of non-empty H-closed spaces and* Θ*-closed bonding mapping* p_{ab}. *Then* $\lim \mathbf{X}$ *is non-empty. Moreover, if* p_{ab} *are surjections, then the projections* $p_a : \lim \mathbf{X} \to X_a, a \in A$, *are surjections.*

Proof. Consider the following diagram

$$\begin{array}{ccccccc} X_a & \overset{p_{ab}}{\longleftarrow} & X_b & \overset{p_{bc}}{\longleftarrow} & X_c & \dots. & \lim \mathbf{X} \\ \downarrow i_a & & \downarrow i_b & & \downarrow i_c & & \downarrow i \\ (X_a)_\Theta & \overset{(p_{ab})_\Theta}{\longleftarrow} & (X_b)_\Theta & \overset{(p_{bc})_\Theta}{\longleftarrow} & (X_c)_\Theta & \dots. & \lim \mathbf{X}_\Theta \end{array}$$

from Proposition 4. By Theorem 2.9 each $(X_a)_\Theta$ is a compact T_1 space. Furthermore, each maping $(p_{ab})_\Theta$ is closed by c) of Lemma 2.12. This means that the inverse system $\mathbf{X}_\Theta = \{(X_a)_\Theta, (p_{ab})_\Theta, A\}$ satisfies the conditions of Theorem 3.1. It follows that $\lim \mathbf{X}_\Theta$ is non-empty. This implies that $\lim \mathbf{X}$ is non-empty. Further, if $p_{ab}, b \geq a$, are onto mappings, then for each $x_a \in X_a$ the sets $Y_b = p_{ab}^{-1}(x_a)$ are non-empty Θ-closed sets (Lemma 2.17). This means that the system $\mathbf{Y}_\Theta = \{(Y_b)_\Theta, (p_{bc})_\Theta|(Y_c)_\Theta, a \leq b \leq c\}$ satisfies Theorem 3.1 and has a non-empty limit. This means $\mathbf{Y} = \{Y_b, p_{bc}|Y_c, a \leq b \leq c\}$ has a non-empty limit. For every $y \in Y$ we have $p_a(y) = x_a$. The proof is completed. ∎

If X and Y are nearly-compact spaces, then each mapping $f : X \to Y$ is Θ-closed (Theorem 2.13). We have the following consequence of Theorem 3.5.

Corollary 3.6. *Let* $\mathbf{X} = \{X_a, p_{ab}, A\}$ *be an inverse system of non-empty nearly-compact spaces. Then* $\lim \mathbf{X}$ *is non-empty. Moreover, if* p_{ab} *are surjections, then the projections* $p_a : \lim \mathbf{X} \to X_a, a \in A$, *are surjections.*

Lemma 3.7. *Let* $\mathbf{X} = \{X_a, p_{ab}, A\}$ *be an inverse system of H-closed spaces and* Θ*-closed surjective bonding mapping* p_{ab}. *The projections* $p_a : \lim \mathbf{X} \to \mathbf{X}_a$, $a \in A$, *are* Θ*-closed if and only if the mapping* $p_\Theta : (\lim \mathbf{X})_\Theta \to \lim \mathbf{X}_\Theta$ *from Proposition 5 is a homeomorphism.*

Proof. The if part. Let $F \subset \lim \mathbf{X}$ be Θ-closed. Then $i_\Theta(F)$ is closed in $(\lim \mathbf{X})_\Theta$. This means that $p_\Theta i_\Theta(F)$ is closed in $\lim \mathbf{X}_\Theta$. Now $q_a(p_\Theta i_\Theta(F))$ is closed in $(X_a)_\Theta$ since each projection $q_a : (\lim \mathbf{X})_\Theta \to (X_a)_\Theta$ is closed (Lemma 3.3). We infer that $i_a^{-1}(q_a(p_\Theta i_\Theta(F)))$ is Θ-closed in X_a. This means that $p_a(F)$ is Θ-closed since $p_a(F) = i_a^{-1}(q_a(p_\Theta i_\Theta(F)))$. Thus, p_a is Θ-closed for every $a \in A$.

The only if part. Suppose that the projections $p_a : \lim \mathbf{X} \to \mathbf{X}_a, a \in A$, are Θ-closed. Let us prove that p_Θ is a homeomorphism. It suffice to prove that p_Θ is closed. Let $F \subset (\lim \mathbf{X})_\Theta$ be closed. This means that F is Θ-closed in $\lim \mathbf{X}$.

For each $a \in A$ the set $p_a(F)$ is Θ-closed since the projections p_a are Θ-closed. Now, $i_a p_a(F)$ is closed in $(X_a)_\Theta$. We have the collection $\{q_a^{-1} i_a p_a(F) : a \in A\}$ with finite intersection property. It is clear that $p_\Theta(F) = \cap\{q_a^{-1} i_a p_a(F) : a \in A\}$ and that $\cap\{q_a^{-1} i_a p_a(F) : a \in A\}$ is closed in $\lim \mathbf{X}_\Theta$. Hence, p_Θ is closed and, consequently, a homeomorphism. ∎

Theorem 3.8. *Let* $\mathbf{X} = \{X_a, p_{ab}, A\}$ *be an inverse system with* HJ *mappings* p_{ab}. *If the projections* $p_a : \lim \mathbf{X} \to X_a, a \in A$, *are surjections, then they are* HJ *mapping and, consequently,* Θ*-closed .*

Proof. By Proposition 1 a mapping $f : X \to Y$ is HJ if and only if the counterimage of the boundary of each regularly open set is nowhere dense. Suppose that p_a is not HJ. Then there exist a regularly open set U_a in X_a such that the boundary of $p_a^{-1}(U_a)$ contains an open set U. From the definition of a base in $\lim \mathbf{X}$ it follows that there is a $b \geq a$ and an open set U_b in X_b such that $p_b^{-1}(U_b) \subset U$. It is clair that $U_b \subset \operatorname{Bd}\ p_{ab}^{-1}(U_a)$. This is impossible since p_{ab} is HJ. Hence, the projections $p_a, a \in A$, are HJ. From Theorem 2.15 it follows that p_a is Θ-closed. ∎

Theorem 3.9. *If* $\mathbf{X} = \{X_a, p_{ab}, A\}$ *is an inverse system of H-closed spaces* X_a *and HJ mappings* p_{ab}, *then* $X = \lim \mathbf{X}$ *is H-closed.*

Proof. If $X = \emptyset$, then Theorem holds. Let $X \neq \emptyset$. Then $X_a \neq \emptyset$ for every $a \in A$ and the projections $p_a : X \to X_a$ onto HJ mappings. Let us prove that X is H-closed. It suffices to prove that each maximal centred family $\mathcal{U} = \{U_\mu : \mu \in M, U_\mu$ is open subset of $X\}$ has the property $\cap\{\operatorname{Cl} U_\mu : \mu \in M\} \neq \emptyset$. For each $a \in A$ we define a centred family $\mathcal{U}_a = \{U_{\mu_a} : U_{\mu_a}$ is open in X_a and there exists $U_\mu \in \mathcal{U}$ such $p_a(U_\mu) \subset U_{\mu_a}, \mu_a \in M_a\}$. Now we shall prove that $\mathcal{U}_a$ is maximal. Let U_a be ope in X_a with property $U_a \cap U_{\mu_a} \neq \emptyset$ for every $U_{\mu_a} \in \mathcal{U}_a$. It is readily seen that $\operatorname{Cl} U_a \cap p_a(U_\mu) \neq \emptyset$ for each $U_\mu \in \mathcal{U}$. Hence, if we denote $\operatorname{Int}\operatorname{Cl} U_a$ by V_a, then we have $\operatorname{Cl} V_a \cap p_a(U_\mu) \neq \emptyset$ for each $U_\mu \in \mathcal{U}$. From the fact that p_a is HJ we conclude that $\operatorname{Cl}(p_a^{-1}(V_a))$ $\cap U_\mu \neq \emptyset$ since $\operatorname{Cl}(p_a^{-1}(V_a))$ $\cap U_\mu = \emptyset$ implies that $X \setminus p_a^{-1}(\operatorname{Cl} V_a) \in \mathcal{U}$; a contradiction. From $p_a^{-1}(V_a) \cap U_\mu \neq \emptyset$ and the maximality of $\mathcal{U}$ it folows that $p_a^{-1}(V_a) \in \mathcal{U}$ and, consequently, $V_a \in \mathcal{U}_a$. This means that $\mathcal{U}_a$ is maximal. In similar way on can prove that if $U_{\mu_a} \in \mathcal{U}_a$, then $p_{ab}^{-1}(U_{\mu_a}) \in \mathcal{U}_b$, where $b > a$. Since X_a is H-closed and $\mathcal{U}_a$ maximal, there exists $x_a \in X_a$ such that $x_a = \cap\{\operatorname{Cl} U_{\mu_a} : U_{\mu_a} \in \mathcal{U}_a\}$. Moreover, $p_{ab}(x_b) = x_a$ if $b \geq a$. It is easely to prove that $x = (x_a : a \in A) \in \cap\{\operatorname{Cl} U_\mu : U_\mu \in \mathcal{U}\}$. The proof is completed. ∎

Corollary 3.10. *If* $\mathbf{X} = \{X_a, p_{ab}, A\}$ *is an inverse system of H-closed spaces* X_a *and semi-open* (*open, closed irreducuble*) *mappings* p_{ab}, *then* $X = \lim \mathbf{X}$ *is H-closed.*

REMARK. If $\mathbf{X} = \{X_a, p_{ab}, A\}$ is an inverse system of H-closed spaces X_a open bonding mappings p_{ab}, then see [5] and [15].

We close this section with result concerning the connectedness of the limit space $\lim \mathbf{X}$.

Theorem 3.11. *Let $\mathbf{X} = \{X_a, p_{ab}, A\}$ be an inverse system of H-closed spaces X_a and surjective Θ-closed mappings p_{ab}. If the projections $p_a : \lim \mathbf{X} \to \mathbf{X}_a, a \in A$, are Θ-closed and $X = \lim \mathbf{X}$ is H-closed , then X is connected if and only if each X_a is connected.*

Proof. If $\lim \mathbf{X}$ is connected, then each X_a is connected since, by Theorem 3.5, the projections $p_a : \lim \mathbf{X} \to \mathbf{X}_a$ are surjective mappings. Let us prove the converse. Suppose that X is not connected. There exists a pair of clopen sets U, V such that $U \cup V = X$. It is clear that U and V are Θ-closed. Now, $p_a(U), p_a(V)$ is a pair of Θ-closed sets since p_a is Θ-closed. Moreover, $X_a = p_a(U) \cup p_a(V)$. Now, $Y_a = p_a(U) \cap p_a(V)$ is non-empty since X_a is connected. Moreover, Y_a is Θ-closed (see (b) of Theorem 2.2). By Lemma 2.17 each $p_a^{-1}(Y_a)$ is Θ-closed. The collection $\{p_a^{-1}(Y_a) : a \in A\}$ has the finite intersection property. By Theorem 2.6 $Y = \cap\{p_a^{-1}(Y_a) : a \in A\}$ is non-empty. This is imposible since U and V are disjoint. ∎

Corollary 3.12. *Let $\mathbf{X} = \{X_a, p_{ab}, A\}$ be an inverse system of H-closed spaces X_a and surjective HJ mappings p_{ab}. Then X is connected if and only if each X_a is connected.*

References

[1] P. S. Aleksandroff et P. S. Urysohn, *Mémoire sur les espaces topologiques compacts*, Verh. Akademie Amsterdam, Deel XIV, Nr. 1 (1929), 1-96.

[2] R. F. Dickman, Jr. and J. R. Porter, *Θ-closed subsets of Hausdorff spaces*, Pac. J. Math. 59 (1975), 407-415.

[3] R. F. Dickman, JR., J. R. Porter, and L. R. Rubin, *Completely regular absolutes and projective objects*, Pacific J. Math, 94 (1981), 277-295.

[4] R. Engelking, *General Topology*, PWN,Warszawa, 1977.

[5] L. M. Friedler and D. H. Pettey, *Inverse limits and mappings of minimal topological spaces*, Pacific J. Math. 71 (1977), 429-448.

[6] L. L. Herrington and P. E. Long, *Characterizations of H-closed spaces*, Proc. Amer. Math. Soc. 48 (1975), 469-475.

[7] L. L. Herrington, *Characterizations of Urysohn-closed spaces*, Proc. Amer. Math. Soc. 55 (1976), 435-439.

[8] L. L. Herrington, Properties of nearly-compact spaces, Proc. Amer. Math. Soc. 45 (1974), 431-436.

[9] J. E. Joseph, *On Urysohn-closed and minimal Urysohn spaces*, Proc. Amer. Math. Soc. 68 (1978), 235-242.

[10] J. E. Joseph, *Multifunctions and cluster sets*, Proc. Amer. Math. Soc. 74 (1979), 329–337.

[11] J. Mioduszewski and L. Rudolf, *H-closed and extremally disconnected Hausdorff spaces*, Diss. Math. 51, 1969.

[12] A. H. Stone, *Inverse limits of compact spaces*, Gen. Topology and Appl. 10 (1979), 203-211.

[13] V. M. Uljanov, *O bikompaktnyh rasširenijah sčetnogo haraktera i absoljutah,* Mat. Sbornik 98 (1975), 223-254.

[14] N. V. Veličko, *H-closed topological spaces*, Mat. Sb., 70 (112) (1966), 98-112.

[15] T. O. Vinson, Jr. and R. F. Dickman, Jr., *Inverse limits and absolutes of H-closed spaces*, Proc. Amer. Math. Soc. 66 (1977), 351-358.

[16] T. H. Yalvaç, *Relations between the topologies τ and τ_Θ*, Hacettepe Bull. Nat. Sci. and Engin. 20 (1991), 29-39.

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