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SOME NONLINEAR DELAY INTEGRAL INEQUALITIES AND THEIR DISCRETE ANALOGUES

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Abstract

In this paper, we investigate some nonlinear delay integral inequalities and their analogues which provide explicit bounds on unknown functions. The inequalities given here can be used as tools in the qualitative theory of certain delay differential equations, delay integral equations and delay difference equations.

1 Introduction

The integral inequalities and the finite difference inequalities play a fundamental role in the development of the theory of differential equations, integral equations and difference equations. During the past few years, many such inequalities have been discovered, which are motivated by certain applications. For example, see the monographes[1, 2, 9, 10], papers[3–7, 11, 12] and the references therein. However, in the qualitative analysis of some classes of delay differential equations, delay integral equations and delay difference equations, the bounds provided by the earlier inequalities are inadequate and it is necessary to seek some new integral inequalities and their discrete analogues in order to achieve a diversity of desired goals. In this paper, we investigate some nonlinear delay integral inequalities and their discrete analogues which provide explicit bounds on unknown functions.

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2 Formulation of the Problem

In what follows, R denotes the set of real numbers, $R_+ = [0, \infty)$ is the given subset of R, $C(M, S)$ denotes the class of all continuous functions defined on set M with range in the set S, and $N_0 = \{0, 1, 2, \cdots\}$ denotes the set of nonnegative integers. We use the usual conventions that empty sums and products are taken to be 0 and 1 respectively. Throughout this paper, all the functions which appear in the inequalities are assumed to be real-valued and all the involved sums exist on the respective domains of their definitions.

In this paper, on the one hand, we study the following nonlinear delay integral inequalities

$$x^p(t) \le a(t) + b(t) \int_0^t [f(s)x^p(s-\tau) + g(s)x(s) + h(s)]ds, \quad t \in R_+, \quad (E1)$$

and

$$x^p(t) \le a(t) + b(t) \int_0^t L(s, x(s-\tau))ds, \quad t \in R_+, \quad (E2)$$

with the initial condition

$$\begin{cases} x(t) = \varphi(t), & t \in [-\tau, 0], \\ \varphi(t-\tau) \le (a(t))^{1/p} & \text{for } \ t \in R_+ \ \text{ with } \ t - \tau \le 0, \end{cases} \quad (I)$$

where $p > 1$ and $\tau \in R_+$ are constants, $\varphi(t) \in C([-\tau, 0], R_+)$, and $L \in C(R_+^2, R_+)$.

On the other hand, we also investigate the following discrete analogues of $(E1)$ and $(E2)$

$$x^p(n) \le a(n) + b(n) \sum_{s=0}^{n-1} [f(s)x^p(s-\sigma) + g(s)x(s) + h(s)], \quad n \in N_0, \quad (E'1)$$

and

$$x^p(n) \le a(n) + b(n) \sum_{s=0}^{n-1} V(s, x(s-\sigma)), \quad n \in N_0, \quad (E'2)$$

with the initial condition

$$\begin{cases} x(n) = \psi(n), & n \in \{-\sigma, \cdots, -1, 0\}, \\ \psi(n-\sigma) \le (a(n))^{1/p} & \text{for } \ n \in N_0 \ \text{ with } \ n - \sigma \le 0, \end{cases} \quad (I')$$

where $p > 1$, $\sigma \in N_0$ are constants, $\psi(n) \in R_+, n \in \{-\sigma, \cdots, -1, 0\}$, and $V : N_0 \times R_+ \to R_+$.

3 Main Results

The following lemmas are useful in our main results.

Lemma 1[8]. *Assume that* $\frac{1}{p}+\frac{1}{q}=1$ *with* $p>1$. *Then*

$$x^{\frac{1}{p}}y^{\frac{1}{q}}\leq\frac{x}{p}+\frac{y}{q}, \tag{1}$$

for $x,y\in R_+$.

Lemma 2[11]. *(i) Assume that* $u(t),a(t),b(t)\in C(R_+,R_+)$, *and* $a(t)$ *is nondecreasing for* $t\in R_+$. *If*

$$u(t)\leq a(t)+\int_0^t b(s)u(s)ds,$$

for $t\in R_+$, *then*

$$u(t)\leq a(t)\exp\bigg(\int_0^t b(s)ds\bigg),$$

for $t\in R_+$.

(ii) Assume that $u(n),a(n),b(n)$ *are nonnegative functions defined for* $n\in N_0$, *and* $a(n)$ *is nondecreasing for* $n\in N_0$. *If*

$$u(n)\leq a(n)+\sum_{s=0}^{n-1}b(s)u(s),\quad n\in N_0,$$

then

$$u(n)\leq a(n)\prod_{s=0}^{n-1}[1+b(s)],\quad n\in N_0.$$

Theorem 1. *Assume that* $x(t),a(t),b(t),f(t),g(t),h(t)\in C(R_+,R_+)$. *If* $a(t)$ *and* $b(t)$ *are nondecreasing in* R_+, *then the inequality* $(E1)$ *with the initial condition* (I) *implies*

$$x(t)\leq\bigg\{a(t)+b(t)B(t)\exp\bigg(\int_0^t b(s)\bigg(f(s)+\frac{g(s)}{p}\bigg)ds\bigg)\bigg\}^{\frac{1}{p}}, \tag{2}$$

for $t\in R_+$, *where*

$$B(t)=\int_0^t\bigg[f(s)a(s)+\frac{a(s)+p-1}{p}g(s)+h(s)\bigg]ds. \tag{3}$$

Proof. Fixing any positive number T, we define a function $z(t)$ by

$$z(t) = \left\{ a(T) + b(t) \int_0^t [f(s)x^p(s-\tau) + g(s)x(s) + h(s)]ds \right\}^{\frac{1}{p}}, \quad t \in [0, T]. \quad (4)$$

It is easy to see that $z(t)$ is a nonnegative and nondecreasing function, and

$$x(t) \le z(t), \quad t \in [0, T].$$

Therefore,

$$x(t-\tau) \le z(t-\tau) \le z(t), \quad t - \tau \ge 0, \ t \in [0, T]. \quad (5)$$

Using the initial condition (I), we have

$$x(t-\tau) = \varphi(t-\tau) \le (a(t))^{1/p} \le (a(T))^{1/p} \le z(t), \quad t-\tau \le 0, \ t \in [0, T]. \quad (6)$$

(5) and (6) guarantee

$$x(t-\tau) \le z(t), \quad t \in [0, T]. \quad (7)$$

It follows from (4) and (7) that

$$z^p(t) \le a(T) + b(t) \int_0^t [f(s)z^p(s) + g(s)z(s) + h(s)]ds, \quad t \in [0, T]. \quad (8)$$

Taking $t = T$ in (8), we obtain

$$z^p(T) \le a(T) + b(T) \int_0^T [f(s)z^p(s) + g(s)z(s) + h(s)]ds. \quad (9)$$

Noting that $T \in R_+$ is arbitrary, from (9), we have

$$z^p(t) \le a(t) + b(t) \int_0^t [f(s)z^p(s) + g(s)z(s) + h(s)]ds, \quad t \in R_+. \quad (10)$$

Similarly, we obtain

$$x(t) \le z(t), \quad t \in R_+. \quad (11)$$

Define a function $u(t)$ by

$$u(t) = \int_0^t [f(s)z^p(s) + g(s)z(s) + h(s)]ds, \quad t \in R_+. \quad (12)$$

Then (10) can be restated as

$$z^p(t) \le a(t) + b(t)u(t), \quad t \in R_+. \quad (13)$$

Using Lemma 1, from (13), we easily obtain

$$z(t) \leq (a(t)+b(t)u(t))^{\frac{1}{p}}(1)^{\frac{p-1}{p}} \leq \frac{a(t)}{p}+\frac{b(t)}{p}u(t)+\frac{p-1}{p}, \quad t \in R_+. \tag{14}$$

Combining (12)–(14), we get

$$u^{'}(t) \leq b(t)\left(f(t)+\frac{g(t)}{p}\right)u(t)+\left[f(t)a(t)+\frac{a(t)+p-1}{p}g(t)+h(t)\right],$$

i.e.

$$u(t) \leq B(t)+\int_0^t b(s)\left(f(s)+\frac{g(s)}{p}\right)u(s)ds, \quad t \in R_+, \tag{15}$$

where $B(t)$ is defined by (3). Using the Part (i) of Lemma 2, from (15), we have

$$u(t) \leq B(t)\exp\left(\int_0^t b(s)\left(f(s)+\frac{g(s)}{p}\right)ds\right), \quad t \in R_+. \tag{16}$$

Clearly, the desired inequality (2) follows from (11), (13) and (16). The proof is complete.

Theorem 2. *Assume that $x(t), a(t), b(t) \in C(R_+, R_+)$, $a(t)$ and $b(t)$ are nondecreasing in R_+. If*

$$0 \leq L(t,x)-L(t,y) \leq K(t,y)(x-y), \tag{17}$$

for $x \geq y \geq 0$, where $K \in C(R_+^2, R_+)$, then the inequality $(E2)$ with the initial condition (I) implies

$$x(t) \leq \left\{a(t)+b(t)E(t)\exp\left(\int_0^t K\left(s, \frac{a(s)+p-1}{p}\right)\frac{b(s)}{p}ds\right)\right\}^{\frac{1}{p}}, \tag{18}$$

for $t \in R_+$, where

$$E(t)=\int_0^t L\left(s, \frac{a(s)+p-1}{p}\right)ds. \tag{19}$$

Proof. Fixing any positive number T, we define a function $z(t)$ by

$$z(t)=\left\{a(T)+b(t)\int_0^t L(s, x(s-\tau))ds\right\}^{\frac{1}{p}}, \quad t \in [0,T].$$

Using a similar way in the proof of Theorem 1 and noting the condition (17), we easily obtain that $z(t)$ is a nonnegative and nondecreasing function, and

$$x(t) \leq z(t), \quad t \in R_+, \tag{20}$$

and

$$z^p(t) \le a(t) + b(t) \int_0^t L(s, z(s))ds, \quad t \in R_+. \tag{21}$$

Define a function $u(t)$ by

$$u(t) = \int_0^t L(s, z(s))ds. \tag{22}$$

Then (21) can be restated as

$$z^p(t) \le a(t) + b(t)u(t), \; t \in R_+. \tag{23}$$

As in the proof of Theorem 1, from (23), we obtain (14). Noting the condition (17), from (22) and (14), we have

$$\begin{aligned} u'(t) &= L(t, z(t)) \\ &\le L\left(t, \frac{a(t)}{p} + \frac{b(t)}{p}u(t) + \frac{p-1}{p}\right) - L\left(t, \frac{a(t)}{p} + \frac{p-1}{p}\right) \\ &\quad + L\left(t, \frac{a(t)}{p} + \frac{p-1}{p}\right) \\ &\le K\left(t, \frac{a(t)}{p} + \frac{p-1}{p}\right)\frac{b(t)}{p}u(t) + L\left(t, \frac{a(t)}{p} + \frac{p-1}{p}\right), \end{aligned}$$

i.e.

$$u(t) \le E(t) + \int_0^t K\left(s, \frac{a(s)+p-1}{p}\right)\frac{b(s)}{p}u(s)ds, \; t \in R_+, \tag{24}$$

where $E(t)$ is defined by (19). Using the Part (i) of Lemma 2, it follows from (24) that

$$u(t) \le E(t) \; \exp\left(\int_0^t K\left(s, \frac{a(s)+p-1}{p}\right)\frac{b(s)}{p}ds\right), \; t \in R_+, \tag{25}$$

We easily see that the desired inequality (18) follows from (20), (23) and (25). This completes the proof of Theorem 2.

Theorem 3. *Assume $x(n), a(n), b(n), f(n), g(n), h(n)$ be nonnegative functions defined for $n \in N_0$. If $a(n)$ and $b(n)$ are nondecreasing in N_0, then the inequality $(E'1)$ with the initial condition (I') implies*

$$x(n) \le \left\{a(n) + b(n)G(n)\prod_{s=0}^{n-1}\left[1 + b(s)\left(f(s) + \frac{g(s)}{p}\right)\right]\right\}^{\frac{1}{p}}, \tag{26}$$

for $n \in N_0$*, where*

$$G(n) = \sum_{s=0}^{n-1} \left[f(s)a(s) + \frac{a(s)+p-1}{p} g(s) + h(s) \right]. \tag{27}$$

Proof. Fixing any positive integer M, we define a function $z(n)$ by

$$z(n) = \left\{ a(M) + b(n) \sum_{s=0}^{n-1} \left[f(s)x^p(s-\sigma) + g(s)x(s) + h(s) \right] \right\}^{\frac{1}{p}}, \quad n \in N_M, \tag{28}$$

where $N_M = \{0, 1, \cdots, M\}$. It is easy to see that $z(n)$ is a nonnegative and nondecreasing function, and

$$x(n) \le z(n), \quad n \in N_M.$$

Therefore, for $n \in N_0$ with $n - \sigma \ge 0$, we have

$$x(n-\sigma) \le z(n-\sigma) \le z(n), \quad n \in N_M. \tag{29}$$

Using the initial condition (I'), for $n \in N_0$ with $n - \sigma \le 0$, we have

$$x(n-\sigma) = \varphi(n-\sigma) \le (a(n))^{1/p} \le (a(M))^{1/p} \le z(n), \quad n \in N_M. \tag{30}$$

Combining (29) and (30), we obtain

$$x(n-\sigma) \le z(n), \quad n \in N_M. \tag{31}$$

Therefore,

$$z^p(n) \le a(M) + b(n) \sum_{s=0}^{n-1} [f(s)z^p(s) + g(s)z(s) + h(s)], \quad n \in N_M. \tag{32}$$

Taking $n = M$ in (32), we obtain

$$z^p(M) \le a(M) + b(M) \sum_{s=0}^{M-1} [f(s)z^p(s) + g(s)z(s) + h(s)]. \tag{33}$$

Noting that $M \in N_0$ is arbitrary, from (33), we observe that

$$z^p(n) \le a(n) + b(n) \sum_{s=0}^{n-1} [f(s)z^p(s) + g(s)z(s) + h(s)], \quad n \in N_0. \tag{34}$$

Using a Similar way, we obtain

$$x(n) \le z(n), \quad n \in N_0. \tag{35}$$

Define a function $u(n)$ by

$$u(n) = \sum_{s=0}^{n-1} [f(s)z^p(s) + g(s)z(s) + h(s)], \quad n \in N_0. \tag{36}$$

Then (34) can be restated as

$$z^p(n) \le a(n) + b(n)u(n), \quad n \in N_0. \tag{37}$$

Using Lemma 1, from (37), we easily obtain

$$z(n) \le \frac{b(n)}{p} u(n) + \frac{a(n) + p - 1}{p}, \quad n \in N_0. \tag{38}$$

Therefore,

$$\begin{aligned} u(n+1) - u(n) \quad \le \quad & \left[f(n)a(n) + \frac{a(n) + p - 1}{p} g(n) + h(n) \right] \\ & + b(n) \left(f(n) + \frac{g(n)}{p} \right) u(n), \quad n \in N_0. \end{aligned} \tag{39}$$

Substituting $n = s$ and taking the sum over s from 0 to $n - 1$, it follows from (39) that

$$u(n) \le G(n) + \sum_{s=0}^{n-1} \left[1 + b(s) \left(f(s) + \frac{g(s)}{p} \right) \right] u(s), \quad n \in N_0, \tag{40}$$

where $G(n)$ is defined by (27). Using the Part (ii) of Lemma 2, we easily see that (40) guarantees

$$u(n) \le G(n) \prod_{s=0}^{n-1} \left[1 + b(s) \left(f(s) + \frac{g(s)}{p} \right) \right], \quad n \in N_0. \tag{41}$$

It is easy to see that the desired inequality (26) follows from (35), (37) and (41). This completes the proof of Theorem 3.

Theorem 4. *Let $x(n), a(n), b(n)$ be nonnegative functions for $n \in N_0$, $a(n)$ and $b(n)$ be nondecreasing in N_0. If*

$$0 \le V(n, x) - V(n, y) \le W(n, y)(x - y), \tag{42}$$

for $x \ge y \ge 0$, where $W : N_0 \times R_+ \to R_+$, then the inequality $(E'2)$ with the initial condition (I') implies

$$x(n) \le \left\{ a(n) + b(n)F(n) \prod_{s=0}^{n-1} \left[1 + W\left(s, \frac{a(s)+p-1}{p} \right) \frac{b(s)}{p} \right] \right\}^{\frac{1}{p}}, \tag{43}$$

for $n \in N_0$, where

$$F(n) = \sum_{s=0}^{n-1} V\left(s, \frac{a(s)+p-1}{p} \right). \tag{44}$$

Proof. Fixing any positive integer M, we define a function $z(n)$ by

$$z(n) = \left\{ a(M) + b(n) \sum_{s=0}^{n-1} V(s, x(s-\sigma)) \right\}^{\frac{1}{p}}, \quad n \in N_M.$$

Using a similar way in the proof of Theorem 3 and noting the condition (42), we easily obtain that $z(n)$ is a nonnegative and nondecreasing function, and

$$x(n) \le z(n), \quad n \in N_0, \tag{45}$$

and

$$z^p(n) \le a(n) + b(n) \sum_{s=0}^{n-1} V(s, z(s)), \quad n \in N_0. \tag{46}$$

Define a function $u(n)$ by

$$u(n) = \sum_{s=0}^{n-1} V(s, z(s)), \quad n \in N_0. \tag{47}$$

Then (46) can be restated as

$$z^p(n) \le a(n) + b(n)u(n), \quad n \in N_0. \tag{48}$$

As in the proof of Theorem 3, from (48), we obtain (38). Noting the condition (42), from (47) and (38), we have

$$\begin{aligned} u(n+1) - u(n) &= V(n, z(n)) \\ &\le V\left(n, \frac{a(n)}{p} + \frac{b(n)}{p}u(n) + \frac{p-1}{p} \right) \\ &\quad - V\left(n, \frac{a(n)}{p} + \frac{p-1}{p} \right) + V\left(n, \frac{a(n)}{p} + \frac{p-1}{p} \right) \\ &\le W\left(n, \frac{a(n)}{p} + \frac{p-1}{p} \right) \frac{b(n)}{p} u(n) \\ &\quad + V\left(n, \frac{a(n)}{p} + \frac{p-1}{p} \right), \; n \in N_0. \end{aligned} \tag{49}$$

Substituting $n = s$ and taking the sum over s from 0 to $n-1$, it follows from (49) that

$$u(n) \le F(n) + \sum_{s=0}^{n-1} W\left(s, \frac{a(s)+p-1}{p}\right) \frac{b(s)}{p} u(s), \ n \in N_0, \tag{50}$$

where $F(n)$ is defined by (44). Using the Part (ii) of Lemma 2, from (50), we have

$$u(n) \le F(n) \prod_{s=0}^{n-1} \left[1 + W\left(s, \frac{a(s)+p-1}{p}\right) \frac{b(s)}{p}\right], \ n \in N_0, \tag{51}$$

The desired inequality (43) follows from (45), (48) and (51). This completes the proof.

Finally, we present an application of Theorem 1.

Example. Consider the delay differential equation

$$(x^p(t))' = P(t, x(t), x(t-\tau)), \quad t \in R_+, \tag{52}$$

with the initial condition

$$\begin{cases} x(t) = \phi(t), & t \in [-\tau, 0], \\ \phi(t-\tau) \le |C|^{\frac{1}{p}} \ \text{ for } \ t \in R_+ & \text{with } \ t - \tau \le 0, \end{cases} \tag{53}$$

where $P \in C(R_+ \times R^2, R)$, $C = x^p(0)$, $p > 1$, $\tau \in R_+$ are constants, and $\phi \in C([-\tau, 0], R)$.

Assume that

$$|P(t, x(t,), x(t-\tau))| \le f(t)|x^p(t-\tau)| + g(t)|x(t)| + h(t), \tag{54}$$

where $f(t), g(t), h(t)$ are as defined in Theorem 1. If $x(t)$ is a solution of the equation (52) satisfying the initial condition (53), then

$$|x(t)| \le \left\{|C| + b(t)\widetilde{B}(t) \exp\left(\int_0^t b(s)\left(f(s) + \frac{g(s)}{p}\right)ds\right)\right\}^{\frac{1}{p}}, \tag{55}$$

for $t \in R_+$, where

$$\widetilde{B}(t) = \int_0^t \left[|C|f(s) + \frac{|C|+p-1}{p} g(s) + h(s)\right] ds. \tag{56}$$

In fact, the solution $x(t)$ of equation (52) satisfying the initial condition (53) satisfies the equivalent delay integral equation

$$x^p(t) = C + \int_0^t P(s, x(s), x(s-\tau)) ds, \quad t \in R_+, \tag{57}$$

with the initial condition (53). Noting the assumption (54), we have

$$|x^p(t)| \leq |C| + \int_0^t [f(s)|x^p(s-\tau)| + g(s)|x(s)| + h(s)]ds \tag{58}$$

with the initial condition (53). Now a suitable application of Theorem 1 to (58) yields (55).

Remark 1. The right–hand side of (55) gives us the bound on the solution $x(t)$ of the equation (52) satisfying the initial condition (53) in terms of the known functions for $t \in R_+$.

Remark 2. We can present some applications of Theorems 2–4. Due to limited space, their statements are omitted here.

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