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ON RANK 2 GEOMETRIES OF THE MATHIEU GROUP M_{24}

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Abstract

In this paper we determine rank 2 geometries for the Mathieu group M_{24} for which object stabilizers are maximal subgroups.

1 Introduction

One of the major open questions nowadays in finite simple groups is to find a unified geometric interpretation of all the finite simple groups. The theory of buildings due to Jacques Tits answers partially this question by associating a geometric object to each of the finite simple groups except the Alternating groups and the sporadic groups. Since the 1970's, Francis Buekenhout introduced diagram geometries, allowing more residues than just generalized polygons and started building geometries for the sporadic groups. In that spirit, he classified with Dehon and Leemans all primitive geometries for the Mathieu group M_{11} (see [6]).

In the 1990's, the team led by Buekenhout decided to change slightly the axioms, replacing the "primitivity" condition by a weaker condition because they were convinced that the primitivity condition was too strong to achieve their goal. They then studied residually weakly primitive geometries and nowadays, 10 of the 26 sporadic groups are fully analyzed under this condition and local two-transitivity (see [7]).

In [18], D. Leemans gave the list of all of the firm and residually connected geometries that satisfy the $(IP)_2$ and $(2T_1)$ conditions on which the

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Mathieu group M_{24} acts flag-transitively and residually weakly primitively, he got $15, 21, 21, 22, 5, 0$ geometries of rank $2, 3, 4, 5, 6, \geq 7$. These results were obtained using a series of MAGMA [1] programs.

In this paper, we give the list of rank 2 primitive geometries for M_{24} that are firm, residually connected and flag transitive. These results were obtained using a series of MAGMA[1] programs. The paper is organized as follows. In section 2, we recall the basic definitions needed in order to understand this paper. In section 3, we give the list of geometries we obtained. Finally, in section 4, we give the diagrams of some rank 2 geometries for M_{24}.

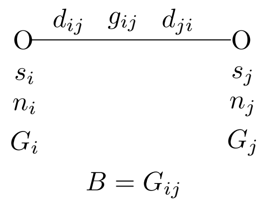
2 Definitions and Notation

We begin by reviewing geometries and some standard notations. A *geometry* is a triple $(\Gamma, I, \star)$ where Γ is a set, I an index set and $\star$ a symmetric incidence relation on Γ which satisfy

(i) $\Gamma = \dot{\bigcup}_{i \in I} \Gamma_i$; and
(ii) if $x \in \Gamma_i$, $y \in \Gamma_j (i, j \in I)$ and $x \star y$, then $i \neq j$.

The elements of Γ_i are called *objects of type* i, and $|I|$ is the *rank* of the geometry Γ (as is usual we use Γ is place of the triple $(\Gamma, I, \star)$). A *flag* F of Γ is a subset of Γ in which every two element of F are incident. The rank of F is $|F|$, the corank of F is $|I \backslash F|$ and the type of F is $\{i \in I | F \cap \Gamma_i \neq \emptyset\}$. A *chamber* of Γ is a flag of type I. All geometries we consider are assumed to contain at least one flag of rank $|I|$. The automorphism group of Γ, $Aut\Gamma$, consists of all permutations of Γ which preserve the sets Γ_i and the incidence relation $\star$. Let G be a subgroup of $Aut\Gamma$. We call Γ a *flag transitive geometry* for G if for any two flags F_1 and F_2 of Γ having the same type, there exists $g \in G$ such that $F_1{}^g = F_2$. For $\Delta \subseteq \Gamma$, the residue of Δ, denoted Γ_Δ, is defined to be $\{x \in \Gamma | x \star y$ for all $y \in \Delta\}$. A geometry Γ is called *residually connected* if for all flags F of Γ of corank at least 2 the incidence graph of Γ_F is connected. We call Γ firm provided that every flag of rank $|I| - 1$ is contained in at least two chambers. The diagram of a firm, residually connected, flag-transitive geometry Γ is a complete graph K, whose vertices are the elements of the set of type I of Γ, provided with some additional structure which is further described as follows. To each vertex $i \in I$, we attach the order s_i which is $|\Gamma_F| - 1$ where F is any flag of type $I \backslash \{i\}$, and the *number* n_i of *varieties* of *type* i, which is the index of G_i in G, and the subgroup G_i. To every edge $\{i, j\}$ of K, we associate three positive integers d_{ij}, g_{ij} and d_{ji} where g_{ij} (the gonality) is equal to half the girth of the incidence graph of a residue Γ_F of type $\{i, j\}$, and d_{ij} (resp. d_{ji}), the $i-diameter$ (resp. $j-diameter$) is the greatest distance from some

fixed $i-$ element (resp. $j-$ element) to any other element in Γ_F.
On a picture of the diagram, this structure will often be depicted as follows.



Now suppose that Γ is a flag transitive geometry for the group G. As it is well-known we may view Γ in terms of certain cosets of G. This is the approach we shall follow here. For each $i \in I$ choose an $x_i \in \Gamma_i$ and set $G_i = Stab_G(x_i)$. Let $\mathcal{F}=\{G_i : i \in I\}$. We now define a geometry $\Gamma(G,\mathcal{F})$ where the objects of type i in $\Gamma(G,\mathcal{F})$ are the right cosets of G_i in G and for $G_i x$ and $G_j y$ $(x, y \in G, i, j \in I)$ $G_i x \star G_j y$ whenever $G_i x \cap G_j y \neq \emptyset$. Also by letting G to act upon $\Gamma(G,\mathcal{F})$ by right multiplication we see that $\Gamma(G,\mathcal{F})$ is a flag transitive geometry for G. Moreover Γ and $\Gamma(G,\mathcal{F})$ are isomorphic geometries for G. So we shall be studying geometries of the form $\Gamma(G,\mathcal{F})$, where $G \cong M_{24}$ and G_i is a maximal subgroup of G for all $i \in I$. For further information about this subsection, see [5].

For the remainder of this paper, G will denote M_{24}, the Mathieu Group of degree 24. Also Ω will denote a 24 element set possessing the Steiner system $S(24, 8, 5)$ as described by Curtis's MOG [9]. We will follow the notation of [9].

So $\Omega =$

O_1	O_2	O_3

$=$

∞	14	17	11	22	19
0	8	4	13	1	9
3	20	16	7	12	5
15	18	10	2	21	6

, where O_1, O_2 and O_3 are the heavy bricks of the MOG. Here M_{24} is the Mathieu group of degree 24 which leaves invariant the Steiner system $S(24, 8, 5)$ on Ω.

An octad of Ω is just an 8-element block of the Steiner system and a subset of Ω is called a dodecad if it is the symmetric difference of two octads of Ω which intersect in a set of size two. Corresponding to each 4 points of Ω there is a partition of the 24 points into 6 tetrads with the property that the union of any two tetrads is an octad, this configuration will be called a *sextet.* Let us call a set of 3 disjoint octads a *trio.* We introduce a further maximal subgroup known as the octern group O^n. O^n may be defined as the centralizer in M_{24} of a certain element of order 3 in $S_{24} \setminus M_{24}$. The following sets will appear when we describe geometries for G.

(i) $\mathcal{D}$=$\{X \subseteq \Omega||X| = 2\}$ (duads of Ω).
(ii) $\mathcal{T}$=$\{X \subseteq \Omega||X| = 3\}$ (triads of Ω).
(iii) $\mathcal{S}$=$\{X_i \subseteq \Omega||X_i| = 4$ (for each $i \in I$), $X_i \cup X_j$ is an octad $(i \neq j)$ and $\Omega = \dot{\bigcup}_{i\in I} X_i$, $i \in I = \{1...6\}\}$ (sextets of Ω).
(iv) $\mathcal{O}$=$\{X \subseteq \Omega|X$ is an octad of $\Omega\}$.
(v) $\mathcal{D}_o$=$\{X \subseteq \Omega|X$ is a dodecad of $\Omega\}$.
(vi) $\mathcal{T}_o$=$\{Y \subseteq \Omega|Y := X_1 \cup X_2 \cup X_3$ for each X_i is an octad of Ω, $i \in I = \{1...3\}\}$ and (trios of Ω).

From the [8], the conjugacy classes of the maximal subgroups of G are as follows:

Order	*Index*	M_i	*Description*
10200960	24	$M_1 \cong M_{23}$	$M_1 = Stab_G\{a\}$, $a \in \Omega$
887040	276	$M_2 \cong M_{22} : 2$	$M_2 = Stab_G\{X\}$, $X \in \mathcal{D}$
120960	2024	$M_3 \cong L_3(4) : S_3$	$M_3 = Stab_G\{X\}$, $X \in \mathcal{T}$
138240	1771	$M_4 \cong 2^6 : 3.S_6$	$M_4 = Stab_G\{X\}$, $X \in \mathcal{S}$
322560	759	$M_5 \cong 2^4 : A_8$	$M_5 = Stab_G\{X\}$, $X \in \mathcal{O}$
190080	1288	$M_6 \cong M_{12} : 2$	$M_6 = Stab_G\{X\}$, $X \in \mathcal{D}_o$
64512	3795	$M_7 \cong 2^6 : L_3(2) : S_3$	$M_7 = Stab_G\{X\}$, $X \in \mathcal{T}_o$
6072	40320	$M_8 \cong L_2(23)$	Projective group
168	1457280	$M_9 \cong L_2(7)$	Octern group

For $i \in \{1, ..., 9\}$, we let $\mathfrak{M}_i$ denote the conjugacy class of M_i, M_i as given in the previous table. We also set $\mathfrak{M}$=$\bigcup_{i=1}^{9} \mathfrak{M}_i$; so $\mathfrak{M}$ consist of all maximal subgroups of G. In [9], we can find further information about projective group and octern group. Also put $\mathfrak{X} = \Omega \cup \mathcal{D} \cup \mathcal{T} \cup \mathcal{S} \cup \mathcal{O} \cup \mathcal{D}_o \cup \mathcal{T}_o$.

Suppose G_1 and G_2 are maximal subgroups of G with $G_1 \neq G_2$. Set $G_{12} = G_1 \cap G_2$. We use $\mathfrak{M}_{ij}(t)$ to describe $\{G_1, G_2, G_1 \cap G_2\}$ according to the following scheme: $G_1 \in \mathfrak{M}_i$, $G_2 \in \mathfrak{M}_j$ (and so $G_1 = Stab_G(X_1)$ and $G_2 = Stab_G(X_2)$ for some appropriate subsets X_1 and X_2 of Ω in $\mathfrak{X}$). Indeed, $\mathfrak{M}_{23}(1)$ means the first case of the intersection of duad and triad, $\mathfrak{M}_{23}(2)$ means the second case of the intersection of duad and triad and $\mathfrak{M}_{23}(3)$ means the third case of the intersection of duad and triad, using the same kind of idea we can define the remaining geometries. In this paper, N denotes the number of geometries.

Below we give certain subsets of Ω which will be encountered frequently in our list.

$S_1 =$

×	△	•	□	+	−
×	△	•	□	+	−
×	△	•	□	+	−
×	△	•	□	+	−

, $D_1 =$

		×	×	×	
×	×				×
×	×				×
×	×				×

, $O_2 =$

		•	•		
		•	•		
		•	•		
		•	•		

where $S_1 \in \mathcal{S}$, $D_1 \in \mathcal{D}_o$ and $O_2 \in \mathcal{O}$. Our notation is as in the [8].

3 Rank 2 geometries of M_{24}

In this section, we give the list of rank 2 primitive geometries for M_{24} that are firm, residually connected and flag transitive.

Γ	G_{12}	$\mid G_{12} \mid$	N	Γ	G_{12}	$\mid G_{12} \mid$	N
$\mathfrak{M}_{11}(1)$	M_{22}	443520	1	$\mathfrak{M}_{12}(1)$	$L_3(4):2$	40320	1
$\mathfrak{M}_{12}(2)$	M_{22}	443520	1	$\mathfrak{M}_{13}(1)$	$2^4:(3\times A_5):2$	5760	1
$\mathfrak{M}_{13}(2)$	$L_3(4):2$	40320	1	$\mathfrak{M}_{14}(1)$	$2^4:(3\times A_5):2$	5760	1
$\mathfrak{M}_{15}(1)$	A_8	20160	1	$\mathfrak{M}_{15}(2)$	$2^4:A_7$	40320	1
$\mathfrak{M}_{16}(1)$	M_{11}	7920	1	$\mathfrak{M}_{17}(1)$	$2^4:L_3(2)$	2688	1
$\mathfrak{M}_{18}(1)$	$23:11$	253	1	$\mathfrak{M}_{19}(1)$	C_7	7	1
$\mathfrak{M}_{22}(1)$	$2^4:S_5$	3840	1	$\mathfrak{M}_{22}(2)$	$L_3(4)$	20160	1
$\mathfrak{M}_{23}(1)$	$2^4:3^2:2^2$	576	1	$\mathfrak{M}_{23}(2)$	$2^4:S_5$	1920	1
$\mathfrak{M}_{23}(3)$	$L_3(4):2$	40320	1	$\mathfrak{M}_{24}(1)$	$2^4:3^2:2^2$	576	1
$\mathfrak{M}_{24}(2)$	$2^5:S_5$	3840	1	$\mathfrak{M}_{25}(1)$	A_7	2520	1
$\mathfrak{M}_{25}(2)$	$2^4:L_3(2)$	2688	1	$\mathfrak{M}_{25}(3)$	$2^4:S_6$	11520	1
$\mathfrak{M}_{26}(1)$	$L_2(11):2$	1320	1	$\mathfrak{M}_{26}(2)$	$A_6:2^2$	1440	1
$\mathfrak{M}_{27}(1)$	$L_3(2)\times 2$	336	1	$\mathfrak{M}_{27}(2)$	$2^4:(2\times S_4)$	768	1
$\mathfrak{M}_{28}(1)$	D_{22}	22	1	$\mathfrak{M}_{29}(1)$	C_2	2	3
$\mathfrak{M}_{29}(2)$	C_7	7	1				
$\mathfrak{M}_{33}(1)$	$3^2:D_{12}$	108	1	$\mathfrak{M}_{33}(2)$	$2^4:D_{12}$	192	1
$\mathfrak{M}_{33}(3)$	$2^4:(3^2:2^2)$	576	1	$\mathfrak{M}_{33}(4)$	$2^4:S_5$	1920	1
$\mathfrak{M}_{34}(1)$	$3^2:D_{12}$	108	1	$\mathfrak{M}_{34}(2)$	$2^4:D_{12}$	192	1
$\mathfrak{M}_{34}(3)$	$2^4:(3:S_5)$	5760	1	$\mathfrak{M}_{35}(1)$	$2\times L_3(2)$	336	1
$\mathfrak{M}_{35}(2)$	$2^4:3^2:2^2$	576	1	$\mathfrak{M}_{35}(3)$	S_6	720	1
$\mathfrak{M}_{35}(4)$	$2^4:(3:S_5)$	5760	1	$\mathfrak{M}_{36}(1)$	S_5	120	1
$\mathfrak{M}_{36}(2)$	$M_9:S_3$	432	1	$\mathfrak{M}_{37}(1)$	$2^3:S_3$	48	1
$\mathfrak{M}_{37}(2)$	$F_{21}\times S_3$	126	1	$\mathfrak{M}_{37}(3)$	$2^4:S_4$	384	1
$\mathfrak{M}_{38}(1)$	C_3	3	1	$\mathfrak{M}_{39}(1)$	1	1	11
$\mathfrak{M}_{39}(2)$	C_3	3	3	$\mathfrak{M}_{39}(3)$	F_{21}	21	1
$\mathfrak{M}_{44}(1)$	$2^4:S_3$	96	1	$\mathfrak{M}_{44}(2)$	$S_4\times S_4$	576	1
$\mathfrak{M}_{44}(3)$	$2^6:(C_2\times D_{12})$	1536	1	$\mathfrak{M}_{45}(1)$	$A_5:S_3$	360	1
$\mathfrak{M}_{45}(2)$	$2^4:S_4$	384	1	$\mathfrak{M}_{45}(3)$	$2^4:(2^4:(S_3\times S_3))$	9216	1
$\mathfrak{M}_{46}(1)$	$3:(3^2:D_8)$	216	1	$\mathfrak{M}_{46}(2)$	$2^3:(2\times S_4)$	384	1
$\mathfrak{M}_{46}(3)$	$A_5:D_8$	480	1	$\mathfrak{M}_{47}(1)$	$2\times S_4$	48	1
$\mathfrak{M}_{47}(2)$	$2^3:S_4$	192	1	$\mathfrak{M}_{47}(3)$	$(2^3:2^2):S_4$	768	1
$\mathfrak{M}_{47}(4)$	$2^6:(S_4\times S_3)$	9216	1	$\mathfrak{M}_{48}(1)$	D_8	8	1
$\mathfrak{M}_{48}(2)$	D_{12}	12	1	$\mathfrak{M}_{48}(3)$	D_{24}	24	1
$\mathfrak{M}_{48}(4)$	S_4	24	1	$\mathfrak{M}_{49}(1)$	1	1	6
$\mathfrak{M}_{49}(2)$	C_2	2	5	$\mathfrak{M}_{49}(3)$	C_3	3	2
$\mathfrak{M}_{49}(4)$	C_4	4	2	$\mathfrak{M}_{49}(5)$	2^2	4	1
$\mathfrak{M}_{49}(6)$	S_3	6	2	$\mathfrak{M}_{49}(7)$	D_8	8	2
$\mathfrak{M}_{49}(8)$	S_4	24	1				
$\mathfrak{M}_{55}(1)$	S_6	720	1	$\mathfrak{M}_{55}(2)$	$2^6:(3^2:2)$	1152	1

$\mathfrak{M}_{55}(3)$	$2^6 : L_3(2)$	10752	1	$\mathfrak{M}_{56}(1)$	$2^3 : (S_4 : 2)$	384	1
$\mathfrak{M}_{56}(2)$	S_6	720	1	$\mathfrak{M}_{57}(1)$	$2^4 : S_3$	96	1
$\mathfrak{M}_{57}(2)$	$2^6 : D_{12}$	768	1	$\mathfrak{M}_{58}(1)$	D_8	8	1
$\mathfrak{M}_{59}(1)$	1	1	1	$\mathfrak{M}_{59}(2)$	C_2	2	1
$\mathfrak{M}_{59}(3)$	C_4	4	1	$\mathfrak{M}_{59}(4)$	C_7	7	1
$\mathfrak{M}_{59}(5)$	D_8	8	1				
$\mathfrak{M}_{66}(1)$	$A_5 \times 2^2$	240	1	$\mathfrak{M}_{66}(2)$	$(2^3 : 2) : S_4$	384	1
$\mathfrak{M}_{67}(1)$	$2^3 : D_{12}$	96	1	$\mathfrak{M}_{67}(2)$	$S_3 : S_4$	144	1
$\mathfrak{M}_{67}(3)$	$2^4 : S_4$	384	1	$\mathfrak{M}_{68}(1)$	D_{12}	12	1
$\mathfrak{M}_{68}(2)$	D_{22}	22	1	$\mathfrak{M}_{68}(3)$	D_{24}	24	1
$\mathfrak{M}_{68}(4)$	S_4	24	1	$\mathfrak{M}_{69}(1)$	1	1	3
$\mathfrak{M}_{69}(2)$	C_2	2	6	$\mathfrak{M}_{69}(3)$	C_3	3	2
$\mathfrak{M}_{69}(4)$	C_4	4	1	$\mathfrak{M}_{69}(5)$	2^2	4	1
$\mathfrak{M}_{69}(6)$	S_3	6	2	$\mathfrak{M}_{69}(7)$	D_8	8	1
$\mathfrak{M}_{69}(8)$	S_4	24	1				
$\mathfrak{M}_{77}(1)$	S_4	24	1	$\mathfrak{M}_{77}(2)$	$2^3 : 2^3$	64	1
$\mathfrak{M}_{77}(3)$	$2^6 : (3^2 : 2)$	1152	1	$\mathfrak{M}_{77}(4)$	$2^6 : (C_2 \times D_{12})$	1536	1
$\mathfrak{M}_{78}(1)$	2^2	4	1	$\mathfrak{M}_{78}(2)$	S_3	6	1
$\mathfrak{M}_{78}(3)$	D_{12}	12	1	$\mathfrak{M}_{78}(4)$	S_4	24	2
$\mathfrak{M}_{78}(5)$	D_{24}	24	1	$\mathfrak{M}_{79}(1)$	1	1	15
$\mathfrak{M}_{79}(2)$	C_2	2	9	$\mathfrak{M}_{79}(3)$	C_3	3	4
$\mathfrak{M}_{79}(4)$	C_4	4	2	$\mathfrak{M}_{79}(5)$	2^2	4	2
$\mathfrak{M}_{79}(6)$	S_3	6	3	$\mathfrak{M}_{79}(7)$	D_8	8	1
$\mathfrak{M}_{79}(8)$	F_{21}	21	1	$\mathfrak{M}_{79}(9)$	S_4	24	2
$\mathfrak{M}_{88}(1)$	C_2	2	9	$\mathfrak{M}_{88}(2)$	2^2	4	3
$\mathfrak{M}_{88}(3)$	S_3	6	5	$\mathfrak{M}_{88}(4)$	D_8	8	1
$\mathfrak{M}_{88}(5)$	A_4	12	2	$\mathfrak{M}_{88}(6)$	D_{22}	22	4
$\mathfrak{M}_{88}(7)$	D_{24}	24	1	$\mathfrak{M}_{88}(8)$	S_4	24	1
$\mathfrak{M}_{89}(1)$	1	1	200	$\mathfrak{M}_{89}(2)$	2	2	65
$\mathfrak{M}_{89}(3)$	C_3	3	15	$\mathfrak{M}_{89}(4)$	C_4	4	2
$\mathfrak{M}_{89}(5)$	2^2	4	3	$\mathfrak{M}_{89}(6)$	S_3	6	5
$\mathfrak{M}_{89}(7)$	D_8	8	1	$\mathfrak{M}_{89}(8)$	A_4	12	2
$\mathfrak{M}_{89}(9)$	S_4	24	3				
$\mathfrak{M}_{99}(1)$	1	1	?	$\mathfrak{M}_{99}(2)$	C_2	2	?
$\mathfrak{M}_{99}(3)$	C_3	3	?	$\mathfrak{M}_{99}(4)$	C_4	4	?
$\mathfrak{M}_{99}(5)$	2^2	4	?	$\mathfrak{M}_{99}(6)$	S_3	6	?
$\mathfrak{M}_{99}(7)$	C_7	7	?	$\mathfrak{M}_{99}(8)$	F_{21}	21	?

4 Diagrams for Rank 2 Geometries

Using a series of MAGMA [1] programs, we can only calculate the diagram of the following geometries. The MAGMA [1] programs perform the calculation of the following list of geometries.

Γ	*Diagrams*	Γ	*Diagrams*	Γ	*Diagrams*
$\mathfrak{M}_{11}(1)$	4.1	$\mathfrak{M}_{12}(2)$	4.2	$\mathfrak{M}_{12}(1)$	4.3
$\mathfrak{M}_{13}(2)$	4.4	$\mathfrak{M}_{13}(1)$	4.5	$\mathfrak{M}_{14}(1)$	4.6
$\mathfrak{M}_{15}(1)$	4.7	$\mathfrak{M}_{15}(2)$	4.8	$\mathfrak{M}_{16}(1)$	4.9
$\mathfrak{M}_{17}(1)$	4.10	$\mathfrak{M}_{22}(1)$	4.11	$\mathfrak{M}_{22}(2)$	4.12
$\mathfrak{M}_{23}(3)$	4.13	$\mathfrak{M}_{23}(2)$	4.14	$\mathfrak{M}_{23}(1)$	4.15
$\mathfrak{M}_{24}(1)$	4.16	$\mathfrak{M}_{24}(2)$	4.17	$\mathfrak{M}_{25}(3)$	4.18
$\mathfrak{M}_{25}(1)$	4.19	$\mathfrak{M}_{25}(2)$	4.20	$\mathfrak{M}_{26}(2)$	4.21
$\mathfrak{M}_{26}(1)$	4.22	$\mathfrak{M}_{27}(2)$	4.23	$\mathfrak{M}_{27}(1)$	4.24
$\mathfrak{M}_{33}(1)$	4.25	$\mathfrak{M}_{33}(4)$	4.26	$\mathfrak{M}_{33}(2)$	4.27
$\mathfrak{M}_{33}(3)$	4.28	$\mathfrak{M}_{34}(1)$	4.29	$\mathfrak{M}_{34}(3)$	4.30
$\mathfrak{M}_{34}(2)$	4.31	$\mathfrak{M}_{35}(3)$	4.32	$\mathfrak{M}_{35}(4)$	4.33
$\mathfrak{M}_{35}(1)$	4.34	$\mathfrak{M}_{35}(2)$	4.35	$\mathfrak{M}_{36}(1)$	4.36
$\mathfrak{M}_{36}(2)$	4.37	$\mathfrak{M}_{37}(3)$	4.38	$\mathfrak{M}_{37}(1)$	4.39
$\mathfrak{M}_{37}(2)$	4.40	$\mathfrak{M}_{44}(3)$	4.41	$\mathfrak{M}_{44}(1)$	4.42
$\mathfrak{M}_{44}(2)$	4.43	$\mathfrak{M}_{45}(3)$	4.44	$\mathfrak{M}_{45}(2)$	4.45
$\mathfrak{M}_{45}(1)$	4.46	$\mathfrak{M}_{46}(2)$	4.47	$\mathfrak{M}_{46}(1)$	4.48
$\mathfrak{M}_{46}(3)$	4.49	$\mathfrak{M}_{47}(4)$	4.50	$\mathfrak{M}_{47}(1)$	4.51
$\mathfrak{M}_{47}(2)$	4.52	$\mathfrak{M}_{47}(3)$	4.53	$\mathfrak{M}_{55}(3)$	4.54
$\mathfrak{M}_{55}(2)$	4.55	$\mathfrak{M}_{55}(1)$	4.56	$\mathfrak{M}_{56}(1)$	4.57
$\mathfrak{M}_{56}(2)$	4.58	$\mathfrak{M}_{57}(1)$	4.59	$\mathfrak{M}_{57}(2)$	4.60
$\mathfrak{M}_{66}(1)$	4.61	$\mathfrak{M}_{66}(2)$	4.62	$\mathfrak{M}_{67}(2)$	4.63
$\mathfrak{M}_{67}(1)$	4.64	$\mathfrak{M}_{67}(3)$	4.65	$\mathfrak{M}_{77}(1)$	4.66
$\mathfrak{M}_{77}(2)$	4.67				

4.1 O—3—2—3—O

22 22

24 24

M_{23} M_{23}

$B = M_{22}$

4.2 O—4—3—3—O

22 1

276 24

$M_{22}:2$ M_{23}

$B = M_{22}$

Due to Buekenhout

(see [4], Truncation of geometry 45).

4.3 O—3 2 3—O

252	21
276	24
$M_{22}:2$	M_{23}

$B = L_3(4):2$

4.4 O—3 2 4—O

2	252
24	2024
M_{23}	$L_3(4):S_3$

$B = L_3(4):2$

4.5 O—3 2 3—O

20	1770
24	2024
M_{23}	$L_3(4):S_3$

$B = 2^4:(3 \times A_5):2$

4.6 O—2 2 2—O

1770	23
1771	24
$2^6:3.S_6$	M_{23}

$B = 2^4:((3 \times A_5):2)$

4.7 O—3 2 3—O

46	15
759	24
$2^4:A_8$	M_{23}

$B = A_8$

4.8 O—4 2 3—O

252	7
759	24
$2^4:A_8$	M_{23}

$B = 2^4:A_7$

4.9 O—2 2 2—O

23	1287
24	1288
M_{23}	$M_{12}:2$

$B = M_{11}$

4.10 O—2 2 2—O

3794	23
3795	24
$2^6:L_3(2):S_3$	M_{23}

$B = 2^4:L_3(2)$

4.11 O—3 2 3—O

230	230
276	276
$M_{22}:2$	$M_{22}:2$

$B = 2^4:S_5$

4.12 O—3 2 3—O

43	43
276	276
$M_{22}:2$	$M_{22}:2$

$B = L_3(4)$

4.13 O —5 3 6— O

2	21
276	2024
$M_{22}:2$	$L_3(4):S_3$

$B = L_3(4):2$

Due to Buekenhout(see [4], Truncation of geometry 45).

4.14 O —3 2 3— O

62	461
276	2024
$L_3(4):S_3$	$M_{22}:2$

$B = 2^4:S_5$

4.15 O —3 2 3— O

209	1539
276	2024
$M_{22}:2$	$L_3(4):S_3$

$B = 2^4:3^2:2^2$

4.16 O —3 2 3— O

239	1539
276	1771
$M_{22}:2$	$2^6:3.S_6$

$B = 2^4:3^2:2^2$

4.17 O —3 2 3— O

35	230
276	1771
$M_{22}:2$	$2^6:3.S_6$

$B = 2^5:S_5$

4.18 O —4 2 3— O

76	27
759	276
$2^4:A_8$	$M_{22}:2$

$B = 2^4:S_6$

4.19 O —3 2 3— O

351	127
759	276
$2^4:A_8$	$M_{22}:2$

$B = A_7$

4.20 O —3 2 3— O

329	119
759	276
$2^4:A_8$	$M_{22}:2$

$B = 2^4:L_3(2)$

4.21 O —3 2 3— O

131	615
276	1288
$M_{22}:2$	$M_{12}:2$

$B = A_6:2^2$

4.22 O —3 2 3— O

143	671
276	1288
$M_{22}:2$	$M_{12}:2$

$B = L_2(11):2$

4.23 O——3 2 3——O

1154	83
3795	276
$2^6 : L_3(2) : S_3$	$M_{22} : 2$

$B = 2^4 : (2 \times S_4)$

4.24 O——3 2 3——O

2639	191
3795	276
$2^6 : L_3(2) : S_3$	$M_{22} : 2$

$B = L_3(2) \times 2$

4.25 O——3 2 3——O

1119	1119
2024	2024
$L_3(4) : S_3$	$L_3(4) : S_3$

$B = 3^2 : D_{12}$

4.26 O——4 2 4——O

62	62
2024	2024
$L_3(4) : S_3$	$L_3(4) : S_3$

$B = 2^4 : S_5$

4.27 O——3 2 3——O

629	629
2024	2024
$L_3(4) : S_3$	$L_3(4) : S_3$

$B = 2^4 : D_{12}$

4.28 O——3 2 3——O

209	209
2024	2024
$L_3(4) : S_3$	$L_3(4) : S_3$

$B = 2^4 : (3^2 : 2^2)$

4.29 O——3 2 3——O

1119	1279
1771	2024
$2^6 : 3.S_6$	$L_3(4) : S_3$

$B = 3^2 : D_{12}$

4.30 O——4 2 4——O

20	23
1771	2024
$2^6 : 3.S_6$	$L_3(4) : S_3$

$B = 2^4 : (3 : S_5)$

4.31 O——3 2 3——O

629	719
1771	2024
$2^6 : 3.S_6$	$L_3(4) : S_3$

$B = 2^4 : D_{12}$

4.32 O——4 2 3——O

167	447
759	2024
$2^4 : A_8$	$L_3(4) : S_3$

$B = S_6$

4.33 O —4— 2 —4— O

20	55
759	2024
$2^4 : A_8$	$L_3(4) : S_3$

$B = 2^4 : (3 : S_5)$

4.34 O —3— 2 —3— O

359	959
759	2024
$2^4 : A_8$	$L_3(4) : S_3$

$B = 2 \times L_3(2)$

4.35 O —3— 2 —3— O

209	559
759	2024
$2^4 : A_8$	$L_3(4) : S_3$

$B = 2^4 : 3^2 : 2^2$

4.36 O —3— 2 —3— O

1583	1007
2024	1288
$L_3(4) : S_3$	$M_{12} : 2$

$B = S_5$

4.37 O —3— 2 —3— O

439	279
2024	1288
$L_3(4) : S_3$	$M_{12} : 2$

$B = M_9 : S_3$

4.38 O —3— 2 —3— O

314	167
3795	2024
$2^6 : L_3(2) : S_3$	$L_3(4) : S_3$

$B = 2^4 : S_4$

4.39 O —3— 2 —3— O

2519	1343
3795	2024
$2^6 : L_3(2) : S_3$	$L_3(4) : S_3$

$B = 2^3 : S_3$

4.40 O —3— 2 —3— O

959	511
3795	2024
$2^6 : L_3(2) : S_3$	$L_3(4) : S_3$

$B = F_{21} \times S_3$

4.41 O —3— 2 —3— O

89	89
1771	1771
$2^6 : 3.S_6$	$2^6 : 3.S_6$

$B = 2^6 : (C_2 \times D_{12})$

4.42 O —3— 2 —3— O

1439	1439
1771	1771
$2^6 : 3.S_6$	$2^6 : 3.S_6$

$B = 2^4 : S_3$

4.43 O—3—2—3—O

239	239
1771	1771
$2^6 : 3.S_6$	$2^6 : 3.S_6$

$B = S_4 \times S_4$

4.44 O—4—2—4—O

14	34
759	1771
$2^4 : A_8$	$2^6 : 3.S_6$

$B = 2^4 : (2^4 : (S_3 \times S_3))$

4.45 O—3—2—3—O

359	839
759	1771
$2^4 : A_8$	$2^6 : 3.S_6$

$B = 2^4 : S_4$

4.46 O—3—2—3—O

383	895
759	1771
$2^4 : A_8$	$2^6 : 3.S_6$

$B = A_5 : S_3$

4.47 O—3—2—3—O

494	359
1771	1288
$2^6 : 3.S_6$	$M_{12} : 2$

$B = 2^3 : (2 \times S_4)$

4.48 O—3—2—3—O

879	639
1771	1288
$2^6 : 3.S_6$	$M_{12} : 2$

$B = 3 : (3^2 : D_8)$

4.49 O—3—2—3—O

395	287
1771	1288
$2^6 : 3.S_6$	$M_{12} : 2$

$B = A_5 : D_8$

4.50 O—5—3—5—O

14	6
3795	1771
$2^6 : L_3(2) : S_3$	$2^6 : 3.S_6$

$B = 2^6 : (S_4 \times S_3)$

4.51 O—3—2—3—O

2879	1343
3795	1771
$2^6 : L_3(2) : S_3$	$2^6 : 3.S_6$

$B = 2 \times S_4$

4.52 O—3—2—3—O

719	335
3795	1771
$2^6 : L_3(2) : S_3$	$2^6 : 3.S_6$

$B = 2^3 : S_4$

4.53 O —3—2—3— O

179	83
3795	1771
$2^6 : L_3(2) : S_3$	$2^6 : 3.S_6$

$B = (2^3 : 2^2) : S_4$

4.54 O —4—2—4— O

29	29
759	759
$2^4 : A_8$	$2^4 : A_8$

$B = 2^6 : L_3(2)$

4.55 O —3—2—3— O

279	279
759	759
$2^4 : A_8$	$2^4 : A_8$

$B = 2^6 : (3^2 : 2)$

4.56 O —3—2—3— O

447	447
759	759
$2^4 : A_8$	$2^4 : A_8$

$B = S_6$

4.57 O —3—2—3— O

494	839
759	1288
$2^4 : A_8$	$M_{12} : 2$

$B = 2^3 : (S_4 : 2)$

4.58 O —3—2—3— O

263	447
759	1288
$2^4 : A_8$	$M_{12} : 2$

$B = S_6$

4.59 O —3—2—3— O

671	3359
759	3795
$2^4 : A_8$	$2^6 : L_3(2) : S_3$

$B = 2^4 : S_3$

4.60 O —3—2—3— O

83	419
759	3795
$2^4 : A_8$	$2^6 : L_3(2) : S_3$

$B = 2^6 : D_{12}$

4.61 O —3—2—3— O

791	791
1288	1288
$M_{12} : 2$	$M_{12} : 2$

$B = A_5 \times 2^2$

4.62 O —3—2—3— O

494	494
1288	1288
$M_{12} : 2$	$M_{12} : 2$

$B = (2^3 : 2) : S_4$

4.63 O—3—2—3—O

1319	447
3795	1288
$2^6 : L_3(2) : S_3$	$M_{12} : 2$

$B = S_3 : S_4$

4.64 O—3—2—3—O

1979	671
3795	1288
$2^6 : L_3(2) : S_3$	$M_{12} : 2$

$B = 2^3 : D_{12}$

4.65 O—3—2—3—O

494	167
3795	1288
$2^6 : L_3(2) : S_3$	$M_{12} : 2$

$B = 2^4 : S_4$

4.66 O—3—2—3—O

2687	2687
3795	3795
$2^6 : L_3(2) : S_3$	$2^6 : L_3(2) : S_3$

$B = S_4$

4.67 O—3—2—3—O

1007	1007
3795	3795
$2^6 : L_3(2) : S_3$	$2^6 : L_3(2) : S_3$

$B = 2^3 : 2^3$

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