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GENERALIZATION OF JOHNSON'S THEOREM FOR WEIGHTED SEMIGROUP ALGEBRAS

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Abstract

This paper attempts to generalize Johnson's techniques to apply them to establish a bijective correspondence between S-derivations and continuous derivations on weighted semigroup algebra $M_a(S,\omega)$, where S is a locally compact foundation semigroup with identity e, and ω is a weight function on S.

Introduction

Amenability theory of Banach algebras was originated by B. E. Johnson in [J], where he proved that a locally compact group G is amenable in the classical sense if and only if the group algebra $L^1(G)$ with the convolution multiplication is amenable. He has established a bijective correspondence between continuous derivations on $L^1(G)$ and G-derivations, and further, he proved that inner derivations correspond to inner G-derivations, and $H^1(L^1(G),E^*) = \{0\}$ if and only if every G-derivations into E^* is inner, where E is a neo-unital Banach $L^1(G)$-bimodule (see [J] and [D]).

We are interested to generalize Johnson's theorem for weighted semigroup algebras. To do this the preliminaries and notations are given in section 1. In section 2, we investigate the strong operator topology (so) with respect to $M_a(S,\omega)$ on $M(S,\omega)$ and we also show that $\ell^1(S,\omega)$ is (so) dense in $M(S,\omega)$. Section 3 is devoted to establish a correspondence between continuous derivations from $M_a(S,\omega)$ and S-derivations into dual modules. In section 4, some applications are indicated.

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1 Preliminaries

This section is preliminary in character. For a Banach algebra $\mathfrak{A}$, an $\mathfrak{A}$-bimodule will always refer to a *Banach* $\mathfrak{A}$*-bimodule* X, that is a Banach space which is algebraically an $\mathfrak{A}$-bimodule, and for which there is a constant $C_{\mathfrak{A},X} > 0$ such that

$$\|a.x\|, \|x.a\| \leq C_{\mathfrak{A},X}\|a\|\|x\| \quad (a \in \mathfrak{A}, x \in X).$$

Let $\mathfrak{A}$ be a Banach algebra and X be a Banach $\mathfrak{A}$-bimodule, then X^*, the conjugate space of X, is a Banach $\mathfrak{A}$-bimodule with the natural module multiplications defined by

$$\langle ax^*, x\rangle = \langle x^*, xa\rangle, \qquad \langle x^*a, x\rangle = \langle x^*, ax\rangle.$$

for $a \in \mathfrak{A}$, $x \in X$ and $x^* \in X^*$. Recall that an $\mathfrak{A}$-bimodule X is *neo-unital* if

$$X = \mathfrak{A}.X.\mathfrak{A} = \{a.x.b : a, b \in \mathfrak{A}, x \in X\};$$

essential if

$$X = \overline{\mathfrak{A}X\mathfrak{A}} = cls\{a.x.b : a, b \in \mathfrak{A}, x \in X\}.$$

Similarly, one defines neo-unital and essential left and right Banach modules. Clearly neo-unital implies essential, and an important consequence of the Cohen-Hewitt factorization theorem is that the converse holds when $\mathfrak{A}$ has a bounded approximate identity. A *derivation* from $\mathfrak{A}$ into an $\mathfrak{A}$-bimodule X is a bounded linear map D, such that $D(ab) = D(a).b + a.D(b)$, for all $(a, b \in \mathfrak{A})$. If $x \in X$, then $ad_x : \mathfrak{A} \longrightarrow X$ defined by

$$ad_x(a) = a.x - x.a \qquad (a \in \mathfrak{A}),$$

is a derivation. Such derivations are called *inner*. Denote by $\mathcal{Z}^1(\mathfrak{A}, X)$ the space of all continuous derivations from $\mathfrak{A}$ into X and by $\mathcal{N}^1(\mathfrak{A}, X)$ the space of all inner derivations from $\mathfrak{A}$ into X. Then $\mathcal{N}^1(\mathfrak{A}, X)$ is a subspace of $\mathcal{Z}^1(\mathfrak{A}, X)$. The quotient space $\mathcal{H}^1(\mathfrak{A}, X) = \mathcal{Z}^1(\mathfrak{A}, X)/\mathcal{N}^1(\mathfrak{A}, X)$ is called *the first cohomology group* with coefficients in X. All the amenability theories are related to the question of whether $\mathcal{H}^1(\mathfrak{A}, X) = \{0\}$ for certain X. A Banach algebra A is amenable if, for every $\mathfrak{A}$-bimodule X, every derivation $D : \mathfrak{A} \longrightarrow X^*$ is inner, equivalently if $\mathcal{H}^1(\mathfrak{A}, X^*) = \{0\}$ for every Banach $\mathfrak{A}$-bimodule X. More information on this matter can be found in [D]. An *approximate identity* in $\mathfrak{A}$ for X is a net (e_α) in $\mathfrak{A}$ such that for each $x \in X$, $\lim_\alpha e_\alpha.x = \lim_\alpha x.e_\alpha = x$.

Throughout, S denotes a semigroup. A topological semigroup S is a locally compact, Hausdorff and jointly continuous semigroup. Let $CB(S)$ denote the set of all bounded continuous complex-valued functions on S. Recall that on a topological semigroup S the algebra $M_a(S)$ (or $\tilde{L}(S)$ as in [BB]) will denote the set of all measures $\mu \in M(S)$ (the Banach algebra of all bounded complex-valued measures on S with the total variation norm and convolution product) for which both the mappings: $x \mapsto |\mu| * \delta_x$ and $x \mapsto \delta_x * |\mu|$ (δ_x denotes the Dirac measure at x) from S into $M(S)$ are weakly continuous. A topological semigroup S is called a *foundation semigroup* if S coincides with the closure of $\bigcup\{supp(\mu) : \mu \in M_a(S)\}$. Note that if S is a foundation semigroup with identity then $M_a(S)$ has a bounded approximate identity [S1]. The space of all bounded complex-valued $M_a(S)$-measurable functions on S will be denoted by $L^\infty(S, M_a(S))$ [L1]. It is well-known that if S is a foundation semigroup with an identity, $L^\infty(S, M_a(S))$ may be identified with $M_a(S)^*$ [S2]. Note that if S is a discrete semigroup then S is a foundation semigroup with $L^\infty(S, M_a(S)) = \ell^\infty(S)$, and $M(S) = M_a(S) = \ell^1(S)$.

We shall assume that there is a *weight function* ω on S, i.e. a continuous function $\omega : S \longrightarrow (0, +\infty)$ such that

$$\omega(st) \leq \omega(s)\omega(t) \qquad (s, t \in S).$$

Furthermore, if S is unital with unit e, then we also insist that $\omega(e) = 1$. A function f from S into a normed space is said to be ω-bounded if there is $k > 0$ such that $\|f(s)\| \leq k\omega(s)$ for all $s \in S$.
Let X be a Banach space of measures or of equivalence classes of functions on a locally compact semigroup S, and let $\omega : S \longrightarrow (0, +\infty)$ be a continuous function. Define the Banach space

$$X(\omega) = \{f : \omega f \in X\},$$

where $\|f\|_{X(\omega)} = \|\omega f\|_X$ (see [DL] for more details). In particular, $M(S, \omega) = M(S)(\omega)$, $M_a(S, \omega) = M_a(S)(\omega)$ and

$$C_0(S, \omega^{-1}) = C_0(S)(1/\omega) \quad , \quad CB(S, \omega^{-1}) = CB(S)(1/\omega),$$

and also, $L^\infty(S; M_a(S, \omega)) = L^\infty(S; M_a(S))(1/\omega)$. When ω is a weight function, with the convolution multiplication of measures, $M(S, \omega)$ becomes a Banach algebra, having $M_a(S, \omega)$ as a closed two-sided L-ideal and $\ell^1(S, \omega)$ as a closed subalgebra [Re1] (see also [Re2]). Recall that, if S is a foundation semigroup with identity then $M_a(S, \omega)$ has a bounded approximate identity [L1], and also the mapping $f \mapsto \tau_f$; defined by

$$\tau_f(\mu) = \int_S f(x) d\mu(x) \qquad (\mu \in M_a(S, \omega)),$$

is an isometric isomorphism from $L^\infty(S; M_a(S,\omega))$ into $M_a(S,\omega)^*$ [L2]. By pairing

$$\langle \mu, \psi\rangle = \int_S \psi(x) d\mu(x) \quad (\mu \in M(S,\omega), \psi \in C_0(S,\omega^{-1})),$$

we have $(C_0(S,\omega^{-1})^* = M(S,\omega)$, and we can define the product of $\mu, \nu \in M(S,\omega)$ by

$$\int_S \psi(x) d(\mu * \nu)(x) = \int_S \int_S \psi(xy) d\mu(x) d\nu(y) \quad (\psi \in C_0(S,\omega^{-1}).$$

2 Strong operator topology with respect to $M_a(S,\omega)$ on $M(S,\omega)$

From now on, we suppose that S is a locally compact foundation semigroup with identity e, and ω is a weight function on S. In this section, we follow B. E. Johnson [J] to define *weighted Banach S-bimodule.*

Definition 2.1. *A left weighted Banach S-module E is a Banach space E which is a left S-module such that,*
(1) for all x in E the map $s \longmapsto s.x$ is continuous from S into E.
(2) There is $k > 0$ with $\| s.x \| \leq k\|x\|\omega(s)$ for all $s \in S$ and $x \in E$.
In a similar way we define right weighted Banach S-module and two-sided weighted Banach S-modules. From now on, we write weighted Banach S-bimodule instead of two-sided weighted Banach S-module.

On $M(S,\omega)$ we consider one topology other than the norm topology and the weak topology; it is the strong operator topology (we write it (so) for short) in which a net $(\mu_\alpha)_\alpha$ in $M(S,\omega)$ converges to $\mu \in M(S,\omega)$ in (so), if $\mu_\alpha * \nu \longrightarrow \mu * \nu$ and $\nu * \mu_\alpha \longrightarrow \nu * \mu$, for every $\nu \in M_a(S,\omega)$ in norm topology. We begin with an elementary result.

Lemma 2.2. $F \in (M(S,\omega), so)^*$ *if and only if there exist subsets $\{\nu_1, ..., \nu_n\}$, $\{\eta_1, ..., \eta_m\}$ in $M_a(S,\omega)$ and $\{\varphi_1, ..., \varphi_n\}$, $\{\psi_1, ..., \psi_m\}$ in $M_a(S,\omega)^*$, such that for each $\mu \in M(S,\omega)$;*

$$F(\mu) = \sum_{i=1}^{n} \varphi_i(\nu_i * \mu) + \sum_{j=1}^{m} \psi_j(\mu * \eta_j).$$

where

$$\Phi(\tau * \mu) = \int_S \Phi d(\tau * \mu).$$

Proof. The sufficient condition is evident. Let $F \in (M(S,\omega), so)^*$. According to the [C, Theorem IV 3.1], there is $\{\nu_1, ..., \nu_n\}$ of $M_a(S,\omega)$ which

$$|F(\mu)| \leq \sum_{i=1}^{n}(\| \nu_i * \mu\|_\omega + \|\mu * \nu_i\|_\omega).$$

We define

$$\Delta = \{(\nu_1 * \mu, ..., \nu_n * \mu, \mu * \nu_1, ..., \mu * \nu_n) : \mu \in M(S,\omega)\},$$

and $\gamma : \Delta \longrightarrow \mathbb{C}$ by $\gamma(T) = F(\mu)$. Clearly $\Delta \subseteq \bigoplus_{i=1}^{2n} M_a(S,\omega)$ and γ is well-defined and bounded. By Hahn-Banach theorem, there is a bounded functional Γ on $\bigoplus_{i=1}^{2n} M_a(S,\omega)$ such that $\Gamma \mid_\Delta = \gamma$.
Now for all $1 \leq i \leq n$ and $1 \leq j \leq 2$, we define $\Phi_{ij} \in M_a(S,\omega)^*$ by

$$\Phi_{ij}(\nu) = \Gamma(0, ..., 0, \underbrace{\nu}_{(j-1)n+i}, 0, ..., 0) \quad (\nu \in M_a(S,\omega)).$$

It follows that

$$\begin{aligned} F(\mu) &= \gamma(\nu_1 * \mu, ..., \nu_n * \mu, \mu * \nu_1, ..., \mu * \nu_n) \\ &= \Gamma(\nu_1 * \mu, ..., \nu_n * \mu, \mu * \nu_1, ..., \mu * \nu_n) \\ &= \sum_{i=1}^{n} \Phi_{i1}(\nu_i * \mu) + \sum_{i=1}^{n} \Phi_{i2}(\mu * \nu_i). \end{aligned}$$

for each $\mu \in M(S,\omega)$ which proves the lemma. □

We are thus led to the following important theorem

Theorem 2.3. *$\ell^1(S,\omega)$ is (so) dense in $M(S,\omega)$.*

Proof. Through $\mu \longrightarrow \hat{\mu} : M(S,\omega) \hookrightarrow CB(S,\omega^{-1})^*$ where

$$\hat{\mu}(f) = \int_S f d\mu \qquad (f \in CB(S,\omega^{-1})).$$

We equip $M(S,\omega)$ with the hereditary topology from the w^*-topology on $CB(S,\omega^{-1})^*$ and denote this topology by $(M(S,\omega), w^*)$. [RUD, Theorem 4.7] yields, $\overline{\ell^1(S,\omega)}^{w^*} = ({}^\perp(\ell^1(S,\omega)))^\perp$, where

$${}^\perp(\ell^1(S,\omega)) = \{\varphi \in CB(S,\omega^{-1})^* : \varphi(f) = 0, \quad \forall f \in \ell^1(S,\omega)\},$$

and hence $\overline{\ell^1(S,\omega)}^{w^*} \cap M(S,\omega) = M(S,\omega)$. Now if $\mu \in M(S,\omega)$ then there exists a net $(\mu_\alpha)_\alpha$ in $\ell^1(S,\omega)$ such that $\mu_\alpha \longrightarrow \mu$ in the w^*-topology. Let $\varphi \in M_a(S,\omega)^*$. We have,

$$\varphi(\mu) = \int_S \varphi d\mu \quad (\varphi \in M_a(S,\omega)^*, \mu \in M_a(S,\omega)).$$

Now for each $\nu \in M_a(S,\omega)$,

$$\begin{aligned} \varphi(\mu_\alpha * \nu) &= \int_S \varphi d\mu_\alpha * \nu = \int_S \int_S \varphi(xy) d\mu_\alpha(x) d\nu(y) \\ &= \int_S \int_S \varphi(xy) d\nu(y) d\mu_\alpha(x) \longrightarrow \int_S \int_S \varphi(xy) d\nu(y) d\mu(x) \\ &= \int_S \int_S \varphi(xy) d\mu(x) d\nu(y) = \int_S \varphi d\mu * \nu = \varphi(\mu * \nu) \end{aligned}$$

Lemma 2.2 now shows that $\mu \in \overline{\ell^1(S,\omega)}^{weak}$. Since $\ell^1(S,\omega)$ is convex, it follows that $\mu \in \overline{\ell^1(S,\omega)}^{(so)}$ [RUD, Theorem 3.12]. □

The following result may be proved in much the same way as [R, Theorem 2.1.7] with applying Theorem 2.3 above.

Corollary 2.4. *Suppose that E is a neo-unital Banach $M_a(S,\omega)$-bimodule, and $D \in \mathcal{Z}^1(M_a(S,\omega), E^*)$. Then E is a Banach $M(S,\omega)$-bimodule, and there is a unique $\tilde{D} \in \mathcal{Z}^1(M(S,\omega), E^*)$ that extends D and is continuous with respect to (so) on $M(S,\omega)$ and the w^*-topology on E^*. In particular, $\tilde{D}$ is uniquely determined by its values on $\{\delta_s : s \in S\}$.*

3 First cohomology of weighted semigroup algebras

The next definition is a generalization of the one given by B. E. Johnson [J].

Definition 3.1. *Suppose that E is a weighted S-bimodule. A map $\varrho : S \longrightarrow E$ is a weighted S-derivation if*
(1) $\varrho(st) = \varrho(s).t + s.\varrho(t)$, for each $s,t \in S$;
(2) ϱ is ω-bounded.
Furthermore, ϱ is inner if there exists $x \in E$ such that $\varrho(s) = s.x - x.s$, for all $s \in S$.

We shall regard an essential Banach $M_a(S,\omega)$-bimodule as a unital Banach $M(S,\omega)$-bimodule. Let E be a unital Banach $M_a(S,\omega)$-bimodule. Then $\delta_s.x$ and $x.\delta_s$ are defined in E for each $s \in S$ and $x \in E$, they are often denoted by $s.x$ and $x.s$ respectively.

Lemma 3.2. *A Banach space X is a weighted S-bimodule if and only if, X is a neo-unital Banach $M_a(S,\omega)$-bimodule.*

Proof. Suppose that X is a weighted S-bimodule. Let $\mu \in M(S,\omega)$ and $x \in X$. Define

$$\mu.x = \int_S s.x d\mu(s) \qquad , \qquad x.\mu = \int_S x.s d\mu(s).$$

That are Bochner integrals. We only check that X is a neo-unital left Banach $M_a(S,\omega)$-module by $\mu.x$ as an action. Since for some $k > 0$

$$\int_S \|s.x\| d|\mu|(s) \leq k\|x\| \int_S \omega(s) d|\mu|d(s) = k\|x\|\|\mu\|_\omega < \infty,$$

then $\mu.x$ is well-defined and $\|\mu.x\| \leq k\|x\|\|\mu\|_\omega$. We have

$$\mu.(\nu.x) = \int_S s \int_S t.x d\nu(t) d\mu(s) = \int_S \int_S (s.t).x d(\mu \otimes \nu)(s,t) = \int_S u.x d\mu\nu(u)$$
$$= (\mu.\nu).x,$$

It is clear that the map $\mu \longmapsto \mu.x$ is continuous from $M(S,\omega)$ into X. Then X is a left Banach $M(S,\omega)$-module, and a left Banach $\ell^1(S,\omega)$-module by restriction of this operation to the discrete measures. Choose a fixed neighborhood U_0 of e. Let Λ denote the collection of all compact neighborhoods of e contained in U_0, ordered by inclusion. Suppose that (ν_λ) is the bounded approximate identity for $M_a(S,\omega)$ by conditions: for each $\lambda \in \Lambda$, $\|\nu_\lambda\|_\omega = 1$ and $\nu_\lambda \geq 0$, and furthermore, its support contained in λ(see [L2]). Let $\varepsilon > 0$ be given and $x \in X$ be fixed. As the map $s \mapsto s.x$ is continuous, there is a neighborhood U_ε such that

$$s \in U_\varepsilon \Rightarrow \|x - s.x\| < \varepsilon.$$

As above for all $\lambda \geq \lambda_\varepsilon$ we have $supp(\nu_\lambda) \subseteq U_\varepsilon$ and consequently,

$$\| \int_S s.x d\nu_\lambda(s) - x\| \leq \int_S \|s.x - x\| d\nu_\lambda(s) = \int_{U_\varepsilon} \|s.x - x\| d\nu_\lambda(s) \leq \varepsilon \int_{U_\varepsilon} d\nu_\lambda(s)$$
$$= \varepsilon,$$

This shows that $x \in \overline{M_a(S,\omega)X}$, and hence $X = \overline{M_a(S,\omega)X}$. Therefore, X is an essential left Banach $M_a(S,\omega)$-module. But $M_a(S,\omega)$ has a bounded approximate identity, then X is neo-unital.

Conversely, let X be a neo-unital Banach $M_a(S,\omega)$-bimodule. As mentioned above, X is a neo-unital $M(S,\omega)$-bimodule. We claim that, X is a weighted

Banach S-bimodule by those actions. It is enough to show that there is $k > 0$ such that $\|s.x\| \leq k\|x\|\omega(s)$, for each $s \in S, x \in X$. But

$$\|s.x\| = \|\delta_s.x\| \leq \|\delta_s\|_\omega \|x\| = \omega(s)\|x\|.$$

□

Let $E = M(S,\omega)$. Then E is a weighted S-bimodule via $s.\mu = \delta_s * \mu$ and, therefore, by 3.2, an $M(S,\omega)$-bimodule. We need the following Lemma for the next results.

Lemma 3.3. *For each $\nu \in M_a(S,\omega)$,*

$$\int_S \delta_{st} d\nu(s) = \nu * \delta_t \qquad (t \in S).$$

Proof. Let $t \in S$. For each $g \in C_0(S,\omega^{-1})$,

$$\langle g, \int_S \delta_{st} d\nu(s)\rangle = \int_S g(st) d\nu(s) = \langle g, \nu * \delta_t\rangle.$$

□

We establish a correspondence between continuous derivations from $M_a(S,\omega)$ and S-derivations into dual modules.

Proposition 3.4. *Let E be a weighted S-bimodule. Suppose that every weighted S-derivation from S into E^* is inner. Then $\mathcal{H}^1(M_a(S,\omega), E^*) = \{0\}$.*

Proof. Assume that E is a neo-unital Banach $M_a(S,\omega)$-bimodule and $D : M_a(S,\omega) \longrightarrow E^*$ is a continuous derivation. According to the Lemma 3.2, E is a weighted S-bimodule. Since $M_a(S,\omega)$ has a bounded approximate identity, 2.4 shows that there exists a unique extension derivation of D on $M(S,\omega)$ which we denote it briefly by D. Define $\varrho : S \longrightarrow E^*$ by $\varrho(s) = D(\delta_s)$. We claim that ϱ is a weighted S-derivation. Let $s, t \in S$,

$$\varrho(st) = D(\delta_{st}) = D(\delta_s * \delta_t) = D(\delta_s).\delta_t + \delta_s.D(\delta_t) = \varrho(s).t + s.\sigma(t).$$

But

$$\frac{\|\varrho(s)\|}{\omega(s)} = \frac{\|D(\delta_s)\|}{\omega(s)} \leq \frac{\|D\|\|\delta_s\|_\omega}{\omega(s)} = \|D\|,$$

and hence

$$\sup\{\frac{\|\varrho(s)\|}{\omega(s)} : s \in S\} < \infty.$$

Therefore, ϱ is a weighted S-derivation and there is $\varphi \in E^*$ such that $\varrho(s) = s.\varphi - \varphi.s$. Now if $f \in \ell^1(S,\omega) \subseteq M(S,\omega)$, hence $f = \sum_{s\in S} \lambda_s.\delta_s$, where $\|f\|_\omega = \sum_{s\in S} |\lambda_s|\omega(s) < \infty$. It follows that

$$\begin{aligned} D(f) &= D(\sum_{s\in S}\lambda_s\delta_s) = \sum_{s\in S}\lambda_s D(\delta_s) \\ &= \sum_{s\in S}\lambda_s\varrho(s) = \sum_{s\in S}\lambda_s(s.\varphi - \varphi.s) \\ &= (\sum_{s\in S}\lambda_s\delta_s)\varphi - \varphi(\sum_{s\in S}\lambda_s\delta_s) \\ &= f.\varphi - \varphi.f, \end{aligned}$$

We conclude from 2.4 that, D is inner. ■ □

Let E be a Banach space. We shall often denote the topology $\sigma(E^*, E)$ (the w^*-topology) on E^* by σ.

Lemma 3.5. *Let E be a neo-unital Banach $M_a(S,\omega)$-bimodule, let $D \in \mathcal{Z}^1(M_a(S,\omega), E^*)$, and let $\varrho : S \longrightarrow E^*$ be the associated weighted S-derivation defined in the proof of 3.4. Then the map $\varrho : S \longrightarrow (E^*,\sigma)$ is continuous.*

Proof. Let $s_\alpha \longrightarrow s$ in S. For each $\nu \in M_a(S,\omega)$ the maps $s \longrightarrow \nu * \delta_s$ and $s \longrightarrow \delta_s * \nu$ are continuous, hence $\nu * \delta_{s_\alpha} \longrightarrow \nu * \delta_s$ and then by definition $\delta_{s_\alpha} \longrightarrow \delta_s$ in (so). According to the above Proposition D is continuous in w^*-topology and hence $D(\delta_{s_\alpha}) \longrightarrow D(\delta_s)$. Therefore $\varrho(s_\alpha) \longrightarrow \varrho(s)$ in (E^*,σ). □

Let $D \in \mathcal{Z}^1(M_a(S,\omega), E^*)$, and again denote its (so) extension to $M(S,\omega)$ by D. Define $\varrho(s) = D(\delta_s) \quad (s \in S)$. The following lemma which may be proved in much the same way as [J], shows how D can be recovered from ϱ.

Lemma 3.6. *Let D and ϱ be as above. Then*

$$\langle x, D\nu\rangle = \int_S \langle x, \varrho(s)\rangle d\nu(s) \quad (\nu \in M_a(S,\omega), x \in E).$$

Let $\varrho : S \longrightarrow E^*$ be a weighted S-derivation, then $D : (M_a(s,\omega), \|.\|_1) \longrightarrow (E^*, \|.\|)$ defined by

$$\langle x, D\nu\rangle = \int_S \langle x, \varrho(s)\rangle d\nu(s) \quad (\nu \in M_a(S,\omega), x \in E),$$

is continuous. From this we obtain the following

Lemma 3.7. *The above map D is a derivation.*

According to the Lemma 3.7 we have established the following main result due to B. E. Johnson. For more details we refer the reader to [D].

Theorem 3.8. *Let E be an neo-unital Banach $M_a(S,\omega)$-module. Then the map $D \longrightarrow \varrho$, where we define $\varrho(s) = D(\delta_s)$ $(s \in S)$, establishes a bijective correspondence between continuous derivations $D : M_a(S,\omega) \longrightarrow E^*$ and S-derivations $\varrho : S \longrightarrow E^*$. Further, inner derivation correspond to inner S-derivations, and $\mathcal{H}^1(M_a(S,\omega), E^*) = \{0\}$ if and only if every S-derivation into E^* is inner.*

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