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FIXED POINT RESULTS FOR MULTI-VALUED NON-EXPANSIVE MAPPINGS ON AN UNBOUNDED SET

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Abstract

Some results regarding the existence of a fixed point for a multi-valued non-expansive mapping defined on an unbounded subset of a reflexive Banach space are established.

1. Introduction and Preliminaries

Let X be an arbitrary real Banach space. We denote by $CB(C)$ the family of all nonempty closed bounded subsets of C, by $K(C)$ the family of all nonempty compact subsets of C and by $KC(C)$ the family of all nonempty convex compact subsets of C. On $CB(X)$, the *Hausdorff metric* is defined by

$$H(A,B) = \max\{\sup_{x\in A} d(x,B), \sup_{y\in B} d(y,A)\}, \quad \forall A,B \in CB(X),$$

where $d(x,E) = \inf\{d(x,y) : y \in E\}$ is the distance from a point $x \in X$ to a subset E of X. A multi-valued mapping $T : C \to CB(X)$ is said to be *contractive* if there exists a constant $k \in [0,1)$ such that if, for any $x, y \in C$,

$$H(Tx,Ty) \le k\,\|x-y\|\,, \quad \forall x,y \in C.$$

The mapping T is said to be *non-expansive* if

$$H(Tx,Ty) \le \|x-y\|\,, \quad \forall x,y \in C.$$

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The mapping T is said to be *asymptotically contractive* on C if, for some x_0 in C, there exists y_0 in Tx_0 such that

$$\lim \sup_{\|x\|\to\infty} \frac{\|y - y_0\|}{\|x - x_0\|} < 1, \quad \forall y \in Tx.$$

Example 1.1. Let $T : [0,\infty) \to 2^{[0,\infty)}$ be a multi-valued mapping defined by

$$Tx = \begin{cases} 0, & x \in Q^+, \\ [1, \frac{x}{2}], & x \in [0,\infty) - Q^+ \text{ and } 1 < \frac{x}{2}, \\ [\frac{x}{2}, 1], & x \in [0,\infty) - Q^+ \text{ and } \frac{x}{2} < 1, \end{cases}$$

where Q^+ is set of all nonnegative rational numbers. Take $x_0 = 0$, then $y_0 = 0$. If $x \in Q^+$, then, for any $y \in Tx$, $\lim \sup_{|x|\to\infty} \frac{|y - y_0|}{|x - x_0|} = 0 < 1$. If $x \in [0,\infty) - Q^+$, and $1 < \frac{x}{2}$, then, for any $y \in Tx$, $\lim \sup_{|x|\to\infty} \frac{|y - 0|}{|x - x_0|} \le \frac{1}{2} < 1$. Hence T is an asymptotical contraction with respect to 0.

A non-self mapping $T : C \to X$ is said to satisfy the *inward condition* on C if

$$Tx \subseteq I_C(x), \quad \forall x \in C,$$

and the mapping T is said to satisfy the *weakly inward condition* on C if

$$Tx \subseteq \overline{I_C(x)}, \quad \forall x \in C,$$

where

$$I_C(x) = \{x + \lambda(y - x) : \lambda \ge 0, y \in C\}$$

is the inward set of C at x and $\overline{E}$ denotes the closure of a set $E \subseteq X$. The mapping T is called *demiclosed* at $y \in C$ if, for any sequence $\{x_n\}$ in C which is weakly convergent to an element x and $y_n \in Tx_n$ with $\{y_n\}$ converging strongly to y, we have $y \in Tx$. A point $x \in C$ is called a *fixed point* of the multi-valued mapping T if $x \in Tx$.

Lim [7] proved the following theorem, which will be very useful to prove results on fixed points for nonself-nonexpansive multi-valued mappings.

Theorem 1.2. *Let C be a nonempty closed subset of a Banach space X and $T : C \to 2^X - \{\emptyset\}$ be a contraction taking closed values. If $Tx \subseteq \overline{I_C(x)}$ for any $x \in C$, then T has a fixed point in C.*

For a bounded sequence $\{x_n\}$ in X, denote $\lim\sup_{n\to\infty} \|x_n - x\|$ by $r(x, \{x_n\})$, where $x \in X$. The number $\inf_{x\in C} r(x, \{x_n\})$ is called the *asymptotic radius* of $\{x_n\}$ with respect to C. A point $z \in C$ is called the *asymptotic center* of the sequence $\{x_n\}$ with respect to C if

$$r(z, \{x_n\}) = \inf_{x\in C} r(x, \{x_n\}).$$

The set of all asymptotic centers of $\{x_n\}$ with respect to C is denoted by $Z(C, \{x_n\})$. A bounded sequence $\{x_n\}$ in X is said to be *regular* with respect to C if

$$\inf_{x\in C} r(x, \{x_n\}) = \inf_{x\in C} r(x, \{x_{n_k}\})$$

for every subsequence $\{x_{n_k}\}$ of $\{x_n\}$. A regular sequence $\{x_n\}$ is said to be *asymptotically uniform* with respect to C if $Z(C, \{x_n\}) = Z(C, \{x_{n_k}\})$ for each subsequence $\{x_{n_k}\}$ of $\{x_n\}$.

Let $G : X \times X \to R$ be a mapping which is linear in the first coordinate and, for any $x, y \in X$, satisfies $\|x\|^2 \leq G(x, x)$ and $|G(x, y)| \leq M \|x\| \|y\|$ for some $M > 0$ ([4]). For the information of the reader, we list some examples of the function G which satisfy conditions mentioned above as follows:

(1) If X is a Hilbert space, then the mapping G can be the inner product of X.

(2) If X is a Banach space, then the semi-inner product in the sense of Lumer [8] can play the role of the mapping G.

(3) If X is a Banach space, $B : X \times X \to R$ is a bilinear mapping and there is a positive constant k such that $B(x, x) \geq k \|x\|^2$ for all $x \in X$, then $G : X \times X \to R$ defined by $G(x, y) = \frac{1}{k} B(x, y)$ satisfies all of the above conditions.

(4) Consider the Banach space $C([0, 1], H)$, where H is a Hilbert space. For the mapping G, we can define

$$G(x, y) = \int_0^1 < x(t), y(t) > dt,$$

where $< \cdot, \cdot >$ is the inner product defined on H.

Definition 1.3. ([11]) A normed space X is said to satisfy *Opial's condition* if, whenever a sequence $\{x_n\}$ converges weakly to a point $x \in X$, then, for any $y \in X$ $(y \neq x)$,

$$\lim\inf \|x_n - x\| < \lim\inf \|x_n - y\|.$$

It is well-known from [11] that all of the l_p spaces for $1 < p < \infty$ have Opial's property. However, the L_p spaces do not have Opial's property unless $p = 2$.

The study of the existence of fixed points for multi-valued contractions and nonexpansive mappings by using the Hausdorff metric was initiated by Markin [10]. Later, an interesting and rich fixed point theory for such mappings was developed, which has applications in control theory, convex optimization, differential inclusion and economics (see [3] and references cited therein).

In 1978, Goebel and Kuczumov [2] proved that, if X is a closed convex subset of l_2 and $T : X \to X$ is non-expansive for which there exists a point $x \in X$ such that the set

$$LS(x, Tx; X) = \{z \in X : \langle z - x, Tx - x\rangle \geq 0\}$$

is bounded, then T has a fixed point in X.

In 1991, Marino [9] extended the results of Goebel and Kuczumov [2] to the multi-valued case and improved some known results.

The following is a very general fixed point theorem for multi-valued non-expansive self-mappings, which is due to Kirk and Massa [5].

Theorem 1.4. *Let C be nonempty closed bounded and convex subset of a Banach space X and $T : C \to KC(C)$ be a nonexpansive mapping. Assume that the asymptotic center in C of each bounded sequence of X is nonempty and compact. Then T has a fixed point in C.*

For the sake of completeness, we state the following theorem, in which Xu [13] gave an extension of Theorem 1.3 for nonself-multi-valued non-expansive mappings satisfying an inwardness condition.

Theorem 1.5. *Let C be a nonempty closed bounded and convex subset of a Banach space X and $T : C \to KC(X)$ a non-expansive mapping satisfying the inwardness condition $Tx \subseteq I_C(x)$ for all $x \in C$. Assume that the asymptotic center in C of each bounded sequence of X is nonempty and compact. Then T has a fixed point in C.*

Xu [13] further proved the following theorem, in which T assumes compact values only.

Theorem 1.6. *Let C be a nonempty closed bounded and convex subset of a uniformly convex Banach space X and $T : C \to K(X)$ be a non-expansive mapping satisfying the weak inwardness condition $Tx \subseteq \overline{I_C(x)}$ for all $x \in C$. Then T has a fixed point in C.*

Theorems 1.4 and 1.5 can be applied to Banach spaces which are uniformly convex. However, they can not be extended to a nearly uniform convex Banach spaces since, in such a space, the asymptotic center of a bounded sequence with respect to a closed bounded and convex subset of X is not necessarily compact. In such cases, T is also assumed to be compact and convex.

In this paper, we present some results which establish the existence of a fixed point for multi-valued non-expansive mappings defined on an unbounded convex set, which in turn generalizes several comparable results valid for bounded sets.

2. Fixed Point Theorems

In [12], Penot, using the notion of asymptotic contraction, obtained some existence theorems for single-valued non-expansive mappings defined on an unbounded set. We extend the notion to multi-valued mappings to obtain a general fixed point result for non-expansive multi-valued mappings, which in turn extends Proposition 2 of [12] to multi-valued mappings.

The following theorem also relaxes the condition of convexity on the domain of the given mapping. It is also noted that, using the notion of the asymptotic contraction, the comparable results in the literature of fixed point theory for multi-valued mappings can be extended to unbounded sets.

Theorem 2.1. *Let $(X, \|\cdot\|)$ be a reflexive Banach space and C be a nonempty unbounded closed star-shaped subset of X. Suppose that the mapping $T : C \to CB(X)$ is a non-expansive and asymptotical contraction with respect to the star-center $x_0 \in C$. If $Tx \subseteq \overline{I_C(x)}$ for any $x \in C$ and $I - T$ is demiclosed, then T has a fixed point in C.*

Proof. Let $\{\lambda_n\}$ be a sequence of real numbers in $(0, 1)$ such that $\lim_{n\to\infty} \lambda_n = 0$. Let $x_0 \in C$. For each $n \geq 1$, define the mapping $T_n : C \to CB(X)$ by

$$T_n x = (1 - \lambda_n)Tx + \lambda_n x_0, \quad \forall x \in C.$$

Then each T_n is a multi-valued contraction with Lipschitz constant $(1 - \lambda_n)$. Since $\underline{I_C(x)}$ is convex for any $x \in C$, it follows that $T_n x \subseteq I_C(x)$ and hence $T_n x \subseteq \overline{I_C(x)}$ for any $x \in C$. By Theorem 1.1, each T_n has a fixed point $x_n \in C$ such that

$$x_n = (1 - \lambda_n)y_n + \lambda_n x_0 \tag{2.1}$$

for some $y_n \in Tx_n$.

Now, we show that $\{x_n\}$ is a bounded sequence. Assume that $\{x_n\}$ is not bounded. Then there exists a subsequence of $\{x_n\}$ whose norm tends to infinity. For notational convenience, denote this subsequence by $\{x_m\}$. Since T is an asymptotical contraction with respect to $x_0 \in C$, for some $y_0 \in Tx_0$, there exist $\alpha \in (0,1)$ and $\beta > 0$ such that $\|y_m - y_0\| \leq \alpha \|x_m - x_0\|$ for some $y_m \in Tx_m$ and $\|x_m\| > \beta$. For m large enough, we have

$$\begin{aligned}
\|x_m\| &= \|(1-\lambda_m)y_m + \lambda_m x_0\| \\
&\leq (1-\lambda_m)\|y_m - y_0\| + \|y_0\| + \lambda_m \|x_0 - y_0\| \\
&\leq (1-\lambda_m)\alpha \|x_m - x_0\| + \|y_0\| + \lambda_m \|x_0 - y_0\| \\
&\leq (1-\lambda_m)\alpha \|x_m\| + (1-\lambda_m)\alpha \|x_0\| + \|y_0\| + \lambda_m \|x_0 - y_0\|.
\end{aligned}$$

Dividing both sides of the above inequality by $\|x_m\|$ and taking the limit as $m \to \infty$, we obtain $1 \leq \alpha$, which is a contradiction. Thus $\|x_n\|$ is bounded. From (2.1), it follows that $\{y_n\}$ is bounded and so is $\|y_n - x_0\|$. Therefore,

$$\|x_n - y_n\| = \lambda_n \|y_n - x_0\|$$

approaches zero as $n \to \infty$. Since a Banach space X is reflexive and $\{x_n\}$ is a bounded sequence, we have a subsequence $\{x_m\}$ which is weakly convergent to an element $p \in C$, $x_m - y_m \in (I-T)x_m$ and $x_m - y_m \to 0$ as $m \to \infty$. The demiclosedness of $I - T$ implies that $0 \in (I-T)p$. Hence $p \in Tp$. This completes the proof.

In Theorems 1 and 2 of [9], the multi-valued non-expansive mapping was assumed to be compact valued, while, in the following theorem, the condition of compactness is replaced by a weak condition of the closedness and boundedness.

Theorem 2.2. *Let $(X, \|\cdot\|)$ be a reflexive Banach space and C be a nonempty unbounded closed star-shaped subset of X. Suppose that the mapping $T : C \to CB(X)$ is a non-expansive multi-valued mapping with $Tx \subseteq I_C(x)$ for any $x \in C$. If*

$$\limsup_{\|x\|\to\infty} \frac{G(y,x)}{\|x\|^2} < 1, \quad \forall y \in Tx, \tag{2.2}$$

and $I - T$ is demiclosed, then T has a fixed point in C.

Proof. Let $\{\lambda_n\}$ be a sequence of real numbers in $(0,1)$ such that $\lim_{n\to\infty} \lambda_n = 0$. For each $n \geq 1$, define a mapping $T_n : C \to CB(X)$ by

$$T_n x = (1-\lambda_n)Tx, \quad \forall x \in C.$$

Then each T_n is a multi-valued contraction with Lipschitz constant $(1-\lambda_n)$. Since $Tx \subseteq \overline{I_C(x)}$ for any $x \in C$, we have $T_n x \subseteq \overline{I_C(x)}$. It follows from Theorem 1.2 that each T_n has a fixed point $x_n \in C$ such that

$$x_n = (1-\lambda_n)y_n$$

for some $y_n \in Tx_n$.

Now, we show that $\{x_n\}$ is a bounded sequence. Assume that $\{x_n\}$ is not a bounded sequence. Then there exists a subsequence $\{x_m\}$ of $\{x_n\}$ whose norm tends to infinity. By (2.2), there exist $\alpha \in (0,1)$ and $\beta > 0$ such that $G(y_m, x_m) \leq \alpha \|x_m\|^2$ for any $x \in C$ and $\|x_m\| > \beta$. For m large enough, we have

$$\begin{aligned} \|x_m\|^2 &\leq G(x_m, x_m) \\ &\leq G((1-\lambda_m)y_m, x_m) \\ &\leq (1-\lambda_m)G(y_m, x_m) \\ &\leq (1-\lambda_m)\alpha \|x_m\|^2 . \end{aligned}$$

Dividing both sides of the above inequality by $\|x_m\|^2$ and taking the limit as $m \to \infty$, we obtain $1 \leq \alpha$, which is a contradiction. Thus $\|x_n\|$ is bounded. The rest of the proof is similar to that given in Theorem 2.1. This completes the proof.

The following theorem offers a simple proof of Corollary 4 in [9].

Theorem 2.3. *Let $(X, \|\cdot\|)$ be a reflexive Banach space and C be a nonempty unbounded closed star-shaped subset of X. Suppose that the mapping $T : C \to CB(X)$ is a non-expansive multi-valued mapping with $Tx \subseteq \overline{I_C(x)}$ for any $x \in C$. If, for the star-center $x_0 \in C$,*

$$\lim \sup_{\|x\| \to \infty} \frac{G(y - x_0, x)}{\|x\|^2} < 1, \quad \forall y \in Tx, \tag{2.3}$$

and $I - T$ is demiclosed. Then T has a fixed point in C.

Proof. Let $\{\lambda_n\}$ be a sequence of real numbers in $(0,1)$ such that $\lim_{n\to\infty} \lambda_n = 0$. For each $n \geq 1$, define a mapping $T_n : C \to CB(X)$ by

$$T_n x = (1-\lambda_n)Tx + \lambda_n x_0, \quad \forall x \in C.$$

Then each T_n is a multi-valued contraction with Lipschitz constant $(1-\lambda_n)$. Since $I_C(x)$ is convex for any x in C, it follows that $T_n x \subseteq \overline{I_C(x)}$ for any $x \in C$. By Theorem 1.2, each T_n has a fixed point $x_n \in C$ such that

$$x_n = (1-\lambda_n)y_n + \lambda_n x_0$$

for some $y_n \in Tx_n$.

Now, we show that $\{x_n\}$ is a bounded sequence. Assume that $\{x_n\}$ not a bounded sequence. Then there exists a subsequence $\{x_m\}$ of $\{x_n\}$ whose norm tends to infinity. By (2.3), there exist $\alpha \in (0,1)$ and $\beta > 0$ such that $G(y_m, x_m) \leq \alpha \|x_m\|^2$ for any $x \in C$ and $\|x_m\| > \beta$. For m large enough, we have

$$\begin{aligned} \|x_m\|^2 &\leq G((1-\lambda_n)y_m + \lambda_m x_0, x_m) \\ &\leq G((1-\lambda_m)y_m - (1-\lambda_n)x_0 + x_0, x_m) \\ &\leq (1-\lambda_m)G(y_m - x_0, x_m) + G(x_0, x_m) \\ &\leq (1-\lambda_m)\alpha \|x_m\|^2 + M \|x_0\| \|x_m\| . \end{aligned}$$

Dividing both sides by $\|x_m\|^2$ and taking the limit as $m \to \infty$, we obtain $1 \leq \alpha$, which is a contradiction. Thus $\|x_n\|$ is bounded. The rest of the proof is similar to that given in Theorem 2.1. This completes the proof.

Remark 2.4. Let $T : R \to 2^R$ be a multi-valued mapping defined by

$$Tx = [0, \frac{x}{2}], \quad \forall x \in R.$$

Take $G(x,y) = xy$ for all x, $y \in R$, then, for $x_0 = 0$, (2.3) is satisfied.

In the following theorem, we assume that every bounded sequence in C is regular and has a unique asymptotic center.

Theorem 2.5. *Let C be a nonempty closed star-shaped subset of a reflexive Banach space X. Suppose that the mapping $T : C \to K(X)$ is a non-expansive and asymptotical contraction with respect to the star-center $x_0 \in C$. If $Tx \subseteq I_C(x)$ for any $x \in C$. Then T has a fixed point in C.*

Proof. Following the proof of Theorem 2.1, we obtain a bounded sequence $\{x_n\}$ in C such that $\lim_{n\to\infty} \|x_n - y_n\| = 0$ for some $y_n \in Tx_n$. Since

$$d(x_n, Tx_n) \leq \|x_n - y_n\| ,$$

$d(x_n, Tx_n) \to 0$ as $n \to \infty$. Also, $\{x_n\}$ is regular with the unique asymptotic center z (say) and hence is asymptotically uniform. We denote the asymptotic radius of $\{x_n\}$ by r. Since Tz is compact, select $z_n \in Tz$ such that

$$\|y_n - z_n\| \leq H(Tx_n, Tz) \leq \|x_n - z\| .$$

Since $\{z_n\} \subseteq Tz$, there exists a subsequence $\{z_m\}$ of $\{z_n\}$ such that $z_m \to z_0$ for some $z_0 \in Tz$. Consider

$$\|x_m - z_0\| \leq \|x_m - y_m\| + \|y_m - z_m\| + \|z_m - z_0\| .$$

Therefore, we have

$$\lim\sup_{m\to\infty} \|x_m - z_0\| \leq \lim\sup_{m\to\infty} \|x_m - z\| = r.$$

By the uniqueness of the asymptotic center of $\{x_n\}$, we conclude that $z = z_0$ and hence $z \in Tz$. This completes the proof.

Remark 2.6. (1) Theorems 2.1, 2.2 and 2.3 can easily be extended to locally convex spaces.

(2) Theorems 2.2 and 2.3 extend Theorems 3.1 and 3.2 of [4] to multi-valued non-expansive nonself-mappings. These theorems are also applicable to the closed convex cones.

(3) Theorems 2.1, 2.2 and 2.3 improve Theorem 1.3 of [5], Theorem 3.4 of [1] and Theorem 1.4 of [13] in the sense that T assumes closed and bounded values instead of compact and convex values. Moreover, our theorems do not require the assumption of compactness of the asymptotic center in C. Also, in our theorems, the domain of the mapping involved is unbounded.

(4) Theorem 2.5 improves Theorem 1.5 of [13].

(5) Theorems 2.1, 2.2 and 2.3 employ simpler techniques to prove fixed point results than those given in [1] and [9]. Moreover, our results extend the results of [9] to nonself-mappings.

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