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DIOPHANTINE QUADRUPLES IN $\mathbb{Z}[\sqrt{-2}\,]$

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Abstract

In this paper, we study the existence of Diophantine quadruples with the property $D(z)$ in the ring $\mathbb{Z}[\sqrt{-2}\,]$. We find several new polynomial formulas for Diophantine quadruples with the property $D(a+b\sqrt{-2}\,)$, for integers a and b satisfying certain congruence conditions. These formulas, together with previous results on this subject by Abu Muriefah, Al-Rashed and Franušić, allow us to almost completely characterize elements z of $\mathbb{Z}[\sqrt{-2}\,]$ for which a Diophantine quadruple with the property $D(z)$ exists.

1. Introduction

Let z be an element of a commutative ring R. A *Diophantine quadruple* with the property $D(z)$, or a $D(z)$-*quadruple*, is a set D of four non-zero elements of R with the property that the product of any two distinct elements of this set increased by z is a square of some element in R. Any set D satisfying this condition is called a *set with the property* $D(z)$.

In the third century, the Greek mathematician Diophantus of Alexandria considered the problem of existence of Diophantine quadruples in the rational field $\mathbb{Q}$. He discovered the set $\{\frac{1}{16}, \frac{33}{16}, \frac{17}{4}, \frac{105}{16}\}$ with the property $D(1)$, i.e. the set $\{1, 33, 65, 105\}$ with the property $D(256)$. The same problem was considered by Fermat. He found a $D(1)$-quadruple in integers $\{1, 3, 8, 120\}$ (see [4, 17]). In 1969, Baker and Davenport [2] proved that Fermat's set cannot be extended to a $D(1)$-quintuple in integers. Recently, it was proved in [8] that in $\mathbb{Z}$ there does not exist a $D(1)$-sextuple and there are only finitely many $D(1)$-quintuples.

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The problem of existence of Diophantine quadruples is almost completely solved in the ring of integers $\mathbb{Z}$. In [3], [16] and [18], it was shown that if $n \in \mathbb{Z}$, $n \equiv 2 \pmod 4$, then there does not exist a $D(n)$-quadruple. In [5], Dujella proved a result in the opposite direction. Namely, if $n \not\equiv 2 \pmod 4$ and $n \notin S = \{-4, -3, -1, 3, 5, 8, 12, 20\}$, then there exists a $D(n)$-quadruple.

In [7], the existence of $D(a + b\sqrt{-1}\,)$-quadruples in the ring $\mathbb{Z}[\sqrt{-1}\,]$ of Gaussian integers was considered. It was shown that if b is odd or $a \equiv b \equiv 2 \pmod 4$, then there does not exist a $D(a + b\sqrt{-1}\,)$-quadruple, and if $z = a+b\sqrt{-1}$ is not of that form and $z \notin \{\pm 2, \pm 1 \pm 2i, \pm 4i\}$, then there exist at least two distinct Diophantine quadruples with the property $D(z)$ (for additional results on Diophantine quadruples in Gaussian integers see [14]).

Franušić (see [12]) gave similar results about the existence of Diophantine quadruples in the ring $\mathbb{Z}[\sqrt{2}\,]$. She proved that there exist infinitely many $D(z)$-quadruples if and only if z has one of the following forms:

$$z = 2a+1+2b\sqrt{2},\ z = 4a+4b\sqrt{2},\ z = 4a+2+4b\sqrt{2},\ z = 4a+2+(4b+2)\sqrt{2},$$

i.e. according to [10], if and only if z can be represented as a difference of squares of two elements in $\mathbb{Z}[\sqrt{2}\,]$. To prove this result, she used the fact that Pellian equations $x^2 - 2y^2 = \pm 1$ and $x^2 - 2y^2 = \pm 2$ are all solvable in integers. In [13, 15], she proved analogous results for more general real quadratic fields $\mathbb{Z}[\sqrt{d}\,]$.

Most of the mentioned results obtained by Franušić deal with real quadratic fields, i.e. with $d > 0$. One important difference between real and complex fields is that in the real case there exist infinitely many units. The methods for the construction of Diophantine quadruples usually use elements with small norm (see e.g. Section 2), which makes a complex case harder to handle.

Apart from mentioning the result by Dujella for Gaussian integers, the only case of a complex quadratic field studied until now is the case of the ring $\mathbb{Z}[\sqrt{-2}\,]$ of integers in the quadratic field $\mathbb{Q}(\sqrt{-2}\,)$. This case was studied by Abu Muriefah and Al-Rashed in [1]. They showed that for any Diophantine quadruple with the property $D(a + b\sqrt{-2}\,)$, b must be an even integer. In the remaining cases there are 8 possibilities for a and b modulo 4. Let $a' = a \bmod 4$, $b' = b \bmod 4$. They completely solved the case $(a', b') = (3, 0)$. The case $(a', b') = (0, 0)$ was considered modulo 16, and they solved 6 out of 16 possible subcases, while for $(a', b') = (2, 0)$ they solved 8 out of 16 possible subcases. The cases $(a', b') = (1, 0)$ and $(a', b') = (1, 2)$ were considered modulo 8, and in both cases they solved 2 out of 4 possible subcases. The case $(a', b') = (2, 2)$ was considered modulo 24 and they solved 8 out of 36 possible subcases. There are no results in [1] for the cases $(a', b') = (0, 2)$ and $(a', b') = (3, 2)$.

In this paper, we will significantly extend the results of Abu Muriefah and Al-Rashed and obtain several new polynomial formulas for Diophantine quadruples with the property $D(a+b\sqrt{-2}\,)$, for integers a and b satisfying certain congruence conditions. In that way, we will solve the problem of existence of $D(z)$-quadruples of a large class of elements z of $\mathbb{Z}[\sqrt{-2}\,]$. The results of Abu Muriefah and Al-Rashed [1] are based on polynomial formulas for Diophantine quadruples derived by Dujella in [6]. These formulas were initially constructed for the analogous problem in $\mathbb{Z}$. One important new ingredient in this paper is considering the derivation of these formulas in a more general setting of quadratic fields. In particular, we will take advantage of the fact that number 3 factorizes in $\mathbb{Z}[\sqrt{-2}\,]$ as $3=(1+\sqrt{-2}\,)(1-\sqrt{-2}\,)$.

Our main result is:

Theorem 1.1. *Let $z\in\mathbb{Z}[\sqrt{-2}\,]$. If z is of the form $z=a+(2b+1)\sqrt{-2}$ or $z=4a+(4b+2)\sqrt{-2}$, $a,b\in\mathbb{Z}$, then there does not exist a $D(z)$-quadruple. If z is not of that form, then there exists at least one $D(z)$-quadruple, except maybe if z has one of the following forms:*

$$z=24a+2+(12b+6)\sqrt{-2},\ \ z=24a+5+(12b+6)\sqrt{-2},\ \ z=48a+44+(24b+12)\sqrt{-2},$$

or if $z\in\{-1,1\pm2\sqrt{-2}\,\}$.

The statement of Theorem 1.1 will follow from the mentioned results by Abu Muriefah, Al-Rashed and Franušić and the propositions which will be presented and proved in Sections 3 and 4.

2. Preliminaries

Let $\{u,v\}$ be an arbitrary pair with the property $D(z)$, for $z\in\mathbb{Z}[\sqrt{-2}\,]$. It means that

$$uv+z=r^2,$$

for $r\in\mathbb{Z}[\sqrt{-2}\,]$. It is easy to check that the set $\{u,v,u+v+2r\}$ also has the property $D(z)$. Indeed,

$$\begin{aligned} u(u+v+2r)+z &= (u+r)^2,\\ v(u+v+2r)+z &= (v+r)^2. \end{aligned}$$

Applying this construction to the Diophantine pair $\{v,u+v+2r\}$ we get the set $\{v,u+v+2r,u+4v+4r\}$. Therefore, the set

$$\{u,v,u+v+2r,u+4v+4r\}$$

has the property $D(z)$ if and only if the product of its first and fourth element increased by z is a perfect square, i.e.

$$u(u+4v+4r)+z=y^2. \tag{2.1}$$

Equation (2.1) is equivalent to

$$3z = (u + 2r - y)(u + 2r + y).$$

This means that there is an element $e \in \mathbb{Z}[\sqrt{-2}\,]$ such that

$$\begin{aligned} u + 2r - y &= e, \\ u + 2r + y &= \frac{3z}{e}, \end{aligned}$$

which gives

$$2u + 4r = \frac{3z\bar{e}}{N(e)} + e, \tag{2.2}$$

where $\bar{e}$ is the conjugate of an element e and $N(e) = e \cdot \bar{e}$ is the norm of e. Suppose that z and e are given. Then, if we look at equation (2.2) modulo 4, we get a condition for the form of u. We choose a u of small norm satisfying this condition. Now, it is easy to find the form of r. It remains to satisfy the condition that $v = (r^2 - z)/u \in \mathbb{Z}[\sqrt{-2}\,]$, which is equivalent to the divisibility condition

$$N(u)|(r^2 - z) \cdot \bar{u},$$

which explains why we choose u to have small norm, preferably (but not always possible) $N(u) = 1$.

This method was described for the first time in [6] (but only for quadruples in $\mathbb{Z}$) and used for systematically finding polynomial formulas for Diophantine quadruples. We will use it to prove some of our results. In the previous applications of this construction (e.g. in Lemma 2.2), the authors usually used $e = 1$ or $e = 3$. Since 3 factorizes in $\mathbb{Z}[\sqrt{-2}\,]$, we can also use $e = \pm 1 \pm \sqrt{-2}$, and (for z of a special form) other factors of $3^g \cdot 2^h$ for small nonnegative values of g and h.

Here we also specify some known results, which will be used later on.

Lemma 2.1. *Let $\{z_1, z_2, z_3, z_4\} \subset \mathbb{Z}[\sqrt{-2}\,]$ be a set with the property $D(z)$ and $w \in \mathbb{Z}[\sqrt{-2}\,]$. Then*

(i) the set $\{\bar{z}_1, \bar{z}_2, \bar{z}_3, \bar{z}_4\}$ has the property $D(\bar{z})$,

(ii) the set $\{z_1 w, z_2 w, z_3 w, z_4 w\}$ has the property $D(zw^2)$.

Lemma 2.2 (see [6], Theorem 1)**.** *The sets*

$$\{m, m(3k+1)^2 + 2k, m(3k+2)^2 + 2k + 2, 9m(2k+1)^2 + 8k + 4\} \tag{2.3}$$
$$\{m, mk^2 - 2k - 2, m(k+1)^2 - 2k, m(2k+1)^2 - 8k - 4\} \tag{2.4}$$

have the property $D(2m(2k+1) + 1)$.

Lemma 2.3 (see [12], Proposition 5). *Let $d \in \mathbb{Z}$ such that $d \equiv 2 \pmod 4$. If $z \in \mathbb{Z}[\sqrt{d}\,]$ is of the form $a + (2b+1)\sqrt{d}$ or $4a + (4b+2)\sqrt{d}$, then there does not exist a $D(z)$-quadruple.*

Lemma 2.4 (see [1]). *Let $z = a + b\sqrt{-2} \in \mathbb{Z}[\sqrt{d}\,]$. If one of the following conditions*

(i) $a \equiv 3 \pmod 4$ *and* $b \equiv 0 \pmod 4$*;*

(ii) $a \equiv 1 \pmod 4$ *and* $b \equiv 0 \pmod 8$*;*

(iii) $a \equiv 1 \pmod 8$ *and* $b \equiv 2 \pmod 4$*;*

(iv) $a \equiv 0 \pmod 8$ *and* $b \equiv 0 \pmod 8$*;*

is satisfied, then there exists at least one $D(z)$-quadruple, except maybe if $z \in \{-1, -3,\, 1 \pm 2\sqrt{-2}\}$.

3. Construction of $D(z)$-quadruples in $\mathbb{Z}[\sqrt{-2}\,]$

3.1. The case $(a', b') = (3, 2)$

We begin with the case $a \equiv 3 \pmod 4$, $b \equiv 2 \pmod 4$, which is the only case modulo 4 for which no results are available until now.

Proposition 3.1. *If $z \in \mathbb{Z}[\sqrt{-2}\,]$ is of the form $z = 8a + 7 + (8b+2)\sqrt{-2}$, $a, b \in \mathbb{Z}$, or $z = 8a + 7 + (8b+6)\sqrt{-2}$, $a, b \in \mathbb{Z}$, then there exists at least one Diophantine quadruple with the property $D(z)$.*

Proof: By Lemma 2.1(i), it suffices to prove the statement for $z = 8a + 7 + (8b+2)\sqrt{-2}$, $a, b \in \mathbb{Z}$. Let $e = 1 - \sqrt{-2}$. We choose $u = 2\sqrt{-2}$ with $N(u) = 8$. Now, from (2.2), it follows that

$$r = 2a - 4b + 1 + (2a + 2b + 1)\sqrt{-2}.$$

It is obvious that the condition

$$N(u) | (r^2 - z) \cdot \bar{u},$$

i.e.

$$\begin{aligned} 32a^2 + 32a - 32ab - 64b^2 - 48b &\equiv 0 \pmod 8, \\ 8a^2 + 24a + 64ab - 16b^2 + 32b + 16 &\equiv 0 \pmod 8, \end{aligned}$$

is satisfied for all $a, b \in \mathbb{Z}$. So, we get the set

$$\begin{aligned}
&\{2\sqrt{-2}, 4a^2+4a-4ab-8b^2-6b+(a^2+3a+8ab-2b^2+4b+2)\sqrt{-2},\\
&4a^2+8a-4ab-8b^2-14b+2+(a^2+7a+8ab-2b^2+8b+6)\sqrt{-2},\\
&16a^2+24a-16ab-32b^2-40b+4+(4a^2+20a+32ab-8b^2+24b+14)\sqrt{-2}\,\}
\end{aligned} \tag{3.1}$$

with the property $D(8a+7+(8b+2)\sqrt{-2}\,)$.

It remains to determine the pairs (a, b) for which the above set has at least two equal elements or some elements equal to zero. It is easy to check that the above cases appear if and only if

$$(a, b) \in \{(-2,-1),\ (-1,0),\ (0,0)\}\,.$$

But, for $(a, b) = (-2,-1)$ the set $\{-4+4\sqrt{-2}, -2+\sqrt{-2}, -4\sqrt{-2}, -\sqrt{-2}\,\}$ has the property $D(-9-6\sqrt{-2}\,)$, for $(a, b) = (-1, 0)$ the set $\{-3-3\sqrt{-2}, -1-\sqrt{-2}, 1+\sqrt{-2}, 3+3\sqrt{-2}\,\}$ has the property $D(-1+2\sqrt{-2}\,)$ and for $(a, b) = (0, 0)$ the set $\{5-2\sqrt{-2}, -4\sqrt{-2}, -1, 8\}$ has the property $D(7+2\sqrt{-2}\,)$.

□

Proposition 3.2. *If $z \in \mathbb{Z}[\sqrt{-2}\,]$ is of the form $z = 8a+3+(4b+2)\sqrt{-2}$, $a, b \in \mathbb{Z}$, then there exists at least one Diophantine quadruple with the property $D(z)$.*

Proof: Let $z = 8a+3+(4b+2)\sqrt{-2}$, $a, b \in \mathbb{Z}$ and $e = -1+\sqrt{-2}$. We choose $u = 2$ with $N(u) = 4$. Now, from (2.2), it follows that

$$r = -2a+2b-1+(-2a-b-1)\sqrt{-2}.$$

The condition $N(u)|(r^2-z)\cdot\bar{u}$ is satisfied for all $a, b \in \mathbb{Z}$. So, we obtain the set

$$\begin{aligned}
&\{2, -2a^2-6a-8ab+b^2-4b-2+(4a^2+4a-2ab-2b^2-3b)\sqrt{-2},\\
&-2a^2-10a-8ab+b^2-2+(4a^2-2ab-2b^2-5b-2)\sqrt{-2},\\
&-8a^2-32a-32ab+4b^2-8b-10+(16a^2+8a-8ab-8b^2-16b-4)\sqrt{-2}\,\}
\end{aligned} \tag{3.2}$$

with the property $D(8a+3+(4b+2)\sqrt{-2}\,)$.

The pairs (a, b) for which the above set has at least two equal elements or some elements equal to zero are now

$$(a, b) \in \{(-1,0),\ (0,-2)\}\,.$$

But, for $(a, b) = (-1, 0)$ the set $\{-6-4\sqrt{-2}, -1-3\sqrt{-2}, -1+\sqrt{-2}, 2\,\}$ has the property $D(-5+2\sqrt{-2}\,)$ and for $(a, b) = (0,-2)$ the set $\{-14+28\sqrt{-2}, -5+5\sqrt{-2}, -1+9\sqrt{-2}, 2\,\}$ has the property $D(3-6\sqrt{-2}\,)$.

□

3.2. The case $(a', b') = (1, 0)$

Proposition 3.3. *If $z \in \mathbb{Z}[\sqrt{-2}\,]$ is of the form $z = 4a + 1 + (8b + 4)\sqrt{-2}$, $a, b \in \mathbb{Z}$, then there exists at least one Diophantine quadruple with the property $D(z)$.*

Proof: Let $z = 4a + 1 + (8b + 4)\sqrt{-2}$, $a, b \in \mathbb{Z}$ and $e = 1 + \sqrt{-2}$. Here we may take $u = 1$ and we get the set

$$\begin{aligned}&\{1, -a^2 + 4a + 16ab + 8b^2 + 8b + 1 - (2a^2 + 2a + 4ab - 16b^2 - 8b)\sqrt{-2},\\ &-a^2 + 6a + 16ab + 8b^2 + 16b + 6 - (2a^2 + 4a + 4ab - 16b^2 - 12b - 2)\sqrt{-2},\\ &-4a^2 + 20a + 64ab + 32b^2 + 48b + 13 - (8a^2 + 12a + 16ab - 64b^2 - 40b - 4)\sqrt{-2}\,\}\end{aligned} \tag{3.3}$$

with the property $D(4a + 1 + (8b + 4)\sqrt{-2}\,)$.

Only for $(a, b) = (0, 0)$ the above set has two equal elements. But, the set $\{-2 + 2\sqrt{-2}, \sqrt{-2}, 4\sqrt{-2}, -4\}$ has the property $D(1 + 4\sqrt{-2}\,)$.

□

3.3. The case $(a', b') = (2, 0)$

Proposition 3.4. *If $z \in \mathbb{Z}[\sqrt{-2}\,]$ is of the form $z = 16a + 6 + (8b + 4)\sqrt{-2}$, $a, b \in \mathbb{Z}$, then there exists at least one Diophantine quadruple with the property $D(z)$.*

Proof: Multiplying by $\sqrt{-2}$ the elements of set (2.3) in Lemma 2.2 for $m = \sqrt{-2}$ and $k = -b - 1 + (2a + 1)\sqrt{-2}/2$ we obtain the set

$$\begin{aligned}&\{-2, 36a^2 + 32a - 18b^2 - 24b - 1 + (24a + 36ab + 16b + 10)\sqrt{-2},\\ &36a^2 + 32a - 18b^2 - 12b + 5 + (12a + 36ab + 16b + 6)\sqrt{-2},\\ &144a^2 + 128a - 72b^2 - 72b + 10 + (72a + 144ab + 64b + 32)\sqrt{-2}\,\}\end{aligned} \tag{3.4}$$

with the property $D(16a + 6 + (8b + 4)\sqrt{-2}\,)$.

□

3.4. The case $(a', b') = (2, 2)$

Proposition 3.5. *If $z \in \mathbb{Z}[\sqrt{-2}\,]$ is of the form $z = 8a + 6 + (4b + 2)\sqrt{-2}$, $a, b \in \mathbb{Z}$, then there exists at least one Diophantine quadruple with the property $D(z)$.*

Proof: Multiplying by $\sqrt{-2}$ the elements of set (2.3) in Lemma 2.2 for $m = -\sqrt{-2}/2$ and $k = b - (a+1)\sqrt{-2}/2$ we get the set

$$\begin{aligned}&\{1, -18a^2 - 32a + 9b^2 + 6b - 13 - (6a + 18ab + 16b + 6)\sqrt{-2},\\ &-18a^2 - 32a + 9b^2 + 12b - 10 - (12a + 18ab + 16b + 10)\sqrt{-2},\\ &-72a^2 - 128a + 36b^2 + 36b - 47 - (36a + 72ab + 64b + 32)\sqrt{-2}\,\}\end{aligned} \tag{3.5}$$

with the property $D(8a + 6 + (4b + 2)\sqrt{-2}\,)$.

It is easy to check that set (3.5) contains two equal elements only for $(a, b) \in \{(-1, -1), (-1, 0)\}$. But, for $(a, b) = (-1, -1)$ the set $\{2 + 2\sqrt{-2}, 3 + 2\sqrt{-2}, 9 + 8\sqrt{-2}, 1\,\}$ has the property $D(-2 - 2\sqrt{-2}\,)$ and for $(a, b) = (-1, 0)$ the set $\{2 - 2\sqrt{-2}, 3 - 2\sqrt{-2}, 9 - 8\sqrt{-2}, 1\,\}$ has the property $D(-2 + 2\sqrt{-2}\,)$.

□

3.5. The case $(a', b') = (0, 0)$

Proposition 3.6. *If $z \in \mathbb{Z}[\sqrt{-2}\,]$ is of the form $z = 16a + (8b + 4)\sqrt{-2}$, $a, b \in \mathbb{Z}$, then there exists at least one Diophantine quadruple with the property $D(z)$.*

Proof: Let $z = 16a + (8b + 4)\sqrt{-2}$, $a, b \in \mathbb{Z}$ and $e = 4 + 2\sqrt{-2}$. We may take $u = 1$. Now, from (2.2), it follows that

$$r = 2a + b + 1 + (-a + b + 1)\sqrt{-2}.$$

Therefore, we obtain the set

$$\begin{aligned}&\{1, 2a^2 - 8a + 8ab - b^2 - 2b - 1 - (4a^2 - 2a - 2ab - 2b^2 + 4b + 2)\sqrt{-2},\\ &2a^2 - 4a + 8ab - b^2 + 2 - (4a^2 - 2ab - 2b^2 + 2b)\sqrt{-2},\\ &8a^2 - 24a + 32ab - 4b^2 - 4b + 1 - (16a^2 - 4a - 8ab - 8b^2 + 12b + 4)\sqrt{-2}\,\}\end{aligned} \tag{3.6}$$

with the property $D(16a + (8b + 4)\sqrt{-2}\,)$.

The pairs (a, b) for which the above set has at least two equal elements or some elements equal to zero are $(a, b) \in \{(0, 1), (1, 2)\}$. But, for $(a, b) = (0, 1)$ the set $\{-7 - 52\sqrt{-2}, -2 - 12\sqrt{-2}, -1 - 14\sqrt{-2}, 1\,\}$ has the property $D(12\sqrt{-2}\,)$, and for $(a, b) = (1, 2)$ the set $\{-63 - 80\sqrt{-2}, -16 - 20\sqrt{-2}, -15 - 20\sqrt{-2}, 1\,\}$ has the property $D(16 + 20\sqrt{-2}\,)$.

□

Proposition 3.7. *If $z \in \mathbb{Z}[\sqrt{-2}\,]$ is of the form $z = 16a + 8 + (8b + 4)\sqrt{-2}$, $a, b \in \mathbb{Z}$, then there exists at least one Diophantine quadruple with the property $D(z)$.*

Proof: Multiplying by $\sqrt{-2}$ the elements of set (2.3) in Lemma 2.2 for $m = \sqrt{-2}/2$ and $k = -b-1+(a+1)\sqrt{-2}/2$, we get a set with the property $D(4a+2+(2b+1)\sqrt{-2}\,)$. Then we multiply the elements of the new set by 2 and get the set

$$\begin{aligned}&\{-1, 18a^2+28a-9b^2-12b+6+(12a+18ab+14b+8)\sqrt{-2},\\&18a^2+28a-9b^2-6b+9+(6a+18ab+14b+6)\sqrt{-2},\\&72a^2+112a-36b^2-36b+31+(36a+72ab+56b+28)\sqrt{-2}\,\}\end{aligned} \tag{3.7}$$

with the property $D(16a+8+(8b+4)\sqrt{-2}\,)$.

Set (3.7) contains two equal elements for $(a,b) \in \{(-1,-1),(-1,0)\}$. But, for $(a,b)=(-1,-1)$ the set $\{7+2\sqrt{-2}, 33+4\sqrt{-2}, 1, 10\}$ has the property $D(-8-4\sqrt{-2}\,)$, and for $(a,b)=(-1,0)$ the set $\{7-2\sqrt{-2}, 33-4\sqrt{-2}, 1, 10\}$ has the property $D(-8+4\sqrt{-2}\,)$.

□

Remark 3.8. Using formula (2.4) instead of (2.3), we can obtain another formula for a set with the property $D(16a+8+(8b+4)\sqrt{-2}\,)$, namely,

$$\begin{aligned}&\{-1, 2a^2+12a-b^2-2b+9+(2a+2ab+6b+2)\sqrt{-2},\\&2a^2+12a-b^2+10+(2ab+6b+4)\sqrt{-2},\\&8a^2+48a-4b^2-4b+39+(4a+8ab+24b+12)\sqrt{-2}\,\}.\end{aligned} \tag{3.8}$$

Therefore, we can prove that there are at least two distinct Diophantine quadruples with the property $D(16a+8+(8b+4)\sqrt{-2}\,)$. A similar extension can also be obtained for some other propositions in this section.

4. Some partial results in the remaining cases

We are not able to solve completely the cases where z has one of the following forms

$$z = 8a+5+(4b+2)\sqrt{-2},\;\; z = 8a+2+(4b+2)\sqrt{-2},\;\; z = 16a+12+(8b+4)\sqrt{-2},$$

and they are also not covered by the results of Abu Muriefah, Al-Rashed and Franušić. However, we can obtain partial results in these cases by considering a and b modulo 3.

Proposition 4.1. *If z is of the form $z = 8a+5+(4b+2)\sqrt{-2}$, then there exists at least one complex Diophantine quadruple with the property $D(z)$, for any $a, b \in \mathbb{Z}$, except maybe for $a \equiv 0 \pmod 3$, $b \equiv 1 \pmod 3$.*

Proof: Let $z = 8a + 5 + (4b + 2)\sqrt{-2}$, $a, b \in \mathbb{Z}$, and $e = 1 + \sqrt{-2}$. We choose $u = 1 + \sqrt{-2}$ and $N(u) = 3$. Now, from (2.2), it follows that

$$r = 2a + 2b + 2 + (-2a + b - 1)\sqrt{-2}.$$

From the condition $uv + z = r^2$, it follows that

$$v = -\frac{20}{3}a^2 - \frac{32}{3}a + \frac{8}{3}ab + \frac{10}{3}b^2 + \frac{4}{3}b - 5 - \left(\frac{4}{3}a^2 + \frac{4}{3}a + \frac{20}{3}ab - \frac{2}{3}b^2 + \frac{16}{3}b + 1\right)\sqrt{-2}.$$

Now, it is easy to compute other elements of the quadruple. We get the set

$$\begin{aligned}
&\{1 + \sqrt{-2}, -\tfrac{20}{3}a^2 - \tfrac{32}{3}a + \tfrac{8}{3}ab + \tfrac{10}{3}b^2 + \tfrac{4}{3}b - 5\\
&-(\tfrac{4}{3}a^2 + \tfrac{4}{3}a + \tfrac{20}{3}ab - \tfrac{2}{3}b^2 + \tfrac{16}{3}b + 1)\sqrt{-2},\\
&-\tfrac{20}{3}a^2 - \tfrac{20}{3}a + \tfrac{8}{3}ab + \tfrac{10}{3}b^2 + \tfrac{16}{3}b\\
&-(\tfrac{4}{3}a^2 + \tfrac{16}{3}a + \tfrac{20}{3}ab - \tfrac{2}{3}b^2 + \tfrac{10}{3}b + 2)\sqrt{-2},\\
&-\tfrac{80}{3}a^2 - \tfrac{104}{3}a + \tfrac{32}{3}ab + \tfrac{40}{3}b^2 + \tfrac{40}{3}b - 11\\
&-(\tfrac{16}{3}a^2 + \tfrac{40}{3}a + \tfrac{80}{3}ab - \tfrac{8}{3}b^2 + \tfrac{52}{3}b + 7)\sqrt{-2}\,\}
\end{aligned} \tag{4.1}$$

with the property $D(8a + 5 + (4b + 2)\sqrt{-2}\,)$ in $\mathbb{Q}(\sqrt{-2}\,)$. It remains to check for which $a \in \{3k, 3k + 1, 3k + 2\}$ and $b \in \{3l, 3l + 1, 3l + 2\}$, $k, l \in \mathbb{Z}$, set (4.1) has the elements in the ring $\mathbb{Z}[\sqrt{-2}\,]$. Equivalently, we have to find all pairs (a, b) which satisfy the condition

$$N(u) | (r^2 - z) \cdot \bar{u},$$

i.e.

$$\begin{aligned}
-20a^2 - 32a + 8ab + 10b^2 + 4b - 15 &\equiv 0 \,(\text{mod } 3),\\
-4a^2 - 4a - 20ab + 2b^2 - 16b - 3 &\equiv 0 \,(\text{mod } 3).
\end{aligned}$$

It is easy to check that the above condition is satisfied for all $(a, b) \in \{(3k, 3l), (3k, 3l+2), (3k+1, 3l+1), (3k+1, 3l+2), (3k+2, 3l), (3k+2, 3l+1)\}$ for any $k, l \in \mathbb{Z}$. For each pair (a, b), the four elements from (4.1) are pairwise distinct.

In a similar way, for $e = -1 + \sqrt{-2}$ we obtain the set

$$\begin{aligned}
&\{1 - \sqrt{-2}, -\tfrac{20}{3}a^2 - \tfrac{28}{3}a - \tfrac{8}{3}ab + \tfrac{10}{3}b^2 + \tfrac{4}{3}b - 2\\
&+(\tfrac{4}{3}a^2 - \tfrac{4}{3}a - \tfrac{20}{3}ab - \tfrac{2}{3}b^2 - \tfrac{14}{3}b - 2)\sqrt{-2},\\
&-\tfrac{20}{3}a^2 - \tfrac{40}{3}a - \tfrac{8}{3}ab + \tfrac{10}{3}b^2 + \tfrac{16}{3}b - 3\\
&+(\tfrac{4}{3}a^2 - \tfrac{16}{3}a - \tfrac{20}{3}ab - \tfrac{2}{3}b^2 - \tfrac{20}{3}b - 5)\sqrt{-2},\\
&-\tfrac{80}{3}a^2 - \tfrac{136}{3}a - \tfrac{32}{3}ab + \tfrac{40}{3}b^2 + \tfrac{40}{3}b - 11\\
&+(\tfrac{16}{3}a^2 - \tfrac{40}{3}a - \tfrac{80}{3}ab - \tfrac{8}{3}b^2 - \tfrac{68}{3}b - 13)\sqrt{-2}\,\}
\end{aligned} \tag{4.2}$$

with the property $D(8a+5+(4b+2)\sqrt{-2}\,)$ in $\mathbb{Q}(\sqrt{-2}\,)$. Set (4.2) has the elements in the ring $\mathbb{Z}[\sqrt{-2}\,]$ for new pairs $(a,b)\in\{(3k+1,3l),(3k+2,3l+2)\}$ for any $k,l\in\mathbb{Z}$ and contains four distinct elements.

The only remaining unsolved case is for $(a,b)=(3k,3l+1)$, i.e. the property $D(24k+5+(12l+6)\sqrt{-2}\,)$, $k,l\in\mathbb{Z}$.

□

Proposition 4.2. *If z is of the form $z=8a+2+(4b+2)\sqrt{-2}$, then there exists at least one complex Diophantine quadruple with the property $D(z)$, for any $a,b\in\mathbb{Z}$, except maybe for $a\equiv 0\,(\mathrm{mod}\ 3)$ and $b\equiv 1\,(\mathrm{mod}\ 3)$.*

Proof: Let $z=8a+2+(4b+2)\sqrt{-2}$, $a,b\in\mathbb{Z}$, and $e=-3\sqrt{-2}$. We can choose $u=1+\sqrt{-2}$ with $N(u)=3$. From (2.2), it follows that

$$r=-b-1+(a-1)\sqrt{-2}.$$

From the condition $uv+z=r^2$, we obtain

$$v=-\frac{2}{3}a^2-\frac{8}{3}a-\frac{4}{3}ab+\frac{1}{3}b^2-\frac{2}{3}b-1+\left(\frac{2}{3}a^2+\frac{2}{3}a-\frac{2}{3}ab-\frac{1}{3}b^2-\frac{4}{3}b+1\right)\sqrt{-2}.$$

Now, it is easy to compute other elements of the quadruple. We get the set

$$\begin{aligned}&\{1+\sqrt{-2},-\tfrac{2}{3}a^2-\tfrac{8}{3}a-\tfrac{4}{3}ab+\tfrac{1}{3}b^2-\tfrac{2}{3}b-1\\&+(\tfrac{2}{3}a^2+\tfrac{2}{3}a-\tfrac{2}{3}ab-\tfrac{1}{3}b^2-\tfrac{4}{3}b+1)\sqrt{-2},\\&-\tfrac{2}{3}a^2-\tfrac{8}{3}a-\tfrac{4}{3}ab+\tfrac{1}{3}b^2-\tfrac{8}{3}b-2\\&+(\tfrac{2}{3}a^2+\tfrac{8}{3}a-\tfrac{2}{3}ab-\tfrac{1}{3}b^2-\tfrac{4}{3}b)\sqrt{-2},\\&-\tfrac{8}{3}a^2-\tfrac{32}{3}a-\tfrac{16}{3}ab+\tfrac{4}{3}b^2-\tfrac{20}{3}b-7\\&+(\tfrac{8}{3}a^2+\tfrac{20}{3}a-\tfrac{8}{3}ab-\tfrac{4}{3}b^2-\tfrac{16}{3}b+1)\sqrt{-2}\,\}\end{aligned}\tag{4.3}$$

with the property $D(8a+2+(4b+2)\sqrt{-2}\,)$ in $\mathbb{Q}(\sqrt{-2}\,)$. It remains to check for which $a\in\{3k,3k+1,3k+2\}$ and $b\in\{3l,3l+1,3l+2\}$, $k,l\in\mathbb{Z}$, set (4.3) has the elements of the ring $\mathbb{Z}[\sqrt{-2}\,]$. Equivalently, we have to find all pairs (a,b) which satisfy the condition

$$N(u)|(r^2-z)\cdot\bar{u},$$

i.e.

$$\begin{aligned}-2a^2-8a-4ab+b^2-2b-3&\equiv 0\,(\mathrm{mod}\ 3),\\2a^2+2a-2ab-b^2-4b+3&\equiv 0\,(\mathrm{mod}\ 3).\end{aligned}$$

It is easy to check that the above condition is satisfied for all $(a,b)\in\{(3k,3l),(3k,3l+2),(3k+1,3l+1),(3k+1,3l+2),(3k+2,3l),(3k+2,3l+1)\}$ for any $k,l\in\mathbb{Z}$. For each pair (a,b) there exists a set with the corresponding property. Let us determine the pairs (a,b) for which set (4.3) has at least two equal elements or some elements equal to zero.

If $(a,b)=(3k,3l)$, the above cases appear iff $(k,l)=(-1,0)$. But for that pair the set $\{1+\sqrt{-2}, 25-7\sqrt{-2}, 36-2\sqrt{-2}, 121-19\sqrt{-2}\}$ has the property $D(-22+2\sqrt{-2})$.

If $(a,b)=(3k,3l+2)$, the above cases appear iff $(k,l)=(0,-1)$. But for that pair the set $\{-1+\sqrt{-2}, 1+\sqrt{-2}, 7+\sqrt{-2}, 2,\}$ has the property $D(2-2\sqrt{-2})$.

If $(a,b)=(3k+1,3l+1)$, then all elements of the quadruple are distinct and nonzero.

If $(a,b)=(3k+1,3l+2)$, the above cases appear iff $(k,l)=(-1,-1)$. But for that pair the set $\{1+\sqrt{-2}, 15+5\sqrt{-2}, 49+9\sqrt{-2}, 10\}$ has the property $D(-14-2\sqrt{-2})$.

If $(a,b)=(3k+2,3l)$, the above cases appear iff $(k,l)=(-1,0)$. But for that pair the set $\{-7-7\sqrt{-2}, -1-3\sqrt{-2}, 1+\sqrt{-2}, -2\}$ has the property $D(-6+2\sqrt{-2})$.

If $(a,b)=(3k+2,3l+1)$, the above cases appear iff $(k,l)\in\{(-2,0),(-1,-1)\}$. But for these pairs, the set $\{-23-71\sqrt{-2}, -9-15\sqrt{-2}, -2-20\sqrt{-2}, 1+\sqrt{-2}\}$ has the property $D(-30+6\sqrt{-2})$ and the set $\{1+\sqrt{-2}, 7+1\sqrt{-2}, 25+1\sqrt{-2}, 6\}$ has the property $D(-6-6\sqrt{-2})$.

In a similar way for $e=3\sqrt{-2}$ we obtain the set

$$\begin{aligned}
&\{1-\sqrt{-2}, -\tfrac{2}{3}a^2-\tfrac{4}{3}a+\tfrac{4}{3}ab+\tfrac{1}{3}b^2+\tfrac{4}{3}b\\
&-(\tfrac{2}{3}a^2+\tfrac{4}{3}a+\tfrac{2}{3}ab-\tfrac{1}{3}b^2+\tfrac{2}{3}b+2)\sqrt{-2},\\
&-\tfrac{2}{3}a^2-\tfrac{4}{3}a+\tfrac{4}{3}ab+\tfrac{1}{3}b^2+\tfrac{10}{3}b+1\\
&-(\tfrac{2}{3}a^2+\tfrac{10}{3}a+\tfrac{2}{3}ab-\tfrac{1}{3}b^2+\tfrac{2}{3}b+1)\sqrt{-2},\\
&-\tfrac{8}{3}a^2-\tfrac{16}{3}a+\tfrac{16}{3}ab+\tfrac{4}{3}b^2+\tfrac{28}{3}b+1\\
&-(\tfrac{8}{3}a^2+\tfrac{28}{3}a+\tfrac{8}{3}ab-\tfrac{4}{3}b^2+\tfrac{8}{3}b+5)\sqrt{-2}\}
\end{aligned}\tag{4.4}$$

with the property $D(8a+2+(4b+2)\sqrt{-2})$ in $\mathbb{Q}(\sqrt{-2})$. Set (4.4) has the elements in the ring $\mathbb{Z}[\sqrt{-2}]$ for new pairs $(a,b)\in\{(3k+1,3l),(3k+2,3l+2)\}$ for any $k,l\in\mathbb{Z}$. Let us also determine the pairs (a,b) for which set (4.4) has at least two equal elements or some elements equal to zero.

If $(a,b)=(3k+1,3l)$, the above cases appear iff $(k,l)=(-1,0)$. But for that pair the set $\{1-\sqrt{-2}, 15-5\sqrt{-2}, 49-9\sqrt{-2}, 10\}$ has the property $D(-14+2\sqrt{-2})$.

If $(a,b)=(3k+2,3l+2)$, the above cases appear iff $(k,l)=(-1,-1)$. But for that pair the set $\{-7+7\sqrt{-2},-1+3\sqrt{-2},1+\sqrt{-2},-2\}$ has the property $D(-6-2\sqrt{-2}\,)$.

The case $(a,b)=(3k,3l+1)$ remains unsolved, i.e. the property $D(24k+2+(12l+6)\sqrt{-2}\,)$, $k,l\in\mathbb{Z}$.

□

Note that by multiplying elements of a $D(z)$-quadruple for $z=8a+2+(4b+2)\sqrt{-2}$ by $\sqrt{-2}$, by Lemma 2.1(ii) we obtain (with obvious substitution) a $D(z)$-quadruple for $z=16a+12+(8b+4)\sqrt{-2}$. Thus, Proposition 4.2 immediately implies

Proposition 4.3. *If z is of the form $z=16a+12+(8b+4)\sqrt{-2}$, then there exists at least one complex Diophantine quadruple with the property $D(z)$, for any $a,b\in\mathbb{Z}$, except maybe for $a\equiv 2\ (\mathrm{mod}\ 3)$ and $b\equiv 1\ (\mathrm{mod}\ 3)$.*

5. The proof of Theorem 1.1

If z is of the form $z=a+(2b+1)\sqrt{-2}$ or $z=4a+(4b+2)\sqrt{-2}$, then by Lemma 2.3 there does not exist a $D(z)$-quadruple (see also [1, Proposition 1]).

Let $z=a+b\sqrt{-2}$ and $a'=a \bmod 4$, $b'=b \bmod 4$. It remains to consider seven possibilities for (a',b'), for which we claim that there exists at least one $D(z)$-quadruple, except maybe for the possible exceptions listed in Theorem 1.1. The proof will follow from Lemma 2.4, i.e. the results of Abu Muriefah and Al-Rashed [1], and the propositions proved in Sections 3 and 4.

By Lemma 2.4, in the case $(a',b')=(3,0)$ $D(z)$-quadruples exist, except maybe for $z=-1$. From Propositions 3.1 and 3.2 we obtain that for z of the form $z=(4a+3)+(4b+2)\sqrt{-2}$ there exists at least one Diophantine quadruple with the property $D(z)$, which solves the case $(a',b')=(3,2)$. Summarizing results for $(a',b')=(3,0)$ and $(3,2)$, we conclude that for all z of the form $z=4a+3+2b\sqrt{-2}$ (with a possible exception of $z=-1$), there exists a $D(z)$-quadruple.

Consider the case $(a',b')=(1,0)$. The subcase corresponding to $b\equiv 0\ (\mathrm{mod}\ 8)$ was solved in [1]. By Lemma 2.4(ii), in that case a $D(z)$-quadruple exists, except maybe for $z=-3$. But the set $\{1-\sqrt{-2},1+\sqrt{-2},2,266\}$ is a $D(-3)$-quadruple. In Proposition 3.3 we solved the other subcase corresponding to $b\equiv 4\ (\mathrm{mod}\ 8)$.

Concerning the case $(a',b')=(1,2)$, in the subcase $a\equiv 1\ (\mathrm{mod}\ 8)$, by Lemma 2.4(iii) we know that there exists a $D(z)$-quadruple, except maybe for $z=1\pm 2\sqrt{-2}$. The subcase $a\equiv 5\ (\mathrm{mod}\ 8)$ is considered in Proposition

4.1, where we proved that a $D(z)$-quadruple exists, except maybe for $a \equiv 0 \pmod 3$, $b \equiv 1 \pmod 3$. Therefore, we conclude that in the case $(a', b') = (1, 2)$ there exists at least one $D(z)$-quadruple, with a possible exception of $z = 1 \pm 2\sqrt{-2}$ and the elements of the form $z = 24k + 5 + (12l + 6)\sqrt{-2}$, $k, l \in \mathbb{Z}$.

Summarizing results for $(a', b') = (1, 0)$ and $(1, 2)$, we conclude that for all z of the form $z = 4a+1+2b\sqrt{-2}$ there exists a $D(z)$-quadruple, except maybe for $z = 1 \pm 2\sqrt{-2}$ and $z = 24k + 5 + (12l + 6)\sqrt{-2}$, $k, l \in \mathbb{Z}$.

In the case $(a', b') = (2, 0)$, we can apply Lemma 2.1(ii) to already proved results for $D(z)$-quadruples for $z = 4a + 3 + 2b\sqrt{-2}$ and $z = 4a + 1 + 2b\sqrt{-2}$. Multiplying elements of these quadruples by $\sqrt{-2}$, we obtain (with obvious substitutions) $D(z)$-quadruples for z of the form $z = 8a + 2 + 4b\sqrt{-2}$ and $z = 8a + 6 + 4b\sqrt{-2}$. We conclude that in the case $(a', b') = (2, 0)$ there exists at least one $D(z)$-quadruple, except maybe for $z = 2$, $z = -2 \pm 4\sqrt{-2}$ and $z = 48k + 38 + (24l + 12)\sqrt{-2}$, $k, l \in \mathbb{Z}$. However, since $38 \equiv 6 \pmod{16}$ and $12 \equiv 4 \pmod 8$, Proposition 3.4 implies that for z of the form $z = 48k + 38 + (24l + 12)\sqrt{-2}$ there certainly exists at least one $D(z)$-quadruple. Furthermore, the set $\{-2, -1, 1, 2\}$ is a $D(2)$-quadruple, the set $\{-2-2\sqrt{-2}, -1-\sqrt{-2}, 1+\sqrt{-2}, 2+2\sqrt{-2}\}$ is a $D(-2+4\sqrt{-2})$-quadruple, while the set $\{-2+2\sqrt{-2}, -1+\sqrt{-2}, 1-\sqrt{-2}, 2-2\sqrt{-2}\}$ is a $D(-2-4\sqrt{-2})$-quadruple.

The case $(a', b') = (2, 2)$ is handled in Propositions 3.5 and 4.2. If $a \equiv 6 \pmod 8$, then by Proposition 3.5 there exists a $D(z)$-quadruple. If $a \equiv 2 \pmod 8$, then by Proposition 4.2 there exists at least one $D(z)$-quadruple, except maybe for $a \equiv 0 \pmod 3$, $b \equiv 1 \pmod 3$, i.e. for z of the form $z = 24k + 2 + (12l + 6)\sqrt{-2}$, $k, l \in \mathbb{Z}$.

It remains to consider the case $(a', b') = (0, 0)$. Here we will consider 4 subcases modulo 8. The subcase $a \equiv 0 \pmod 8$, $b \equiv 0 \pmod 8$ is exactly the statement of Lemma 2.4(iv). The subcase $a \equiv 0 \pmod 8$, $b \equiv 4 \pmod 8$ is handled in Propositions 3.6 and 3.7 and we proved that in this case there exists at least one $D(z)$-quadruple. In the remaining cases we will apply Lemma 2.1(ii). Multiplying elements of a $D(z)$-quadruple for $z = (4a + 2) + 4b\sqrt{-2}$ by $\sqrt{-2}$, we obtain (with obvious substitution) a $D(z)$-quadruple for $z = (8a + 4) + 8b\sqrt{-2}$, so our result already proved for the case $(a', b') = (2, 0)$ implies that if $a \equiv 4 \pmod 8$, $b \equiv 0 \pmod 8$, then there exists a $D(z)$-quadruple. Similarly, multiplying elements of a $D(z)$-quadruple for $z = (4a + 2) + (4b + 2)\sqrt{-2}$ by $\sqrt{-2}$, we obtain (with obvious substitution) a $D(z)$-quadruple for $z = (8a + 4) + (8b + 4)\sqrt{-2}$. Therefore, if $a \equiv 4 \pmod 8$, $b \equiv 4 \pmod 8$, then there exists a $D(z)$-quadruple, except maybe for z of the form $z = 48k + 44 + (24l + 12)\sqrt{-2}$, $k, l \in \mathbb{Z}$.

□

Remark 5.1. Concerning the three unsolved cases

$$z = 24a+2+(12b+6)\sqrt{-2},\ z = 24a+5+(12b+6)\sqrt{-2},\ z = 48a+44+(24b+12)\sqrt{-2},$$

by [10, Theorem 1], each of these elements can be represented as a difference of two squares of elements in $\mathbb{Z}[\sqrt{-2}]$. Thus, in an analogy to what is known in $\mathbb{Z}$ (see [5]) and certain quadratic fields (see [7, 12, 13, 15]), we might expect that for such z's there exists at least one $D(z)$-quadruple (with perhaps finitely many exceptions). On the other hand, it is possible to check that such quadruples cannot contain elements with very small norm (as in the formulas in Sections 3 and 4). We have found some formulas for quadruples for these z's containing e.g. the element $3 + \sqrt{-2}$ with norm 11. These formulas then necessarily involve some congruence conditions modulo 11 on a and b. At present, we have only some partial results, which are too technical to be presented here.

Remark 5.2. In Theorem 1.1 we also have three sporadic possible exceptions $z = -1$, $z = 1 + 2\sqrt{-2}$ and $z = 1 - 2\sqrt{-2}$. Note that $1 \pm 2\sqrt{-2} = -1 \cdot (1 \mp \sqrt{-2})^2$, thus the existence of $D(-1)$-quadruples would imply the existence of $D(1 + 2\sqrt{-2})$ and $D(1 - 2\sqrt{-2})$-quadruples. The problem of the existence of a $D(-1)$-quadruple in the ring $\mathbb{Z}$ has been studied by many authors. There is a conjecture that such a quadruple does not exist. In [11] and [9], it was proved that there does not exist a $D(-1)$-quintuple in $\mathbb{Z}$, and there are at most finitely many such quadruples.

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