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HARNACK INEQUALITY FOR NONLINEAR WEIGHTED EQUATIONS

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Abstract

In this paper, we prove the Harnack inequality for nonnegative weak solutions of the following nonlinear subelliptic equation

$$-\mathrm{div}\, A(x,u,\nabla u) = f(x,u,\nabla u)\,.$$

1 Introduction

In this paper, we prove Harnack inequality for nonnegative (weak) solutions of some class of nonlinear subelliptic equations
More precisely, we consider the equation

$$-\mathrm{div}\, A(x,u,\nabla u) = f(x,u,\nabla u) \tag{1}$$

in an open set $\Omega \subseteq \mathbb{R}^n$.

Throughout this paper we assume that A and f satisfy the following structural conditions with respect to the weight ω: there exist $a, b \geq 0$ and measurable functions f_1, f_2, f_3, g_2, g_3 and h_3 on $\mathbb{R}^n$ such that for a.e. $x \in \mathbb{R}^n$, $u \in \mathbb{R}$ and $\xi \in \mathbb{R}^n$

$$(\mathcal{S}) \begin{cases} |A(x,u,\xi)| \leq \omega(x)[b|\xi|^{p-1} + g_2(x)|u|^{p-1} + g_3(x)], \\ |f(x,u,\xi)| \leq \omega(x)[f_1(x)|\xi|^{p-1} + f_2(x)|u|^{p-1} + f_3(x)], \\ A(x,u,\xi)\xi \geq \omega(x)[a|\xi|^p + f_2(x)|u|^p - h_3(x)]. \end{cases}$$

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The weight ω is supposed to be p–admissible satisfying a property $(\mathcal{P})$ (which will be given in section 2).

In section 2, we point out the essential properties of p-admissible weight, satisfying the $\mathcal{P}$ property and we illustrate with some examples.

The main result is given in section 3, it concerns the Harnack inequality to the case of general operators in (1.1). We generalize the results given in [3], [5],[7] and [8].

We note the equation (1.1) covers equations of the form

$$-\mathrm{div}\,((\theta(x)\nabla u.\nabla u)^{\frac{p-2}{2}}\theta(x)\nabla u) = f(x,u,\nabla u)\theta(x), \tag{2}$$

where $\theta : \mathbb{R}^n \to GL(n,\mathbb{R})$ is a measurable matrix function satisfying for some $\lambda > 0$, the ellipticity conditions:

$$\lambda^{-1}|\xi|^2 \leq \theta(x)\xi.\xi \leq \lambda|\xi|^2 \; for \; x,\xi \in \mathbb{R}^n.$$

We remark that, if $\omega = 1$, $(\mathcal{S})$ is a condition required by Serrin in [9].

Fabes, Kenig and Serapioni proved in [5] the Harnack principle for non-negative (weak) solutions for the linear equation $-\partial_j(a_{i,j}\partial_i u) = 0$, where Ω is bounded and the coefficients $a_{i,j}$ satisfy the following ellipticity condition:

$$\lambda\omega(x)|\xi|^2 \leq a_{i,j}(x)\xi_i\xi_j \leq \Lambda\omega(x)|\xi|^2, \;\; 0 < \lambda < \Lambda,$$

with respect to a weight ω belonging to the Muckenhoupt class A_2 or the (Q.C) class.

De Cicco and Vivaldi proved in [3] the Harnack inequality in the case

$$-\partial_j(a_{i,j}\partial_i u + d_i u) + (b_i\partial_i u + cu) = 0,$$

where the matrix $(a_{i,j})$ satisfies the above ellipticity condition and the coefficients b_i,d_i and c belong to suitable Lebesgue spaces with respect to ω. The weight ω is supposed belonging either to the class A_2 or to (Q.C).

In [7] Heinonen, Kilpelanen and Martio proved the same result when $A(x,u,\nabla u) = A(x,\nabla u)$ and $f \equiv 0$.

In [2] Capponia, Danielli and Carofalo give a similar result when $\omega = 1$ for the equation $\sum_{j=1}^{j=m} X_j^* A_j(x,u,X_1u,...,X_mu) = f(x,u,X_1u,...,X_mu)$, where $X_1,...,X_m$ are $\mathcal{C}^\infty$ vector fields in $\mathbb{R}^n$, satisfying Hörmander's condition for hypoellipticity.

2 p–admissible weights

Throughout this paper Ω will denote an open subset of $\mathbb{R}^n$, $n \geq 2$ and $1 < p < \infty$.

Let ω be a locally integrable, nonnegative function in $\mathbb{R}^n$. Then a Radon measure μ is canonically associated with the weight ω,

$$\mu(E) = \int_E \omega(x)dx.$$

Thus $d\mu(x) = \omega(x)dx$, where dx is the n-dimentional Lebesgue measure.

Definition 2.1. *We say that ω (or μ) is p-admissible if the following four conditions are satisfied:*

1. $0 < \omega < \infty$ *almost everywhere in* $\mathbb{R}^n$ *and the measure* μ *is doubling, i.e. there is a constant* $c_1 > 0$ *such that*

$$\mu(2B) \leq c_1\mu(B),$$

whenever B *is a ball in* $\mathbb{R}^n$,

2. *If* D *is an open set and* $\varphi_i \in \mathcal{C}^\infty(D)$ *is a sequence of functions such that*

$$\int_D |\varphi_i|^p d\mu \to 0 \text{ and } \int_D |\varphi_i - v|^p d\mu \to 0 \ \text{ as } i \to \infty,$$

where v *is a vector-valued measurable function in* $L^p(D, \mu, \mathbb{R}^n)$, *then* $v = 0$.

3. ***The weighted Sobolev embedding Theorem :***
There are constants $\kappa > 1$ *and* $c_3 > 0$ *such that*

$$\left(\frac{1}{\mu(B)}\int_B |\varphi|^{\kappa p} d\mu\right)^{\frac{1}{\kappa p}} \leq c_3 r\left(\frac{1}{\mu(B)}\int_B |\nabla\varphi|^p d\mu\right)^{\frac{1}{p}},$$

whenever $B = B(x_0, r)$ *is a ball in* $\mathbb{R}^n$ *and* $\varphi \in \mathcal{C}_0^\infty(B)$.

4. ***The weighted Poincaré inequality:***
There is a constant $c_4 > 0$ *such that*

$$\int_B |\varphi - \varphi_B|^p d\mu \leq c_4 r^p \int_B |\nabla\varphi|^p d\mu,$$

whenever $B = B(x_0, r)$ *is a ball in* $\mathbb{R}^n$ *and* $\varphi \in \mathcal{C}^\infty(B)$ *is bounded. (Here* $\varphi_B = \frac{1}{\mu(B)}\int_B \varphi d\mu) = \oint_B \varphi d\mu$.*)*

Remark 2.1. *We note that in the classical situation (i.e.* $\omega = 1$*), the constant* c_1 *in* (1) *is equal to* 2^n.

Example 2.1. 1. *If $\omega = 1$ and μ is the Lebesgue measure. Then (1) is obvious, (3) is the ordinary Sobolev inequality and condition (4) is the classical Poincaré inequality.*

2. *Consider the Muckenhoupt class A_p, $(p > 1)$ which consists of all non-negative locally integrable functions ω in $\mathbb{R}^n$ such that:*

$$sup(\fint_B \omega dx)(\fint_B \omega^{\frac{1}{1-p}} dx)^{p-1} < +\infty,$$

where the supremum is taken over all balls B in $\mathbb{R}^n$. If ω belongs to A_p, then ω is p-admissible (see [7]).

3. *The weight ω is said to be in A_1 if there is a constant c such that:*

$$(\fint_B \omega dx) \leq c \; ess\, inf_B \omega,$$

for all balls B in $\mathbb{R}^n$.
Since $A_1 \subset A_p$ whenever $p > 1$, an A_1-weight is p-admissible for every $p > 1$.

4. *Consider the (QC) class of weights $\omega := |detF'|^{1-\frac{2}{n}}$, associated with the quasi-conformal map F in $\mathbb{R}^n$ ($detF'$ denotes the Jacobian determinant of F), by [7], ω is p-admissible.*

Definition 2.2. *For a function $\varphi \in \mathcal{C}^\infty(\Omega)$, we let*

$$\|\varphi\|_{1,p} = (\int_\Omega |\varphi|^p d\mu)^{\frac{1}{p}} + (\int_\Omega |\nabla\varphi|^p d\mu)^{\frac{1}{p}}.$$

The Sobolev space $H^{1,p}(\Omega, \mu)$ is defined to be the completion of $\{\varphi \in \mathcal{C}^\infty(\Omega) : \|\varphi\|_{1,p} < \infty\}$ with respect to the norm $\|\varphi\|_{1,p}$.

In other words, a function u is in $H^{1,p}(\Omega, \mu)$ if and only if u is in $L^p(\Omega, \mu)$ and there is a vector-valued function v in $L^p(\Omega, \mu)$ such that for some sequence $\varphi_i \in \mathcal{C}^\infty(\Omega)$, $\int_\Omega |\varphi_i - u|^p d\mu \to 0$ and $\int_\Omega |\nabla\varphi_i - v|^p d\mu \to 0$ as $i \to \infty$.
The function v is called the gradient of u in $H^{1,p}(\Omega, \mu)$ and denoted by $v = \nabla u$.
The space $H_0^{1,p}(\Omega, \mu)$ is the closure of $\mathcal{C}_0^\infty(\Omega)$ in $H^{1,p}(\Omega, \mu)$.
The corresponding local space $H_{Loc}^{1,p}(\Omega, \mu)$ is defined in the obvious manner: a function u is in $H_{Loc}^{1,p}(\Omega, \mu)$ if and only if u is in $H^{1,p}(\Omega', \mu)$ for each open set $\Omega' \subset \overline{\Omega} \subset \Omega$.
In all the next, let $Q > p$, such that $\kappa = \frac{Q}{Q-p}$ where κ is the constant satisfying the condition (3).

Definition 2.3. *We say that μ satisfies condition $(\mathcal{P})$ if for some $\widetilde{R_0} > 0$, there exists $M > 0$ such that for every $R \leq \frac{1}{2}\widetilde{R_0}$ we have $R\mu(B_{2R})^{\frac{-1}{Q}} \leq M$*

Example 2.2. 1. *If μ is the Lebesgue measure, then μ satisfies $(\mathcal{P})$. In fact we choose*

$$Q = \begin{cases} n & \text{if } p < n, \\ 2p & \text{if } p \geq n. \end{cases}$$

2. *We recall that by the open-end property (see [7]) , if $\omega \in A_p, p > 1$, then $\omega \in A_q$ for some $q < p$. Let $p_0 = \{q > 1 : \omega \in A_q\}$. We suppose that $p < np_0$. Thus by [7], any κ such that $1 < \kappa < \frac{n}{n-1} < \frac{n}{n-\frac{p}{p_0}}$ satisfies the weighted Sobolev embedding theorem and therefore ω is p-admissible. Using the strong doubling of A_p weights (see 15.5 in [7]):*

$$\mu(B_R) \geq c(\frac{2R}{\widetilde{R_0}})^{np}\mu(B_{\widetilde{R_0}}),\ for\ R \leq \frac{1}{2}\widetilde{R_0},$$

we get $R\mu(B_R)^{\frac{-1}{Q}} \leq c(\frac{2R}{\widetilde{R_0}})^{\frac{-np}{Q}+1}\mu(B_{\widetilde{R_0}})^{\frac{-1}{Q}}$. Hence, the p-admissible weight ω satisfies $(\mathcal{P})$.
In particular, if $\omega \in A_2$, then ω satisfies $(\mathcal{P})$.

3. *If $\omega \in A_1$, then by [7], $\omega \in A_p$ for $1 < p \leq n$. Using the last assertion we conclude that ω is p-admissible and satisfies $(\mathcal{P})$.*

4. *Let $\omega \in (QC)$, then by [4] and [1] there exists a positive constant ν such that for $0 < R < \frac{1}{2}\widetilde{R_0}$ we have:*

$$\mu(B_R) \geq \frac{1}{2}(\frac{2R}{R_0})^{\nu}\mu(B_{\frac{R_0}{2}}).$$

Note that for $\nu > 2$ we have $\kappa = \frac{\nu}{\nu-2}$ and so the property $(\mathcal{P})$ is satisfied. The same statement holds for $\nu \leq 2$.

3 Harnack inequality

In all the next, we suppose that ω is a p-admissible weight in $\mathbb{R}^n$ satisfying the property $(\mathcal{S})$ and $(\mathcal{P})$ with the resolvent integrability requirements on the functions f_i, g_i, h_i

(i) $g_2, g_3 \in L^r_{Loc}(\Omega, \mu)$ for $r = \frac{Q}{p-1}$,

(ii) $f_2, f_3, h_3 \in L^s_{Loc}(\Omega, \mu)$ for $s > \frac{Q}{p}$,

(iii) $f_1 \in L^t_{Loc}(\Omega, \mu)$ for $t > Q$.

Assumptions (ii) and (iii) allow to write for some $0 < \epsilon < 1$, $s = \frac{Q}{p-\epsilon}$ and $t = \frac{Q}{1-\epsilon}$.
From now on the letter ϵ will be only used with this meaning.
We propose then to prove Harnack inequality of nonnegative (weak) solution of nonlinear subelliptic equations (1.1).

Definition 3.1. *A function u in $H^{1,p}_{Loc}(\Omega, \mu)$ is a (weak) solution of the equation:*

$$-div\, A(x, u, \nabla u) = f(x, u, \nabla u)$$

if u is a solution of:

$$\int_\Omega A(x, u(x), \nabla u(x))\nabla\varphi(x)dx = \int_\Omega f(x, u(x), \nabla u(x))\varphi(x)dx$$

whenever $\varphi \in H^{1,p}_{Loc}(\Omega, \mu)$.

Remark 3.1. *Let $u \in H^{1,p}_{Loc}(\Omega, \mu)$, for $R > 0$ and $K = K(R)$, where*

$$K(R) = [(\mu(B_{2R}))^{\frac{\epsilon}{Q}} \|f_3\|_{L^s(B_R)} + \|g_3\|_{L^r(B_R)}]^{\frac{1}{p-1}} + [(\mu(B_{2R}))^{\frac{\epsilon}{Q}} \|h_3\|_{L^s(B_{2R})}]^{\frac{1}{p}}.$$

Then the function $\overline{u} = |u| + K$ satisfies $\nabla\overline{u} = \nabla|u|$ a.e. in Ω (see [7]). The assumptions $(\mathcal{S})$ may be written as follows.

$$(\mathcal{S}') \begin{cases} |A(x, u, \xi)| \leq \omega(x)(b|\xi|^{p-1} + \overline{g}_2(x)|\overline{u}|^{p-1}), \\ |f(x, u, \xi)| \leq \omega(x)(f_1(x)|\xi|^{p-1} + \overline{f}_2(x)|\overline{u}|^{p-1}), \\ A(x, u, \xi)\xi \geq \omega(x)(a|\xi|^p - \overline{f}_2(x)|\overline{u}|^p). \end{cases}$$

With $\overline{g}_2 = g_2 + K^{1-p}g_3$ and $\overline{f}_2 = f_2 + K^{1-p}f_3 + K^{-p}h_3$ satisfying

$$\|\overline{f}_2\|_{L^s(B_{2R})} \leq \|f_2\|_{L^s(B_{2R})} + 2(\mu(B_{2R}))^{\frac{-\epsilon}{Q}}$$

and

$$\|\overline{g}_2\|_{L^r(B_{2R})} \leq \|g\|_{L^r(B_{2R})} + 1.$$

Theorem 3.1. *Let u be a nonnegative (weak) solution of (1.1) in Ω and $\widetilde{R}_0$ for which μ satisfies the condition $(\mathcal{P})$ and $B(x, \widetilde{R}_0) \subset \Omega$, then there exists a positive constant c, such that, for any $2R \leq \widetilde{R}_0$, we have*

$$ess\, sup_{B_R} u \leq c(ess\, sup_{B_R} u + K(R))$$

Proof. **Step 1**
We define, for $q \geq 1$, $l > K$ and $\beta = pq - p + 1$

$$F(t) = \begin{cases} t^q & \text{if } K \leq t \leq l, \\ ql^{q-1}t - (q-1)l^q & \text{if } l \leq t. \end{cases}$$

and

$$G(t) = sgn(t)[F(|t| + K)(F'(|t| + K))^{p-1} - q^{p-1}K^\beta], \quad t \in \mathbb{R}.$$

Since $F \in \mathcal{C}^1(\mathbb{R})$, F' is bounded and $F(\overline{u}) \in L^p(\Omega, \mu)$, hence, by [7], Theorem 1.18 and Lemma 1.11, we get that $F(\overline{u})$ and $G \in H^{1,p}(\Omega, \mu)$.
Now, let $\eta \in \mathcal{C}_0^\infty(B_{2R}), 0 \leq \eta \leq 1$ and put $\varphi = \eta^p G(u)$.
Taking φ in (1.1) and using $(\mathcal{S}')$, we obtain:

$$\begin{aligned} 0 = \int_\Omega A(x, u(x), \nabla u(x))\nabla\varphi(x)dx - \int_\Omega f(x, u(x), \nabla u(x))\varphi(x)dx \\ \geq -p \textstyle\int_\Omega \eta^{p-1}|\nabla\eta||G(u)|(b|\nabla u|^{p-1} + \overline{g}_2|\overline{u}|^{p-1})d\mu + \\ \textstyle\int_\Omega \eta^p|G'(u)|(a|\nabla u|^p - \overline{f}_2\overline{u}^p)d\mu - \int_\Omega(f_1|\nabla u|^{p-1} + \overline{f}_2\overline{u}^{p-1})\eta^p|G(u)|d\mu. \end{aligned}$$

Using the fact that $|G(u)| \leq F(\overline{u})(F'(\overline{u}))^{p-1}$, we get $|G'(u)| \leq \beta q^{-1}(F'(\overline{u}))^p$ and $|G'(u)| \geq |F'(\overline{u})|^p$. Therefore

$$\begin{aligned} 0 \geq \int_\Omega & -pb|\nabla\eta F(\overline{u})|(\eta F'(\overline{u})|\nabla u|)^{p-1} - p\overline{g}_2(\eta\overline{u}F'(\overline{u}))^{p-1}|\nabla\eta F(\overline{u})| \\ & + a\eta^p|F'(\overline{u})\nabla u|^p - f_1F(\overline{u})\eta(|\nabla u|\eta F'(\overline{u}))^{p-1} \\ & -\overline{f}_2(\beta q^{-1}|\eta\overline{u}F'(\overline{u})|^p + (\overline{u}F(\overline{u})\eta)^{p-1}\eta F(\overline{u})d\mu. \end{aligned}$$

By setting $v = F(\overline{u})$ and since $\overline{u}F'(\overline{u}) \leq qF(\overline{u})$, we get

$$\begin{aligned} \int_\Omega \eta^p|\nabla v|^p d\mu \leq \frac{1}{a}(pb\int_\Omega(|\nabla\eta|v)(\eta|\nabla v|)^{p-1}d\mu + pq^{p-1}\int_\Omega \overline{g}_2(\eta v)^{p-1}|(\nabla\eta)v|d\mu + \\ + \int_\Omega f_1(\eta v)(|\nabla v|\eta)^{p-1}d\mu + (1+\beta)q^{p-1}\int_\Omega \overline{f}_2(\eta v)^p d\mu \end{aligned}$$

Hölder inequality yields

$$\int_\Omega(|\nabla\eta|v)(\eta|\nabla v|)^{p-1}d\mu \leq (\int_\Omega \eta^p|\nabla v|^p d\mu)^{\frac{p-1}{p}}(\int_\Omega v^p|\nabla\eta|^p)^{\frac{1}{p}}.$$

Step 2
Let $\alpha > 0$ such that $\frac{1}{\alpha} = 1 - \frac{p-1}{p} - \frac{1-\epsilon}{Q}$. Hölder's inequality gives us

$$\int_\Omega f_1(\eta v)(|\nabla v|\eta)^{p-1}d\mu \leq (\int_\Omega f_1^{\frac{Q}{1-\epsilon}}d\mu)^{\frac{1-\epsilon}{Q}}(\int_\Omega \eta^p|\nabla v|)^p d\mu)^{\frac{p-1}{p}}(\int_\Omega(\eta v)^\alpha d\mu)^{\frac{1}{\alpha}}$$

and

$$(\int_\Omega (\eta v)^\alpha d\mu)^{\frac{1}{\alpha}} \leq (\int_\Omega (\eta v)^p d\mu)^{\frac{\epsilon}{p}} (\int_\Omega (\eta v)^{\kappa p} d\mu)^{\frac{1-\epsilon}{\kappa p}}.$$

Using the Sobolev embedding property, we get

$$\int_\Omega (\eta v)^{\kappa p} d\mu)^{\frac{1-\epsilon}{\kappa p}} \leq (2c_3)^{1-\epsilon} R^{1-\epsilon} (\mu(B_{2R}))^{\frac{\epsilon-1}{Q}} [(\int_{B_{2R}} \eta^p |\nabla v|)^p d\mu)^{\frac{1-\epsilon}{p}} +$$

$$+(\int_{B_{2R}} v^p |\nabla \eta|)^p d\mu)^{\frac{1-\epsilon}{p}}]$$

Finally, we get:

$$\int_\Omega f_1(\eta v)(|\nabla v|\eta)^{p-1} d\mu \leq c_R (\int_{B_{2R}} f_1^{\frac{Q}{1-\epsilon}} d\mu)^{\frac{1-\epsilon}{Q}} (\int_\Omega \eta^p |\nabla v|^p d\mu)^{\frac{p-1}{p}}.$$

$$(\int_\Omega (\eta v)^p d\mu)^{\frac{\epsilon}{p}} [(\int_\Omega \eta^p |\nabla v|)^p d\mu)^{\frac{1-\epsilon}{p}} + (\int_\Omega v^p |\nabla \eta|^p d\mu)^{\frac{1-\epsilon}{p}}]$$

By analogous arguments we can estimate the terms $\int_\Omega \overline{f}_2 (\eta v)^p d\mu$ and $\int_\Omega \overline{g}_2 |v||\nabla \eta|(\eta v)^{p-1} d\mu$ as follows: For $\frac{1}{\alpha} = 1 - \frac{p-\epsilon}{Q}$

$$\int_\Omega \overline{f}_2 (\eta v)^p d\mu \leq (\int_{B_{2R}} \overline{f}_2^{\frac{Q}{p-\epsilon}})^{\frac{p-\epsilon}{Q}} (\int_{B_{2R}} (\eta v)^{p\alpha} d\mu)^{\frac{1}{\alpha}}$$

and

$$(\int_{B_{2R}} (\eta v)^{p\alpha} d\mu)^{\frac{1}{\alpha}} \leq (\int_{B_{2R}} (\eta v)^{\kappa p} d\mu)^{\frac{p-\epsilon}{\kappa p}} (\int_{B_{2R}} (\eta v)^p d\mu)^{\frac{\epsilon}{p}}.$$

Then, by Sobolev embedding property, we obtain

$$\int_\Omega (\eta v)^{\kappa p} d\mu)^{\frac{p-\epsilon}{\kappa p}} \leq (2c_3 R)^{p-\epsilon} (\mu(B_{2R}))^{\frac{-p+\epsilon}{Q}} [(\int_{B_{2R}} (|\nabla \eta| v)^p d\mu)^{\frac{p-\epsilon}{p}} +$$

$$+(\int_{B_{2R}} (|\nabla v|\eta)^p d\mu)^{\frac{p-\epsilon}{p}}].$$

Finally

$$\int_\Omega \overline{f}_2 (\eta v)^p d\mu \leq c'_R (\int_{B_{2R}} \overline{f}_2^{\frac{Q}{p-\epsilon}})^{\frac{p-\epsilon}{Q}} (\int_\Omega (\eta v)^p d\mu)^{\frac{\epsilon}{p}} [(\int_\Omega \eta^p |\nabla v|^p d\mu)^{\frac{p-\epsilon}{p}} +$$

$$+(\int_\Omega |\nabla \eta|^p v^p d\mu)^{\frac{p-\epsilon}{p}}].$$

Where $c'_R = (2c_3R)^{p-\epsilon}(\mu(B_{2R}))^{\frac{-p+\epsilon}{Q}}$.
Similarly

$$\int_\Omega \overline{g}_2 v|\nabla\eta|(\eta v)^{p-1}d\mu \leq (\int_\Omega \overline{g}_2^{\frac{Q}{p-1}}d\mu)^{\frac{p-1}{Q}}(\int_\Omega v^p|\nabla\eta|^p d\mu)^{\frac{1}{p}}(\int_\Omega |v\eta|^{\kappa p}d\mu)^{\frac{p-1}{\kappa p}} \leq$$

$$\leq c''_R(\int_\Omega \overline{g}_2^{\frac{Q}{p-1}}d\mu)^{\frac{p-1}{Q}}(\int_\Omega v^p|\nabla\eta|^p d\mu)^{\frac{1}{p}}[(\int_\Omega v^p|\nabla\eta|^p d\mu)^{\frac{p-1}{p}}+(\int_\Omega \eta^p|\nabla v|^p d\mu)^{\frac{p-1}{p}}],$$

where $c''_R = (2c_3R)^{p-1}(\mu(B_{2R}))^{\frac{1-p}{Q}}$. Therefore we get

$$\|\eta\nabla v\|_p^p \leq c(\|\eta\nabla v\|_p^{p-1}\|v\nabla\eta\|_p + \|\eta v\|_p^\epsilon\|\eta\nabla v\|_p^{p-1}(\|\eta\nabla v\|_p^{1-\epsilon} + \|v\nabla\eta\|_p^{1-\varepsilon})+$$
$$+ q^{p-1}\|v\nabla\eta\|_p(\|v\nabla\eta\|_p^{p-1} + \|\eta\nabla v\|_p^{p-1}) + (1+\beta)q^{p-1}\|v\eta\|_p^\epsilon(\|v\nabla\eta\|_p^{p-\epsilon}+$$
$$\|\eta\nabla v\|_p^{p-\epsilon})).$$

By setting $z = \frac{\|\eta\nabla v\|_p}{\|v\nabla\eta\|_p}$ and $\zeta = \frac{\|v\eta\|_p}{\|v\nabla\eta\|_p}$, we get that

$$z^p \leq c[z^{p-1} + q^{p-1}(1+z^{p-1}) + z^{p-1}\zeta^\epsilon(1+z^{\epsilon-1}) + (1+\beta)q^{p-1}\zeta^\epsilon(1+z^{p-\epsilon})].$$

So, using the results in ([9]), we get $z \leq cq^{\frac{p}{\epsilon}}(1+\zeta)$. Hence we obtain

$$\|\eta(\nabla v)\|_p \leq cq^{\frac{p}{\epsilon}}(\|\eta v\|_p + \|v\nabla\eta\|_p),$$

for $c = c(p, c_3, \|f_1\|_{L^t(\Omega)}, \|f_2\|_{L^s(\Omega)}, \|g_2\|_{L^r(\Omega)})$.
We use the embedding theorem which gives us

$$\int_{B_{2R}} |\eta v|^{\kappa p}d\mu)^{\frac{1}{\kappa p}} \leq 2c_3R(\mu(B_{2R}))^{\frac{-1}{Q}}(\int_{B_{2R}} |\nabla(\eta v)|^p d\mu)^{\frac{1}{p}} \leq$$
$$\leq (2c_3R)(\mu(B_{2R}))^{\frac{-1}{Q}}[cq^{\frac{p}{\epsilon}}(\|\eta v\|_p + \|v\nabla\eta\|_p) + \|v\nabla\eta\|_p].$$

Then

$$(\int_{B_{2R}} |\eta v|^{\kappa p}d\mu)^{\frac{1}{\kappa p}} \leq c_R(\mu(B_{2R}))^{\frac{-1}{Q}}q^{\frac{p}{\epsilon}}[(\int_{B_{2R}} |\eta v|^p d\mu)^{\frac{1}{p}} + (\int_{B_{2R}} |v|^p|\nabla\eta|^p d\mu)^{\frac{1}{p}}].$$

Step 3
Let a and b be real numbers satisfying $1 < a < b \leq 2$.
Let the function η be chosen so that $\eta \in \mathcal{C}_0^\infty(B_{bR})$ with $\eta = 1$ in B_{aR} and $|\nabla\eta| \leq \frac{c}{(b-a)R}$. Setting this function in [9], it yields

$$(\int_{B_{aR}} v^{\kappa p}d\mu)^{\frac{1}{\kappa p}} \leq \frac{cq^{\frac{p}{\epsilon}}}{(b-a)}(\mu(B_{2R}))^{-\frac{1}{Q}}(\int_{B_{bR}} v^p d\mu)^{\frac{1}{p}}.$$

At this point, if we let $l \to \infty$ in the definition of F, we get $v = F(\overline{u})$ tends to $\overline{u}^q$ monotonically. Hence, we obtain

$$(\int_{B_{aR}} \overline{u}^{\kappa pq} d\mu)^{\frac{1}{\kappa pq}} \leq \frac{cq^{\frac{p}{q\epsilon}}}{(b-a)^{\frac{1}{q}}} (\mu(B_{2R}))^{-\frac{1}{qQ}} (\int_{B_{bR}} \overline{u}^{pq} d\mu)^{\frac{1}{pq}}.$$

By Moser's iteration technique, we easily infer

$$\operatorname{ess\,sup}_{B_R} \overline{u} \leq c(\frac{1}{\mu(B_{2R})} \int_{B_{2R}} \overline{u}^p d\mu)^{\frac{1}{p}}.$$

An extrapolation argument shows that the exponent p can be replaced by any positive number α.
Using the fact that $\overline{u} = |u| + K = |u| + K(R)$, we have the conclusion.
Step 4
We assume at first that $u \geq \alpha \geq 0$ in Ω. We set $K = K(R)$ and $\overline{u} = |u| + K$. Let $\eta \in \mathcal{C}_0^\infty(B_{2R})$. Then the function $\varphi = \eta^p \overline{u}^{1-p}$ belongs to $H_0^{1,p}(\Omega, \mu)$. Using $(\mathcal{S}')$, we have

$$\begin{aligned}0 = p \int_\Omega A(x,u,\nabla u)\eta^{p-1}\overline{u}^{1-p}\nabla\eta dx + (1-p)\int_\Omega A(x,u,\nabla u)\eta^p\overline{u}^{-p}\nabla u dx - \\ - \textstyle\int_\Omega f(x,u,\nabla u)\eta^p\overline{u}^{1-p}dx \leq \\ \leq p\textstyle\int_\Omega \eta^{p-1}\overline{u}^{1-p}\nabla\eta(b|\nabla u|^{p-1} + \overline{g}_2|\overline{u}|^{p-1})d\mu + \\ +(1-p)\textstyle\int_\Omega \eta^p\overline{u}^{-p}(a|\nabla u|^p - \overline{f}_2|\overline{u}|^p)d\mu + \int_\Omega(f_1|\nabla u|^{p-1} + \overline{f}_2|\overline{u}|^{p-1})\eta^p\overline{u}^{1-p}d\mu.\end{aligned}$$

Denoting $v = \ln \overline{u}$, we obtain:

$$(p-1)a\int_\Omega \eta^p|\nabla v|^p d\mu \leq pb\int_\Omega(|\nabla v|\eta)^{p-1}\nabla\eta d\mu + p\int_\Omega \overline{g}_2\eta^{p-1}|\nabla\eta|d\mu +$$

$$+p\int_\Omega \overline{f_2}\eta^p d\mu + \int_\Omega (f_1(\eta|\nabla v|)^{p-1}\eta d\mu.$$

Let $\eta \in \mathcal{C}_0^\infty(B_{2R})$ such that $\eta = 1$ on B_R and $|\nabla\eta| \leq \frac{C}{R}$.
Using Hölder inequality and the Sobolev embedding Theorem, we get the following inequalities:

$$\begin{aligned}\int_\Omega \overline{g}_2\eta^{p-1}|\nabla\eta|d\mu \leq \|\overline{g}_2\|_{L^r(B_{2R})}(\int_{B_{2R}}\eta^{\kappa p}d\mu)^{\frac{p-1}{\kappa}}(\int_{B_{2R}}|\nabla\eta|^p)^{\frac{1}{p}} \leq \\ \leq \|\overline{g}_2\|_{L^r(B_{2R})}cR^{p-1}(\mu(B_{2R}))^{\frac{1-p}{Q}}(\int_{B_{2R}}|\nabla\eta|^p) \leq \\ \leq \frac{c}{R}(\mu(B_{2R}))^{\frac{Q+1-p}{Q}}\|\overline{g}_2\|_{L^r(B_{2R})},\end{aligned}$$

$$\int_\Omega \overline{f_2}\eta^p d\mu \le \|\overline{f}_2\|_{L^{\frac{Q}{p}}(B_{2R})}(\int_{B_{2R}} \eta^{\kappa p} d\mu)^{\frac{1}{\kappa}} \le$$
$$\le c\|\overline{f}_2\|_{L^{\frac{Q}{p}}(B_{2R})}(\mu(B_{2R}))^{\frac{1}{\kappa}},$$

$$\int_\Omega f_1\eta(\eta|\nabla v|)^{p-1}d\mu \le \|f_1\|_{L^Q(B_{2R})}(\int_{B_{2R}} \eta^{\kappa p}d\mu)^{\frac{1}{\kappa p}}(\int_{B_{2R}} \eta^p|\nabla v|^p d\mu)^{\frac{p-1}{p}} \le$$

$$\le c\|f_1\|_{L^Q(B_{2R})}(\mu(B_{2R}))^{\frac{1}{\kappa p}}(\int_{B_{2R}} \eta^p|\nabla v|^p d\mu)^{\frac{p-1}{p}}$$

and

$$\int_\Omega (\eta|\nabla v|)^{p-1}\nabla\eta d\mu \le (\int_{B_{2R}} |\nabla\eta|^p d\mu)^{\frac{1}{p}}(\int_{B_{2R}} (\eta|\nabla v|)^p d\mu)^{\frac{p-1}{p}} \le$$
$$\le \frac{c}{R}(\mu(B_{2R}))^{\frac{1}{p}}(\int_{B_{2R}} (\eta|\nabla v|)^p d\mu)^{\frac{p-1}{p}}.$$

Finally

$$\int_\Omega \eta^p|\nabla v|^p d\mu \le c[R^{-1}(\mu(B_{2R})^{\frac{1}{p}}(\int_{B_{2R}} (\eta|\nabla v|)^p d\mu)^{\frac{p-1}{p}}+$$

$$+R^{-1}(\mu(B_{2R})^{\frac{Q+1-p}{Q}}\|\overline{g}_2\|_{L^r(B_{2R})}+$$

$$+\|\overline{f}_2\|_{L^{\frac{Q}{p}}(B_{2R})}(\mu(B_{2R})^{\frac{1}{\kappa}} + \|f_1\|_{L^Q(B_{2R})}(\mu(B_{2R})^{\frac{1}{\kappa p}}(\int_{B_{2R}} \eta^p|\nabla v|^p d\mu)^{\frac{p-1}{p}}].$$

If we set $z = (\int_{B_{2R}} \eta^p|\nabla v|^p d\mu)^{\frac{1}{p}}$ we obtain

$$z^p \le c[R^{-1}(\mu(B_{2R})^{\frac{1}{p}}z^{p-1} + R^{-1}(\mu(B_{2R})^{\frac{Q+1-p}{Q}}\|\overline{g}_2\|_{L^r(B_{2R})}+$$

$$+(\mu(B_{2R})^{\frac{1}{\kappa}}\|\overline{f}_2\|_{L^{\frac{Q}{p}}(B_{2R})} + +(\mu(B_{2R})^{\frac{1}{\kappa p}}\|f_1\|_{L^Q(B_{2R})}z^{p-1}].$$

Using [9], we get

$$z \le \frac{c}{R}(\mu(B_{2R})^{\frac{1}{p}}[1 + R^{\frac{p-1}{p}}(\mu(B_{2R})^{\frac{1-p}{pQ}} + R(\mu(B_{2R})^{\frac{-1}{Q}}].$$

So

$$\int_{B_R} |\nabla v|^p d\mu \le c\mu(B_{2R})R^{-p}.$$

Step 5
The Poincaré inequality yields:

$$\oint_{B_R} |v - v_B|^p d\mu \le c.$$

It follows, from the John-Nirenberg Lemma 3.6 in [7], there are constants c_1 and c_2 such that

$$(\oint_{B_R} \exp(c_1 v) d\mu)(\oint_{B_R} \exp(-c_1 v) d\mu) \le (\oint_{B_R} \exp(c_1(v_B - v)) d\mu).$$

$$(\oint_{B_R} \exp(c_1(v - v_B)) d\mu) \le c_2^2.$$

Hence $(\oint_{B_R} \overline{u}^{c_1} d\mu)^{\frac{1}{c_1}} \le c(\oint_{B_R} \overline{u}^{-c_1} d\mu)^{\frac{1}{-c_1}}$.
At this point, we choose $\beta \le 1-p \le 0$ and set $q = \frac{p+\beta-1}{p}$. Let $\varphi \in H_0^{1,p}(\Omega, \mu)$. Proceeding as in the proof of Step 1, we get

$$\begin{aligned} 0 = p \int_\Omega A(x,u,\nabla u)\eta^{p-1}(\nabla\eta)\overline{u}^\beta dx + \beta \int_\Omega A(x,u,\nabla u)\eta^p \overline{u}^{\beta-1}\nabla u dx \le \\ - \int_\Omega f(x,u,\nabla u)\eta^p \overline{u}^\beta dx \le \\ \le pb \int_\Omega (|\nabla u|\eta)^{p-1}\nabla\eta\overline{u}^\beta d\mu + p\int_\Omega \overline{g}_2(\overline{u}\eta)^{p-1}\nabla\eta\overline{u}^\beta d\mu + \beta a \int_\Omega (|\nabla u|\eta)^p \overline{u}^{\beta-1} d\mu - \\ - \beta \int_\Omega \overline{f}_2(|\overline{u}|\eta)^p \overline{u}^{\beta-1} d\mu + \int_\Omega f_1 |\nabla u|^{p-1}\eta^p \overline{u}^\beta d\mu + \int_\Omega \overline{f}_2 |\overline{u}|^{p+\beta-1}\eta^p d\mu. \end{aligned}$$

By setting $v = \overline{u}^q$ we have

$$\begin{aligned} |\beta| a \int_\Omega \eta^p |\nabla v|^p d\mu \le pb|q| \int_\Omega (\eta|\nabla v|)^{p-1} v|\nabla\eta| d\mu + p|q|^p \int_\Omega \overline{g}_2 v|\nabla\eta|(v\eta)^{p-1} d\mu \\ +(1+\beta)q^p \int_\Omega \overline{f}_2(\eta v)^p d\mu + |q| \int_\Omega f_1(\eta v)(\eta|\nabla v|)^{p-1} d\mu. \end{aligned}$$

As in the estimates in the proof of Step 1, we get

$$(\int_{B_{2R}} |\eta v|^{\kappa p})^{\frac{1}{\kappa p}} \le cR(\mu(B_{2R}))^{\frac{-1}{Q}}(1+|q|)^{\frac{p}{\epsilon}}[\int_{B_{2R}} |\eta v|^p)^{\frac{1}{p}} + \int_{B_{2R}} v|\nabla\eta|^p)^{\frac{1}{p}}].$$

Let $1 \le a < b \le 2$ and $\eta \in \mathcal{C}_0^\infty(B_{bR})$ such that $\eta = 1$ on B_{aR} and $|\nabla\eta| \le \frac{c}{(b-a)R}$. We finally obtain

$$\left(\int_{B_{aR}} \overline{u}^{\kappa pq} d\mu\right)^{\frac{1}{\kappa pq}} \ge c\frac{(1+|q|)^{\frac{p}{\epsilon q}}}{(b-a)^{\frac{1}{q}}}(\mu(B_{2R}))^{\frac{-1}{Qq}} \left(\int_{B_{bR}} \overline{u}^{pq} d\mu\right)^{\frac{1}{pq}}.$$

Then, by Moser's iteration procedure, we get

$$\operatorname{ess\,inf}_{B_R} \overline{u} \geq c \left(\frac{1}{\mu(B_{2R})} \int_{B_{2R}} \overline{u}^{-p_0} d\mu \right)^{\frac{-1}{p_0}}$$

for some constant $p_0 > 0$. The proof of the theorem is achieved by getting previous steps. □

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