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FIXED POINT RESULTS FOR φ-CONTRACTIONS ON A SET WITH TWO SEPARATING GAUGE STRUCTURES

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Abstract

The purpose of this article is to present some fixed point theorems for Ćirić-type generalized φ-contractions on a set with two separating gauge structures. Fixed point theorems and a homotopy result are given in Section 2. Then, as applications, some existence results for a multivalued Cauchy problem and a Volterra-type integral inclusion are presented in Section 3. Our theorems extend and generalize some previous results in the literature, such as: [1], [3], [7], [10], [11], [13].

1 Introduction

Throughout this paper X will denote a gauge space endowed with a separating gauge structure $\mathcal{P} = \{p_\alpha\}_{\alpha \in A}$, where A is a directed set (see [8] for definitions).

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A sequence (x_n) of elements in X is said to be Cauchy if for every $\varepsilon > 0$ and $\alpha \in A$, there is an N with $p_\alpha(x_n, x_{n+p}) \leq \varepsilon$ for all $n \geq N$ and $p \in \mathbb{N}$. The sequence (x_n) is called convergent if there exists an $x_0 \in X$ such that for every $\varepsilon > 0$ and $\alpha \in A$, there is an N with $p_\alpha(x_0, x_n) \leq \varepsilon$ for all $n \geq N$.

A gauge space is called sequentially complete if any Cauchy sequence is convergent. A subset of X is said to be sequentially closed if it contains the limit of any convergent sequence of its elements.

If $\mathcal{P} = \{p_\alpha\}_{\alpha \in A}$ and $\mathcal{Q} = \{q_\beta\}_{\beta \in B}$ are two separating gauge structures $(A,\ B$ are directed sets), then for $r = \{r_\beta\}_{\beta \in B} \in (0, \infty)^B$ and $x_0 \in X$ we will denote by $\overline{B}_q^p(x_0, r)$ the closure of $B_q(x_0, r)$ in $(X, \mathcal{P})$, where

$$B_q(x_0, r) = \{x \in X | q_\beta(x, x_0) < r_\beta \text{ for all } \beta \in B\}.$$

Let $P((X, \mathcal{P}))$ be the set of all nonempty subsets of X regarding to the separating gauge structure $\mathcal{P}$. We will use the following symbols where is no place to confusion:

$$P(X) := \{Y \in \mathcal{P}(X) |\ Y \neq \emptyset\}; P_b(X) := \{Y \in P(X) |\ Y \text{ is bounded }\};$$

$$P_{cl}(X) := \{Y \in P(X) |\ Y \text{ is closed }\}.$$

Let us define the gap functional between Y and Z in the $(X, \mathcal{P})$ gauge space

$$D_\alpha : P(X) \times P(X) \to \mathbb{R}_+ \cup \{+\infty\},\ D_\alpha(Y, Z) = \inf\{p_\alpha(y, z) \mid y \in Y,\ z \in Z\}$$

(in particular, if $x_0 \in X$ then $D_\alpha(x_0, Z) := D_\alpha(\{x_0\}, Z))$ and the (generalized) Pompeiu-Hausdorff functional

$$H_\alpha : P(X) \times P(X) \to \mathbb{R}_+ \cup \{+\infty\}, H_\alpha(Y, Z) = \max\{\sup_{y \in Y} D_\alpha(y, Z), \sup_{z \in Z} D_\alpha(Y, z)\}.$$

If $F : X \to P(X)$ is a multivalued operator, then $x \in X$ is called fixed point for F if and only if $x \in F(x)$. The set $FixF := \{x \in X | x \in F(x)\}$

is called the fixed point set of F. The multivalued operator F is said to be closed if $GraphF := \{(x, y) \in X \times X |\ y \in F(x)\}$ is closed in $X \times X$.

The aim of this paper is to give some (local and global) fixed point theorems for multivalued operators on a set endowed with two separating gauge structures. As a consequence we also obtain a homotopy result. Then, as applications, some existence results for a multivalued Cauchy problem and a Volterra-type integral inclusion are presented in Section 3. Our theorems extend and generalize some previous results (in metric spaces as well as in gauge spaces) given by: R.P. Agarwal, J. Dshalalow, D. O'Regan [1], L.B. Ćirić [7], M. Frigon [10], [11], T. Lazăr, D. O'Regan, A. Petruşel [13], R.P. Agarwal, D. O'Regan, M. Sambandham [3].

2 The main results

Ćirić ([7]) proved that if (X, d) is a complete metric space, $F : X \to P_{cl}(X)$ is a multivalued operator and there exists $\alpha \in [0, 1]$ such that $H(F(x), F(y)) \leq \alpha \cdot M_d^F(x, y)$, for every $x, y \in X$ (where $M_d^F(x, y) = \max\{d(x, y), D(x, F(x)), D(y, F(y)), \frac{1}{2}[D(x, F(y))+D(y, F(x))]\}$). Then $FixF \neq \emptyset$ and for every $x \in X$ and $y \in F(x)$ there exists a sequence $(x_n)_{n\in\mathbb{N}}$ such that

(1) $x_0 = x,\ x_1 = y$;

(2) $x_{n+1} \in F(x_n),\ n \in \mathbb{N}$;

(3) $x_n \overset{d}{\to} x^* \in F(x^*)$, for every $n \to \infty$.

V.G. Angelov [4] introduced the notion of generalized φ-contractive single-valued map in gauge spaces in 1987, meanwhile the concept for multivalued operators was given in 1998 (see V.G. Angelov [5]). In what follows we will give a local version of Ćirić's theorem ([7]) for generalized φ-contractions on a set with two separating gauge structures.

Theorem 2.1. *Let X be a nonempty set endowed with two separating gauge structures $\mathcal{P} = \{p_\alpha\}_{\alpha \in A}$, $\mathcal{Q} = \{q_\beta\}_{\beta \in B}$ (A, B are directed sets), $r = \{r_\beta\}_{\beta \in B} \in (0, \infty)^B$, $x_0 \in X$ and $F : \overline{B}^p_q(x_0, r) \to P(X)$. We suppose that:*

(i) $(X, \mathcal{P})$ is a sequentially complete gauge space;

(ii) there exists a function $\psi : A \to B$ and $c = \{c_\alpha\}_{\alpha \in A} \in (0, \infty)^A$ such that

$$p_\alpha(x, y) \leq c_\alpha \cdot q_{\psi(\alpha)}(x, y), \text{ for every } \alpha \in A \text{ and } x, y \in \overline{B}^p_q(x_0, r).$$

(iii) $F : \overline{B}^p_q(x_0, r) \to P(X)$ has closed graph;

(iv) Suppose that for each $\beta \in B$ there exists a continuous function $\varphi_\beta : [0, \infty) \to [0, \infty)$, with $\varphi_\beta(t) < t$, for every $t > 0$ and φ_β is strictly increasing on $(0, r_\beta]$ such that for $x, y \in \overline{B}^p_q(x_0, r)$ we have

$$H_\beta(F(x), F(y)) \leq \varphi_\beta(M^F_\beta(x, y)),$$

where $M^F_\beta(x, y) = \max\{q_\beta(x, y), D_\beta(x, F(x)), D_\beta(y, F(y)), \frac{1}{2}[D_\beta(x, F(y)) + D_\beta(y, F(x))]\}$.

In addition assume for each $\beta \in B$ that

$$\Phi_\beta \text{ is strictly increasing on } [0, \infty), \text{ where } \Phi_\beta(x) = x - \varphi_\beta(x), \tag{2.1}$$

$$\sum_{i=1}^{\infty} \varphi^i_\beta(t) < \infty, \text{ for } t \in (0, r_\beta - \varphi(r_\beta)] \tag{2.2}$$

and

$$\sum_{i=1}^{\infty} \varphi^i_\beta(r_\beta - \varphi_\beta(r_\beta)) \leq \varphi_\beta(r_\beta) \tag{2.3}$$

hold. Finally suppose the following two conditions are satisfied:

(i) For each $\beta \in B$, we have: $D_\beta(x_0, F(x_0)) < r_\beta - \varphi_\beta(r_\beta)$ (2.4)

and

(ii) For every $x \in \overline{B}^p_q(x_0, r)$ and every $\varepsilon = \{\varepsilon_\beta\}_{\beta \in B} \in (0, \infty)^B$, (2.5)

there exists $y \in F(x)$ with $q_\beta(x, y) \leq D_\beta(x, F(x)) + \varepsilon_\beta$, for every $\beta \in B$.

Then F has a fixed point.

Proof. From (2.4) we may choose $x_1 \in F(x_0)$ with

$$q_\beta(x_0, x_1) < r_\beta - \varphi_\beta(r_\beta), \text{ for every } \beta \in B. \tag{2.6}$$

Then $x_1 \in \overline{B}^p_q(x_0, r)$.

For $\beta \in B$ choose $\varepsilon_\beta > 0$ with $\Phi_\beta^{-1}(\varepsilon_\beta) < r_\beta$ so that

$$\varphi_\beta(q_\beta(x_0, x_1) + \varepsilon_\beta) + \varepsilon_\beta + \varphi_\beta(\Phi_\beta^{-1}(\varepsilon_\beta)) < \varphi_\beta(r_\beta - \varphi_\beta(r_\beta)). \tag{2.7}$$

This is possible from (2.6) and the fact that φ_β is strictly increasing on $(0, r_\beta]$.

From (2.16) we can choose $x_2 \in F(x_1)$ so that for every $\beta \in B$ we have

$$q_\beta(x_1, x_2) \leq D_\beta(x_1, F(x_1)) + \varepsilon_\beta \leq H_\beta(F(x_0), F(x_1)) + \varepsilon_\beta. \tag{2.8}$$

We want to see if

$$q_\beta(x_1, x_2) \leq \varphi_\beta(q_\beta(x_0, x_1) + \varepsilon_\beta) + \varepsilon_\beta + \varphi_\beta(\Phi_\beta^{-1}(\varepsilon_\beta)). \tag{2.9}$$

We can notice that

$$H_\beta(F(x_0), F(x_1)) + \varepsilon_\beta \leq \varphi_\beta(M_\beta(x_0, x_1)) + \varepsilon_\beta. \tag{2.10}$$

Let us consider $\gamma_\beta = \max\{q_\beta(x_0, x_1), D_\beta(x_0, F(x_0)), D_\beta(x_1, F(x_1)), \frac{1}{2}[D_\beta(x_0, F(x_1)) + D_\beta(x_1, F(x_0))]\}$.

If $\gamma_\beta = q_\beta(x_0, x_1)$ then from (2.8) and (2.10) we have

$$\begin{aligned} q_\beta(x_1, x_2) &\leq H_\beta(F(x_0), F(x_1)) + \varepsilon_\beta \leq \varphi_\beta(q_\beta(x_0, x_1)) + \varepsilon_\beta \leq \\ &\leq \varphi_\beta(q_\beta(x_0, x_1) + \varepsilon_\beta) + \varepsilon_\beta + \varphi_\beta(\Phi_\beta^{-1}(\varepsilon_\beta)). \end{aligned}$$

So (2.9) is true.

If $\gamma_\beta = D_\beta(x_0, F(x_0))$ then $\gamma_\beta \leq q_\beta(x_0, x_1)$ so (2.9) is true again.

If $\gamma_\beta = D_\beta(x_1, F(x_1))$ then (2.8) implies

$$\begin{aligned} D_\beta(x_1, F(x_1)) &\leq q_\beta(x_1, x_2) \leq H_\beta(F(x_0), F(x_1)) + \varepsilon_\beta \leq \\ &\leq \varphi_\beta(D_\beta(x_1, F(x_1))) + \varepsilon_\beta, \end{aligned}$$

from where we have $D_\beta(x_1, F(x_1)) - \varphi_\beta(D_\beta(x_1, F(x_1))) \le \varepsilon_\beta$, so

$$D_\beta(x_1, F(x_1)) \le \Phi_\beta^{-1}(\varepsilon_\beta).$$

Thus, $q_\beta(x_1, x_2) \le \varphi_\beta(\Phi_\beta^{-1}(\varepsilon_\beta)) + \varepsilon_\beta$ and (2.9) is true.

If $\gamma_\beta = \frac{1}{2}[D_\beta(x_0, F(x_1)) + D_\beta(x_1, F(x_0))]$ then

$$\begin{aligned} q_\beta(x_1, x_2) &\le \frac{1}{2}[D_\beta(x_0, F(x_1)) + D_\beta(x_1, F(x_0))] + \varepsilon_\beta \le \\ &\le \frac{1}{2}[q_\beta(x_0, x_1) + q_\beta(x_1, x_2)] + \varepsilon_\beta, \end{aligned}$$

from where $\frac{1}{2}q_\beta(x_1, x_2) \le \frac{1}{2}q_\beta(x_0, x_1) + \varepsilon_\beta$. So

$$\begin{aligned} q_\beta(x_1, x_2) &\le \varphi_\beta(\frac{1}{2}[D_\beta(x_0, F(x_1)) + D_\beta(x_1, F(x_0))]) + \varepsilon_\beta \le \\ &\le \varphi_\beta(\frac{1}{2}[q_\beta(x_0, x_1) + q_\beta(x_1, x_2)]) + \varepsilon_\beta \le \\ &\le \varphi_\beta(q_\beta(x_0, x_1) + \varepsilon_\beta) + \varepsilon_\beta. \end{aligned}$$

Thus, (2.9) is true again, which means that it holds in all cases. We now have from (2.7) that

$$q_\beta(x_1, x_2) < \varphi_\beta(r_\beta - \varphi_\beta(r_\beta)). \tag{2.11}$$

Also we can point out that

$$\begin{aligned} q_\beta(x_0, x_2) &\le q_\beta(x_0, x_1) + q_\beta(x_1, x_2) < \\ &< [r_\beta - \varphi_\beta(r_\beta)] + \varphi_\beta(r_\beta - \varphi_\beta(r_\beta)) \le \\ &\le r_\beta - \varphi_\beta(r_\beta) + \varphi_\beta(r_\beta) = r_\beta, \end{aligned}$$

Thus, $x_2 \in \overline{B}_q^p(x_0, r)$.

Next, for $\beta \in B$, we choose $\delta_\beta > 0$, with $\Phi_\beta^{-1}(\delta_\beta) < r_\beta$ so that

$$\varphi_\beta(q_\beta(x_1, x_2) + \delta_\beta) + \delta_\beta + \varphi(\Phi_\beta^{-1}) < \varphi_\beta^2(r_\beta - \varphi_\beta(r_\beta)). \tag{2.12}$$

This is possible from (2.11).

From (2.16) we can choose $x_3 \in F(x_2)$ so that for every $\beta \in B$ we have

$$q_\beta(x_2, x_3) \le D_\beta(x_2, F(x_2)) + \delta_\beta \le H_\beta(F(x_1), F(x_2)) + \delta_\beta.$$

As above, we can easily prove that

$$q_\beta(x_2, x_3) \le \varphi_\beta(q_\beta(x_2, x_3) + \delta_\beta) + \delta_\beta + \varphi_\beta(\Phi_\beta^{-1}(\delta_\beta)). \tag{2.13}$$

From (2.12) and (2.13) we have that $q_\beta(x_2, x_3) < \varphi_\beta^2(r_\beta - \varphi_\beta(r_\beta))$.

For $\beta \in B$ we have

$$\begin{aligned} q_\beta(x_0, x_3) &\le q_\beta(x_0, x_1) + q_\beta(x_1, x_2) + q_\beta(x_2, x_3) \le \\ &\le [r_\beta - \varphi_\beta(r_\beta)] + \varphi_\beta(r_\beta - \varphi(r_\beta)) + \varphi_\beta^2(r_\beta - \varphi_\beta r_\beta) \le \\ &\le r_\beta + \left[\sum_{i=1}^{\infty} \varphi_\beta^i(r_\beta - \varphi_\beta(r_\beta)) - \varphi_\beta(r_\beta)\right] \le r_\beta. \end{aligned}$$

Proceeding in the same way, we obtain $x_{n+1} \in F(x_n)$, for $n \in \{3, 4, ...\}$, with $x_{n+1} \in \overline{B}_q^p(x_0, r)$ and

$$q_\beta(x_n, x_{n+1}) < \varphi_\beta^n(r_\beta - \varphi_\beta(r_\beta)), \text{ for every } \beta \in B.$$

From (2.2) it is immediate that $\{x_n\}$ is a Cauchy sequence with respect to q_β, for each $\beta \in B$. (ii) implies that $\{x_n\}$ is also $\mathcal{P}$-Cauchy, hence it is $\mathcal{P}$-convergent to some $x \in \overline{B}_q^p(x_0, r)$. It only remains to show that $x \in F(x)$.

$$\begin{aligned} D_\beta(x, F(x)) &\le q_\beta(x, x_n) + D_\beta(x_n, F(x)) \le \\ &\le q_\beta(x, x_n) + H_\beta(F(x_{n-1}), F(x)) \le \\ &\le q_\beta(x, x_n) + \varphi_\beta(\max\{q_\beta(x_{n-1}, x), D_\beta(x_{n-1}, F(x_{n-1})), D_\beta(x, F(x)), \\ &\qquad \frac{1}{2}[D_\beta(x_{n-1}, F(x)) + D_\beta(x, F(x_{n-1}))]\}). \end{aligned}$$

Since $D_\beta(x, F(x_{n-1})) \le q_\beta(x, x_n) \to 0$, $D_\beta(x_{n-1}, F(x_{n-1})) \le q_\beta(x_{n-1}, x_n) \to 0$ and $|D_\beta(x_{n-1}, F(x)) - D_\beta(x, F(x))| \le q_\beta(x_{n-1}, x) \to 0$, then, letting $n \to \infty$, we obtain:

$$D_\beta(x, F(x)) \le 0 + \varphi_\beta(\{0, 0, D_\beta(x, F(x)), \frac{1}{2} D_\beta(x, F(x))\}).$$

Thus, $D_\beta(x, F(x)) = 0$, so $x \in F(x)$. □

We continue with a global version of Ćirić's theorem ([7]) for generalized φ-contractions on a set with two separating gauge structures.

Theorem 2.2. *Let X be a nonempty set endowed with two separating gauge structures $\mathcal{P} = \{p_\alpha\}_{\alpha \in A}$, $\mathcal{Q} = \{q_\beta\}_{\beta \in B}$ (A, B are directed sets), $x_0 \in X$ and $F : (X, \mathcal{P}) \to P((X, \mathcal{P}))$ be a multivalued operator with closed graph. We suppose that:*

(i) $(X, \mathcal{P})$ is a sequentially complete gauge space;

(ii) there exists a function $\psi : A \to B$ and $c = \{c_\alpha\}_{\alpha \in A} \in (0, \infty)^A$ such that

$$p_\alpha(x, y) \leq c_\alpha \cdot q_{\psi(\alpha)}(x, y), \text{ for every } \alpha \in A \text{ and } x, y \in X;$$

(iii) suppose for each $\beta \in B$, there exists a continuous function $\varphi_\beta : [0, \infty) \to [0, \infty)$, with $\varphi_\beta(t) < t$, for every $t > 0$ and φ_β is strictly increasing such that for $x, y \in X$ we have

$$H_\beta(F(x), F(y)) \leq \varphi_\beta(M_\beta^F(x, y)),$$

where

$$M_\beta^F(x, y) = \max\{q_\beta(x, y), D_\beta(x, F(x)), D_\beta(y, F(y)), \frac{1}{2}[D_\beta(x, F(y)) + D_\beta(y, F(x))]\}.$$

In addition assume for each $\beta \in B$ that

$$\Phi_\beta \text{ is strictly increasing } [0, \infty), \text{ where } \Phi_\beta(x) = x - \varphi_\beta(x), \tag{2.14}$$

and

$$\sum_{i=1}^{\infty} \varphi_\beta^i(t) < \infty, \text{ for } t > 0 \tag{2.15}$$

$$\text{for every } x \in X \text{ and every } \varepsilon = \{\varepsilon_\beta\}_{\beta \in B} \in (0, \infty)^B \text{ there exists } y \in F(x), \text{ with } q_\beta(x, y) \leq D_\beta(x, F(x)) + \varepsilon_\beta, \text{ for every } \beta \in B. \tag{2.16}$$

Then F has a fixed point.

Proof. Let $r = \{r_\beta\}_{\beta\in B} \in (0,\infty)^B$. We claim that we can choose $x_0 \in X$ and $x_1 \in F(x_0)$ such that

$$q_\beta(x_1, x_0) < r_\beta - \varphi(r_\beta). \tag{2.17}$$

If (2.17) is true then as in Theorem 2.1 we can choose $x_{n+1} \in F(x_n)$, for $n \in \{1, 2, ...\}$, with

$$q_\beta(x_n, x_{n+1}) < \varphi_\beta^n(r_\beta - \varphi_\beta(r_\beta)), \text{ for every } \beta \in B.$$

The same reasonings guarantees that $\{x_n\}$ is a $\mathcal{P}$-Cauchy sequence to some $x \in X$, hence it is $\mathcal{P}$-convergent to some $x \in X$. So as in Theorem 2.1, we have $D_\beta(x, F(x)) = 0$, thus $x \in F(x)$.

It remains to show (2.17).

We can observe that (2.17) is immediate if we could show that for any $\beta \in B$ we have

$$\inf_{x\in X} D_\beta(x, F(x)) = 0. \tag{2.18}$$

Assuming that (2.18) is true there exists $x \in X$ with $D_\beta(x, F(x)) < r_\beta - \varphi(r_\beta)$, so there exists $y \in F(x)$, with $q_\beta(x, y) < r_\beta - \varphi(r_\beta)$.

Suppose that (2.18) is false, i.e. suppose that there exists $\beta \in B$ such that

$$\inf_{x\in X} D_\beta(x, F(x)) = \delta_\beta. \tag{2.19}$$

Since $\varphi_\beta(\delta_\beta) < \delta_\beta$ and φ_β is continuous, we have that there exists $\varepsilon_\beta > 0$ such that

$$\varphi_\beta(t) < \delta_\beta, \text{ for } t \in [\delta_\beta, \delta_\beta + \varepsilon_\beta). \tag{2.20}$$

We can choose $v \in X$ such that

$$\delta_\beta \le D_\beta(v, F(v)) < \delta_\beta + \varepsilon_\beta. \tag{2.21}$$

Then there exists $y \in F(v)$ such that

$$\delta_\beta \le q_\beta(v, y) < \delta_\beta + \varepsilon_\beta. \tag{2.22}$$

Thus,

$$\begin{aligned} D_\beta(y,F(y)) &\leq H_\beta(F(v),F(y)) \leq \\ &\leq \varphi_\beta(\max\{q_\beta(v,y), D_\beta(v,F(v)), D_\beta(y,F(y)), \\ &\qquad \frac{1}{2}[D_\beta(v,F(y))+D_\beta(y,F(v))]\}) \end{aligned}$$

Let

$$\begin{aligned} \gamma &= \max\{q_\beta(v,y), D_\beta(v,F(v)), D_\beta(y,F(y)), \\ &\qquad \frac{1}{2}[D_\beta(v,F(y))+D_\beta(y,F(v))]\}. \end{aligned}$$

If $\gamma = q_\beta(v,y)$ then (2.20) and (2.22) yields

$$D_\beta(y,F(y)) \leq \varphi_\beta(q_\beta(v,y)) < \delta_\beta.$$

If $\gamma = D_\beta(v,F(v))$ then (2.20) and (2.21) yields

$$D_\beta(y,F(y)) \leq \varphi_\beta(D_\beta(v,F(v))) < \delta_\beta.$$

If $\gamma = D_\beta(y,F(y))$ then $\gamma = 0$, since $\gamma \neq 0$ results the following inequality

$$D_\beta(y,F(y)) \leq \varphi_\beta(D_\beta(y,F(y))) < D_\beta(y,F(y))$$

which is a contradiction.

If $\gamma = \frac{1}{2}[D_\beta(v,F(y))+D_\beta(y,F(v))]$ and $\gamma \neq 0$ then

$$\begin{aligned} D_\beta(y,F(y)) &\leq \varphi_\beta(\gamma) < \gamma = \frac{1}{2}[D_\beta(v,F(y))+D_\beta(y,F(v))] \leq \\ &\leq \frac{1}{2}[q_\beta(v,y)+D_\beta(y,F(y))+0], \end{aligned}$$

so $\frac{1}{2}D_\beta(y,F(y)) \leq \frac{1}{2}q_\beta(v,y)$. Thus, $\gamma = \frac{1}{2}[D_\beta(v,F(y))+D_\beta(y,F(v))] \leq \frac{1}{2}[q_\beta(v,y)+D_\beta(y,F(y))] < \frac{1}{2}q_\beta(v,y)+\frac{1}{2}q_\beta(v,y) = q_\beta(v,y)$, which contradicts the definition of γ. So we have proved that in this case $\gamma = 0$, which implies $D_\beta(y,F(y)) \leq \varphi_\beta(\gamma) = \varphi(0) = 0$.

We can notice that in all cases we have $D_\beta(y, F(y)) < \delta_\beta$, which contradicts (2.19), thus, (2.18) is true, so (2.17) is immediate and the proof is complete. □

In what follows we will present a homotopy result for Ćirić-type generalized φ-contractions on a set with two separating gauge structures.

Theorem 2.3. *Let X be a nonempty set endowed with two separating gauge structures $\mathcal{P} = \{p_\alpha\}_{\alpha\in A}$, $\mathcal{Q} = \{q_\beta\}_{\beta\in B}$ (A, B are directed sets), $(X, \mathcal{P})$ is a sequentially complete gauge space, there exists a function $\psi : A \to B$ and $c = \{c_\alpha\}_{\alpha\in A} \in (0,\infty)^A$ such that $p_\alpha(x, y) \le c_\alpha \cdot q_{\psi(\alpha)}(x, y)$ for every $\alpha \in A$ and $x, y \in X$. Let U be an open subset of $(X, \mathcal{Q})$. Let $G : \overline{U} \times [0, 1] \to P(X, \mathcal{P})$ be a multivalued operator such that the following assumptions are satisfied:*

(i) $x \notin G(x, t)$, for each $x \in \partial U$ and each $t \in [0, 1]$;

(ii) suppose for each $\beta \in B$ there exists a continuous and strictly increasing function $\varphi_\beta : [0,\infty) \to [0,\infty)$, with $\varphi_\beta(t) < t$, for every $t > 0$, such that for $x, y \in X$ we have

$$H_\beta(G(x, t), G(y, t)) \le \varphi_\beta(M_\beta^{G(\cdot,t)}(x, y)),$$

where

$$M_\beta^{G(\cdot,t)}(x, y) = \max\{q_\beta(x, y), D_\beta(x, G(x, t)), D_\beta(y, G(y, t)), \frac{1}{2}[D_\beta(x, G(y, t)) + D_\beta(y, G(x, t))]\};$$

(iii) there exists a continuous increasing function $\gamma : [0, 1] \to \mathbb{R}$ such that

$$H_\beta(G(x, t), G(x, s)) \le |\gamma(t) - \gamma(s)|, \text{ for all } t, s \in [0, 1] \text{ and each } x \in \overline{U};$$

(iv) $G : (\overline{U}, \mathcal{P})\times [0, 1] \to P(X, \mathcal{P})$ has closed graph;

(v) Φ_β *is strictly increasing on* $[0,\infty)$ *for each* $\beta \in B$*, where* $\Phi_\beta(x) = x - \varphi_\beta(x)$*;*

(vi) $\sum_{i=1}^{\infty} \varphi^i(t) < \infty$*, for* $t > 0$*;*

(vii) for every $x \in X$ *and every* $\varepsilon = \{\varepsilon_\beta\}_{\beta\in B} \in (0,\infty)^B$ *there exists* $y \in F(x)$ *with* $q_\beta(x,y) \leq D_\beta(x, F(x)) + \varepsilon_\beta$*, for every* $\beta \in B$*.*

Then $G(\cdot,0)$ *has a fixed point if and only if* $G(\cdot,1)$ *has a fixed point.*

Proof. Suppose that $z \in FixG(\cdot,0)$. From (i) we have that $z \in U$. We will define the following set:

$$E := \{(x,t) \in U \times [0,1] | x \in G(x,t)\}.$$

Since $(z,0) \in E$, we have that $E \neq \emptyset$. We introduce a partial order defined on E

$$(x,t) \leq (y,s) \text{ if and only if } t \leq s \text{ and } q_\beta(x,y) \leq \Phi_\beta^{-1}(2[\gamma(s) - \gamma(t)]).$$

Let M be a totally ordered subset of E, $t^* := \sup\{t \,|(x,t) \in M\}$ and $(x_n,t_n)_{n\in\mathbb{N}^*} \subset M$ be a sequence such that $(x_n,t_n) \leq (x_{n+1},t_{n+1})$ and $t_n \to t^*$, as $n \to \infty$. Then

$$q_\beta(x_m,x_n) \leq \Phi_\beta^{-1}(2[\gamma(t_m) - \gamma(t_n)]), \text{ for each } m,n \in \mathbb{N}^*,\ m > n,$$

from where we can conclude that $q_\beta(x_m,x_n) - \varphi_\beta(q_\beta(x_m,x_n)) \leq 2[\gamma(t_m) - \gamma(t_n)]$.

Letting $m,n \to +\infty$, we obtain that $q_\beta(x_m,x_n) - \varphi_\beta(q_\beta(x_m,x_n)) \to 0$, so $\varphi_\beta(q_\beta(x_m,x_n)) \to q_\beta(x_m,x_n)$, as $m,n \to +\infty$. Therefore $q_\beta(x_m,x_n) \to 0$, as $m,n \to +\infty$. Thus, $(x_n)_{n\in\mathbb{N}^*}$ is $\mathcal{Q}$-Cauchy, so is $\mathcal{P}$-Cauchy too. Denote by $x^* \in (X,\mathcal{P})$ its limit. We know that $x_n \in G(x_n,t_n)$, $n \in \mathbb{N}^*$ and G is $\mathcal{P}$-closed. Therefore we have that $x^* \in G(x^*,t^*)$. From (i) we can notice that $x^* \in U$. So $(x^*,t^*) \in E$.

From the fact that M is totally ordered we have that $(x,t) \leq (x^*,t^*)$, for each $(x,t) \in M$. Thus, (x^*,t^*) is an upper bound of M. We can apply Zorn's Lemma, so E admits a maximal element $(x_0,t_0) \in E$. We want to prove that $t_0 = 1$.

Suppose that $t_0 < 1$. Let $r = \{r_\beta\}_{\beta\in B} \in (0,\infty)^B$ and $t \in]t_0,1]$ such that $B_q(x_0,r_\beta) \subset U$ and $r_\beta := \Phi_\beta^{-1}(2[\gamma(t)-\gamma(t_0)])$, for every $\beta \in B$. Then for each $\beta \in B$ we have

$$\begin{aligned} D_\beta(x_0,G(x_0,t)) &\leq D_\beta(x_0,G(x_0,t_0)) + H_\beta(G(x_0,t_0),G(x_0,t)) \leq \\ &\leq \gamma(t)-\gamma(t_0) = \frac{\Phi_\beta(r_\beta)}{2} = \frac{r_\beta - \varphi_\beta(r_\beta)}{2} < r_\beta - \varphi_\beta(r_\beta). \end{aligned}$$

Since $\overline{B}_q^p(x_0,r_\beta) \subset U \subset \overline{U}$, the closed multivalued operator $G(\cdot,t) : \overline{B}_q^p(x_0,r_\beta) \to P(X,\mathcal{P})$ satisfies the assumptions of Theorem 2.1, for all $t \in [0,1]$. Hence there exists $x \in \overline{B}_q^p(x_0,r_\beta)$ such that $x \in G(x,t)$. Thus, $(x,t) \in E$. But we know that

$$q_\beta(x_0,x) \leq r_\beta = \Phi_\beta^{-1}(2[\gamma(t)-\gamma(t_0)]),$$

so we have that $(x_0,t_0,) \leq (x,t)$, which is a contradiction with the maximality of (x_0,t_0). Thus, $t_0 = 1$ and the proof is complete. □

3 Applications

The following result is a particular case of Theorem 2.2, namely the case where the complete gauge space is endowed with one separating gauge structure and the multivalued operator is a φ-contraction.

Theorem 3.1. *Let X be a sequentially complete gauge space endowed with a separating gauge structure and let $F : X \to P(X)$ be a φ-contraction with closed graph, i.e. for each $\alpha \in A$ (A is a directed set) there exists a continuous strict comparison function $\varphi_\alpha : [0,\infty) \to [0,\infty)$ such that for $x,y \in X$ we have*

$$H_\alpha(F(x), F(y)) \leq \varphi_\alpha(d_\alpha(x, y)).$$

We assume that for every $x \in X$ and every $\varepsilon \in (0,\infty)^A$ there exists $y \in F(x)$ such that

$$d_\alpha(x, y) \leq D_\alpha(x, F(x)) + \varepsilon_\alpha, \text{ for every } \alpha \in A.$$

Then F has a fixed point.

Remark 3.1. *Some well-known examples of continuous strict comparison functions are:*

a) $\varphi(t) = at$, *with* $a \in [0, 1)$;

b) $\varphi(t) = \frac{t}{1+t}$, $t \in [0,\infty)$.

Definition 3.1. *Let E be a Hilbert space. The multivalued operator $F : [0,\infty) \times E \to P_{b,cl}(E)$ is said to be locally Carathéodory if*

(i) $t \mapsto F(t, x)$ *is measurable, for all* $x \in E$;

(ii) $x \mapsto F(t, x)$ *is continuous, for a.e.* $t \in [0,\infty)$;

(iii) for all $R > 0$, there exists a function $h_R \in L^1_{loc}[0,\infty)$ such that for a.e. $t \in [0,\infty)$ and for every $x \in E$, with $\|x\| \leq R$, we have $H(\{0\}, F(t, x)) \leq h_R(t)$.

Throughout E is a Hilbert space. As usual, $L^1([a, b], E)$ denotes the Banach space of measurable functions $u : [a, b] \to E$ such that $|u|$ is Lebesgue integrable with $\|u\|_1 = \int_a^b |u(t)|dt$. We define the Sobolev class $W^{1,1}([a, b], E)$ as follows: a function $u \in W^{1,1}([a, b], E)$ if it is continuous and there exists $v \in L^1[a, b]$ such that $u(t) - u(a) = \int_a^t v(s)ds$, for all $t \in [a, b]$. Notice that if $u \in W^{1,1}([a, b], E)$ then u is differentiable almost everywhere on $[a, b]$, $u' \in L^1([a, b], E)$ and $u(t) - u(a) = \int_a^t u'(s)ds$, for almost every $t \in [a, b]$.

Let us consider the following Cauchy-problem

$$\begin{cases} x'(t) \in F(t, x(t)) \text{ a.e } t \in [0, \infty], \\ x(0) = 0 \in E, \end{cases} \tag{3.23}$$

where E is also a Hilbert space and the locally Carathéodory multivalued operator F is a φ-contraction.

Theorem 3.2. *Let $(E, \|\cdot\|)$ be a Hilbert space and $F : [0, \infty) \times E \to P_{b,cl}(E)$ be a locally Carathéodory multivalued operator. We suppose that*

(a) for every $R > 0$, there exists $l_R \in L^1_{loc}[0, \infty)$ and a continuous, strict comparison function $\varphi_R \in L^1_{loc}[0, \infty)$, with $\varphi_R(at) \le a \cdot \varphi(t)$, for every $a > 1$, such that for a.e. $t \ge 0$ and for every $x, y \in E$, with $\|x\|, \|y\| \le R$, we have

$$H(F(t, x), F(t, y)) \le l_R(t) \cdot \varphi_R(\|x - y\|);$$

(b) there exists $\theta \in L^1_{loc}[0, \infty)$ and $\psi : [0, \infty) \to [0, \infty)$ an increasing and Borel measurable function such that

(b1) $H(\{0\}, F(t, v)) \le \theta(t) \cdot \psi(\|v\|)$, for a.e. $t \in [0, \infty)$ and every $v \in E$ such that $1/\psi \in L^1_{loc}[0, \infty)$;

(b2) $\int\limits_0^\infty \frac{dz}{\psi(z)} > \|\theta\|_{L^1[0,r]}$, for all $r > 0$.

Then (3.23) has a solution in $W^{1,1}_{loc}([0, \infty), E)$.

Proof. For the proof of our theorem let $M : [0, \infty) \to [0, \infty)$ be a continuous and increasing function such that

$$\int\limits_0^\infty \frac{ds}{\psi(s)} > \int\limits_0^{M(t)} \frac{ds}{\psi(s)} \ge \|\theta\|_{L^1[0,t]},$$

which is possible by assumption (b2). Let $\widetilde{F} : [0, \infty) \times E \to P_{b,cl}(E)$ be defined by

$$\widetilde{F}(t, u) = \begin{cases} F(t, u), \ \|u\| \le M(t), \\ F(t, \frac{M(t)u}{\|u\|}), \ \|u\| > M(t). \end{cases} \tag{3.24}$$

Define $T : C([0,\infty), E) \to P(C([0,\infty), E))$, $T(x)(t) := \int\limits_0^t \widetilde{F}(s, x(s))ds$. Suppose x is a fixed point for T, thus, x is continuous and $x \in T(x)$, which means that $x(t) \in T(x)(t)$, for every $t \in [0,\infty)$, so $x(t) \in \int\limits_0^t \widetilde{F}(s, x(s))ds$, for every $t \in [0,\infty)$. Since

$$\int\limits_0^t \widetilde{F}(s, x(s))ds := \left\{ \int\limits_0^t v_x(s)ds \mid v_x(s) \in \widetilde{F}(s, x(s)), \forall s \in [0,t], v_x \in L^1([0,t], E) \right\},$$

it follows that there exists $v_x \in L^1([0,t], E)$ such that $x(t) := \int\limits_0^t v_x ds$, for every $t \in [0,\infty)$, with $v_x(s) \in \widetilde{F}(s, x(s))$, for every $s \in [0,t]$. Hence we obtain that there exist $x'(t) = v_x(t)$ for a.e. $t \in [0,\infty)$ and $x \in W^{1,1}([0,\infty), E)$. Thus, $x'(t) \in \widetilde{F}(t, x(t))$, for a.e. $t \in [0,\infty)$ and $x(0) = 0$.

We will show that $x'(t) \in F(t, x(t))$, for a.e. $t \in [0,\infty)$.

Suppose that there exists $t > 0$ such that $\|x(t)\| > M(t)$. Then we have that $x'(t) \in F\left(t, \frac{M(t)x'(t)}{\|x'(t)\|}\right)$. By assumption (b1) we have

$$\begin{aligned} \|x'(t)\| &\leq \theta(t) \cdot \psi\left(\left\| \frac{M(t) \cdot x'(t)}{\|x'(t)\|} \right\|\right) = \theta(t) \cdot \psi(M(t)) \\ &\leq \theta(t) \cdot \psi(\|x(t)\|). \end{aligned}$$

Thus,

$$\frac{\|x'(t)\|}{\psi(\|x(t)\|)} \leq \theta(t),$$

which means that

$$\frac{\|x(t)\|'}{\psi(\|x(t)\|)} \leq \theta(t).$$

Integrating from 0 to t and via change of variables theorem ($v = \|x(s)\|$) we obtain

$$\int\limits_0^{\|x(t)\|} \frac{dv}{\psi(v)} \leq \|\theta\|_{L^1[0,t]} \leq \int\limits_0^{M(t)} \frac{ds}{\psi(s)},$$

thus $\|x(t)\| \leq M(t)$, which is a contradiction.

Hence $\|x(t)\| \leq M(t)$, for a.e. $t \in [0, \infty)$ and thus

$$\begin{cases} x'(t) \in F(t, x(t)) \text{ a.e } t \in [0, \infty], \\ x(0) = 0, \end{cases}$$

so x is a solution for (3.23).

Let $l_R(t) = l_{M(n)}(t)$ in assumption (a), for $t \in [0, n]$, $n \in \mathbb{N}^*$. Define on $C([0, \infty), E)$ the Bielecki-type semi-norm:

$$|x|_n = \sup_{t \in [0,n]} \left\{ e^{-\int\limits_0^t l_{M(n)}(s)ds} \cdot \|x(t)\| \right\}.$$

Then T is an admissible φ-contraction if:

(i) $H_{M(n)}(T(x), T(y)) \leq \varphi_{M(n)}(|x - y|_n)$, for every $x, y \in C([0, \infty), E)$;

(ii) for every $x \in C([0, \infty), E)$ and for every $\varepsilon \in (0, \infty)^{\mathbb{N}^*}$ there exists $y \in T(x)$ such that $|x - y|_n \leq D_n(x, T(x)) + \varepsilon_n$.

For (i) let $t \in [0, n]$, $x, y \in C([0, n], E)$ and $u_1 \in T(x)$ such that $\|x(t)\| \leq M(t)$, $\|y(t)\| \leq M(t)$. Then there exists $v_{u_1} \in F(s, x(s))$, $s \in [0, t]$, such that $v_{u_1} \in L^1([0, n], E)$ and $u_1(t) = \int\limits_0^t v_{u_1}(s)ds$. From the inequality below

$$H(F(t, x), F(t, y)) \leq l_{M(n)}(t) \cdot \varphi_{M(n)}(\|x - y\|),$$

it follows that there exists $w \in F(t, y(s))$, $s \in [0, t]$, $w \in L^1([0, n], E)$ such that

$$\|v_{u_1} - w\| \leq l_{M(n)}(s) \cdot \varphi_{M(n)}(\|x - y\|).$$

Thus, the multivalued operator G defined by

$$G(t) = F(s, y(s)) \cap \left\{ w \mid \|v_{u_1} - w\| \leq l_{M(n)}(s) \cdot \varphi_{M(n)}(\|x - y\|) \right\}$$

has nonempty values and is measurable. By Kuratowski and Ryll Nardzewski's selection theorem (see [12]) there exists $v_{u_2}(s)$ a measurable selection for G.

Then $v_{u_2}(s) \in F(s,y(s))$, $s \in [0,t]$, $v_{u_2} \in L^1([0,n],E)$. Define $u_2(t) = \int_0^t v_{u_2}(s)ds \in T(y)(t)$, $t \in [0,n]$. We have:

$$
\begin{aligned}
\|u_1(t) \quad - \quad u_2(t)\| &\leq \int_0^t \|v_{u_1}(s) - v_{u_2}(s)\|ds \\
&\leq \int_0^t l_{M(n)}(s) \cdot \varphi_{M(n)}(\|x(s) - y(s)\|)ds \\
&\leq \int_0^t l_{M(n)}(s) \cdot \varphi_{M(n)}\Big(\|x(s) - y(s)\| \ e^{-\int_0^s l_{M(n)}(z)dz} \cdot e^{\int_0^s l_{M(n)}(z)dz}\Big)ds \\
&\leq \int_0^t l_{M(n)}(s) \cdot e^{\int_0^s l_{M(n)}(z)dz} \cdot \varphi_{M(n)}\Big(\|x(s) - y(s)\| \ e^{-\int_0^s l_{M(n)}(z)dz}\Big)ds \\
&\leq \varphi_{M(n)}(|x-y|_n) \cdot \int_0^t l_{M(n)}(s) \cdot e^{\int_0^s l_{M(n)}(z)dz} ds \\
&\leq \varphi_{M(n)}(|x-y|_n) \cdot e^{\int_0^t l_{M(n)}(s)ds}.
\end{aligned}
$$

Thus, we obtained that $|u_1 - u_2|_n \leq \varphi_{M(n)}(|x-y|_n)$, for a.e. $t \in [0,\infty)$. By the analogous relation obtained by interchanging the roles of x and y it follows that

$$H_{M(n)}(T(x),T(y)) \leq \varphi_{M(n)}(|x-y|_n).$$

For (ii) we will suppose the contrary, i.e. there exists $\varepsilon \in (0,\infty)^{\mathbb{N}^*}$ and exists $x \in C([0,\infty),E)$ such that for all $y \in T(x)$ we have $|x-y|_n > D_n(x,T(x)) + \varepsilon_n$. It follows that $D_n(x,T(x)) \geq D_n(x,T(x)) + \varepsilon_n$, thus $\varepsilon_n \leq 0$, for every $n \in \mathbb{N}^*$. This is a contradiction.

Thus, by Theorem 3.1, the proof is complete.

□

Definition 3.2. *Let* (Ω, Σ), (Φ, Γ) *be two measurable spaces and* X *be a topo-*

logical space. Then a mapping $F : \Omega \times \Phi \to P(X)$ is said to be jointly measurable if for every closed subset B of X, $F^{-1}(B) \in \Sigma \bigotimes \Gamma$, where $\Sigma \bigotimes \Gamma$ denotes the smallest σ-algebra on $\Omega \times \Phi$, which contains all the sets $A \times B$ with $A \in \Sigma$ and $B \in \Gamma$.

Let us consider the following Volterra-type inclusion

$$x(t) \in \int_0^t K(t,s,x(s))ds + g(t) \text{ a.e. } t \in [0,\infty). \tag{3.25}$$

Theorem 3.3. *Let $K : [0,\infty) \times [0,\infty) \times \mathbb{R}^m \to P_{cl,b}(\mathbb{R}^m)$ be a multivalued operator and $g : [0,\infty) \to \mathbb{R}^m$ be a continuous function such that $g(0) = 0$. We suppose that*

(i) K is jointly measurable for all $x \in C[0,\infty)$;

(ii) for almost every $(t,s) \in [0,\infty) \times [0,\infty)$ $K(t,s,\cdot) : \mathbb{R}^m \to P(\mathbb{R}^m)$ is continuous;

(iii) for every $R > 0$, there exists $l_R \in L^1_{loc}[0,\infty)$ and a continuous, strict comparison function $\varphi_R \in L^1_{loc}[0,\infty)$ with $\varphi_R(at) \le a \cdot \varphi_R(t)$, for $a > 1$, such that

$$H_R(K(t,s,x), K(t,s,y)) \le l_R(s) \cdot \varphi_R(\|x-y\|),$$

for every $s \le t$ and every $x, y \in \mathbb{R}^m$, with $\|x\|, \|y\| \le R$;

(iv) there exists $\theta \in L^1_{loc}[0,\infty)$ and $\psi : [0,\infty) \to [0,\infty)$ a Borel measurable function such that

$$H(\{0\}, K(t,s,x(s))) \le \theta(s) \cdot \psi(\|x\|),$$

for a.e. $t \in [0,\infty)$ with $s \le t$ and every $x \in \mathbb{R}^m$, where $1/\psi \in L^1_{loc}[0,\infty)$ and

$$\int_0^\infty \frac{dz}{\psi(z)} > \|\theta\|_{L^1[0,r]}, \text{ for all } r > 0.$$

Then (3.25) has a solution.

Proof. Let $M : [0,\infty) \to [0,\infty)$ be a continuous nondecreasing function such that

$$\int_0^{M(t)} \frac{ds}{\psi(s)} \geq \|\theta\|_{L^1[0,t]}.$$

Suppose that there exists a solution x such that $\|x\| \geq M(t)$, for some $t \in [0,\infty)$. Then there exists $0 \leq t_1 < \infty$ such that

$$\|x(t_1)\| = M(t_1) \text{ and } 0 < \|x(t)\| \leq M(t_1), \text{ for every } t \in (0,t_1).$$

The function $t \mapsto \|x(t)\|$ is differentiable on $(0,t_1)$ and

$$\Big|\|x(t)\|'\Big| = \Big\langle \frac{x(t)}{\|x(t)\|}, x'(t)\Big\rangle \leq \|x'(t)\|.$$

From assumption (iv) we have that $H(0, K(t,s,x(s))) \leq \theta(s)\cdot\psi(\|x(t)\|)$ a.e. $t \in [0,\infty)$ and every $x \in \mathbb{R}^m$. Since $x'(t) \in K(t,s,x(s))$ we have that $\|x'(t)\| \leq \theta(t)\cdot\psi(\|x\|)$. Thus we obtain that $\|x(t)\|' \leq \theta(t)\cdot\psi(\|x\|)$, from where we have that

$$\frac{\|x(t)\|'}{\psi(\|x\|)} \leq \theta(t).$$

Integrating from 0 to t_1 and via Change of variables Theorem we obtain

$$\int_0^{\|x(t_1)\|=M(t_1)} \frac{ds}{\psi(s)} = \int_0^{t_1} \frac{\|x(s)\|'}{\psi(\|x\|)} \leq \int_0^{t_1} \theta(s)ds < \int_0^{M(t_1)} \frac{ds}{\psi(s)},$$

which is a contradiction.

Let $l_R(s) = l_{M(n)}(s)$ in assumption (iii). For $n \in \mathbb{N}$ we consider the Bielecki-type semi-norm:

$$|x|_n = \sup_{t\in[0,n]} \Big\{ e^{-\int_0^t l_{M(n)}(s)ds} \cdot \|x(t)\| \Big\}.$$

Let $X = \{x \in C([0,\infty),\mathbb{R}^m) : \ \|x(t)\| \leq M(t) \text{ for } t \in [0,n]\}$.

We define $F : X \to C([0,\infty),\mathbb{R}^m)$, $F(x)(t) = \int_0^t K(t,s,x(s))ds + g(t)$. We want to show that F is a φ-contraction.

Let $x_1, x_2 \in C([0,n],\mathbb{R}^m)$ and $u_1 \in F(x_1)$. Then $u_1 \in C([0,n],\mathbb{R}^m)$ and $u_1(t) \in \int_0^t K(t,s,x_1(s))ds + g(t)$. Thus, there exists $k_1(t,s) \in K(t,s,x_1(s))$ such that $u_1(t) = \int_0^t k_1(t,s)ds + g(t)$. Since

$$H_{M(n)}(K(t,s,x_1(s)),K(t,s,x_2(s))) \leq l_{M(n)}(s) \cdot \varphi_{M(n)}(\|x_1 - x_2\|),$$

for $s \leq t$ and $\|x_1\|, \|x_2\| \leq M(n)$, follows that there exists $v \in K(t,s,x_2(s))$ such that

$$\|k_1(t,s) - v\| \leq l_{M(n)}(s) \cdot \varphi_{M(n)}(\|x_1 - x_2\|).$$

Thus, the multivalued operator G defined by

$$G(t) = K(t,s,x_2(s)) \cap \Big\{ v \ \Big| \ \|k_1(t,s) - v\| \leq l_{M(n)}(s) \cdot \varphi_{M(n)}(\|x_1 - x_2\|) \Big\}$$

has nonempty values and is measurable. By Kuratowski and Ryll Nardzewski's selection theorem (see [12]) there exists $k_2(t,s)$ a measurable selection for G. Then $k_2(t,s) \in K(t,s,x_2(s))$ and

$$\|k_1(t,s) - k_2(t,s)\| \leq l_{M(n)}(s) \cdot \varphi_{M(n)}(\|x_1 - x_2\|), \text{ for a.e. } t \in [0,\infty), \ s \leq t.$$

Define $u_2(t)=\int\limits_0^t k_2(t,s)ds+g(t)\in F(x_2)$. We have:

$$\begin{aligned}\|u_1(t) &- u_2(t)\| \leq \int\limits_0^t \|k_1(t,s)-k_2(t,s)\|ds \\ &\leq \int\limits_0^t l_{M(n)}(s)\cdot\varphi_{M(n)}(\|x_1-x_2\|)ds \\ &\leq \int\limits_0^t l_{M(n)}(s)\cdot\varphi_{M(n)}\Big(\|x_1-x_2\|e^{-\int\limits_0^s l_{M(n)}(z)dz}\cdot e^{\int\limits_0^s l_{M(n)}(z)dz}\Big)ds \\ &\leq \int\limits_0^t l_{M(n)}(s)\cdot\varphi_{M(n)}\Big(\|x_1-x_2\|e^{-\int\limits_0^s l_{M(n)}(z)dz}\Big)\cdot e^{\int\limits_0^s l_{M(n)}(z)dz}ds \\ &\leq \varphi_{M(n)}(|x_1-x_2|_n)\cdot\int\limits_0^t l_{M(n)}(s)\cdot e^{\int\limits_0^s l_{M(n)}(z)dz}ds \\ &\leq \varphi_{M(n)}(|x_1-x_2|_n)\cdot e^{\int\limits_0^t l_{M(n)}(s)ds}\end{aligned}$$

Thus, we obtained that $|u_1(t)-u_2(t)|_n\leq\varphi(|x_1-x_2|_n)$, for a.e. $t\in[0,\infty)$. By the analogous relation obtained by interchanging the roles of x_1 and x_2 it follows that

$$H_{M(n)}(F(x_1),F(x_2))\leq\varphi(\|x_1-x_2\|_n).$$

In order to see if F is an admissible φ-contraction we have to prove that for every $\varepsilon\in(0,\infty)^{\mathbb{N}^*}$ and for every $x\in C([0,\infty),H)$ there exists $y\in F(x)$ such that $|x-y|_n>D_n(x,F(x))+\varepsilon_n$. We will suppose the contrary, i.e. there exists $\varepsilon\in(0,\infty)^{\mathbb{N}^*}$ and exists $x\in C([0,\infty),H)$ such that for all $y\in F(x)$ we have $|x-y|_n>D_n(x,F(x))+\varepsilon_n$. It follows that $D_n(x,F(x))\geq D_n(x,F(x))+\varepsilon_n$, thus, $\varepsilon_n\leq 0$, for every $n\in\mathbb{N}^*$. Which is a contradiction.

Thus, by Theorem 3.1, the proof is complete. □

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