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COINCIDENCE POINTS FOR MULTIVALUED MAPPINGS

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Abstract

In this paper we prove some coincidence theorems for two hybrid pairs satisfying an implicit contractive condition in a metric space which is not necessarily compact and for (possibly) non continuous mappings. The class of implicit relations used here is different from the one considered in [9]. Our work provides generalizations of the main results of [2], [3] and [4]. Also our work may be considered as a natural continuation of the papers [2], [3], [4] and [9].

1 Introduction

Let (X, d) be a metric space. We denote by $CL(X)$ the set of all nonempty closed subsets of X. We denote by $CB(X)$ the set of all nonempty closed bounded subsets of X and by $CC(X)$ the set of all nonempty compact subsets of X. Obviously, $CB(X) = CL(X) = CC(X)$ if (X, d) is a compact metric space.

We denote by H the Hausdorff-Pompeiu metric on $CB(X)$ i.e.

$$H(A, B) = \max\{\sup_{x\in A} D(x, B), \sup_{x\in B} D(x, A)\},$$

where $A, B \in CB(X)$ and $D(x, A) = \inf_{y\in A} d(x, y)$.

It is well-known that $(CB(X), H)$ is a metric space and the completeness of X implies the completeness of $(CB(X), H)$. Also we denote

$$\delta(A, B) = \sup\{d(a, b) : a \in A \text{ and } b \in B\}.$$

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Let S be a multivalued mapping of X into $CL(X)$ and f be a self-mapping of X. Then the pair (f, S) is said to be a hybrid pair. A point $x \in X$ is called a coincidence point of f and S if $fx \in Sx$. A point x is called a fixed point of S if $x \in Sx$.

Kubiak [2] and Kubiaczyc [3] proved some fixed point theorems for contractive type multivalued mappings in compact metric spaces. Z. Liu and T.S. Ume [4] proved some coincidence theorems for multivalued mappings in a compact metric space which extend properly the results of Kubiak and Kubiaczyc. The following theorems are proved in [4].

Theorem 1.1. *Let (X, d) be a compact metric space and S and T be mappings of X into $CL(X)$. Suppose that f and g are self-mappings of X satisfying:*

$$\delta(Sx, Ty) < \max\{d(fx, gy), H(fx, Sx), H(gy, Ty), \frac{1}{2}[D(fx, Ty) + D(gy, Sx)],$$

$$\frac{H(fx, Sx).H(gy, Ty)}{d(fx, gy)}, \frac{D(fx, Ty).D(gy, Sx)}{d(fx, gy)}, \} \tag{1.1}$$

for all $x, y \in X$ with $fx \neq gy$.

Let $S(X) \subset g(X)$ and $T(X) \subset f(X)$. If either f and S or g and T are continuous, then either f and S or g and T have a coincidence point u with $Su = \{fu\}$ or $Tu = \{gu\}$.

Theorem 1.2. *Let (X, d) be a compact metric space and let S and T be mappings of X into $CL(X)$. Assume that f and g are self mappings of X satisfying:*

$$H(Sx, Ty) < \max\{d(fx, gy), D(fx, Sx), D(gy, Ty), \frac{1}{2}[D(fx, Ty) + D(gy, Sx)],$$

$$\frac{D(fx, Sx).D(gy, Ty)}{d(fx, gy)}, \frac{D(fx, Ty).D(gy, Sx)}{d(fx, gy)}, \} \tag{1.2}$$

for all $x, y \in X$ with $fx \neq gy$.

Let $S(X) \subset g(X)$ and $T(X) \subset f(X)$. If f and S or g and T are continuous, then either f and S or g and T have a coincidence point.

In [9], V. Popa has extended the results of [4], for hybrid pairs which satisfy an implicit contractive condition.

Let $Q_2 := \{t = (t_1, \dots, t_6) \in \mathbb{R}^6 : t_2 = 0\}$. Let F (see [9]) be the set of all real functions $F(t_1, \dots, t_6) : \mathbb{R}^6 \setminus Q_2 \to \mathbb{R}$ satisfying the following conditions:
(F1): F is increasing in variable t_1 and non-increasing in variable t_5 and t_6.
(F2): For every $u \geq 0$, $v > 0$

(Fa): $F(u, v, v, u, u + v, 0) < 0$
or
(Fb): $F(u, v, u, v, 0, u + v) < 0$,
we have $u < v$.

Examples of such functions are given in [9].

The following result is proved in [9].

Theorem 1.3. *Let (X, d) be a compact metric space, $S, T : X \to CL(X)$ and $f,\ g : X \to X$ such that*

$$F(\delta(Sx, Ty), d(fx, gy), H(fx, Sx), H(gy, Ty), D(fx, Ty), D(gy, Sx)) < 0 \tag{1.3}$$

for all x, y in X with $fx \neq gy$, where $F \in \mathcal{F}$.

Let $S(X) \subset g(X)$ and $T(X) \subset f(X)$. If either f and S or g and T are continuous, then either f and S or g and T have a coincidence point u with $Su = \{fu\}$ or $Tu = \{gu\}$.

The implicit functions in metric fixed point theory are extensively used by V. Popa (see the papers [6], [7], [8] and [7]). Implicit relations are now basic tools to provide new results (and unify many old ones) in this theory. An early use of such implicit methods may be found in two papers by M. Turinici (see [10] and [11]).

The purpose of this paper is to give some extensions of Theorem 1.1 and Theorem 1.2 to the general case of metric spaces which are not necessarily compact and for mappings which may be non continuous. The contractive conditions considered here for two hybrid pairs are defined by implicit functions which are in a class called $\mathcal{G}$ different from the class $\mathcal{F}$ of [9].

The analogous of Theorem 1.3 is also given for implicit relations using the class $\mathcal{G}$. So our work gives a continuation to the papers [2], [3], [4] and [9].

In Section 2, we introduce the class $\mathcal{G}$ and provide some examples of its elements. The main results of this paper are presented in Section 3 (see Theorem 3.1 and Theorem 3.2 below).

2 Implicit relations

Let $Q_2 := \{t = (t_1, \dots, t_6) \in \mathbb{R}^6 : t_2 = 0\}$. Let $\mathcal{G}$ be the set of all real functions $G(t_1, \dots, t_6) : \mathbb{R}^6 \setminus Q_2 \to \mathbb{R}$ satisfying the following conditions:
(G1): G is increasing in variable t_1 and
(G2): For every $u, w \geq 0$, $v > 0$,

(Ga): $G(u, v, v, u, w, 0) < 0$
or
(Gb): $G(u, v, u, v, 0, w) < 0$,
we have $u < v$.

Remark 2.1. The condition (G1) is weaker than condition (F1). The condition (F2) is weaker than (G2).

It is easy to see that all functions G defined below are in the class $\mathcal{G}$.

Example 2.1. $G(t_1, \ldots, t_6) := t_1 - \max\{t_2, t_3, t_4, \dfrac{t_3t_4}{t_2}, \dfrac{t_5t_6}{t_2}\}$.

Example 2.2. $G(t_1, \ldots, t_6) := t_1 - \max\{t_2, t_3, t_4, \dfrac{t_2^2 + t_5t_6}{t_2 + t_3 + t_4}\}$.

Example 2.3. $G(t_1, \ldots, t_6) := t_1^3 - t_2t_3t_4 - bt_5^2t_6 - ct_5t_6^2$, where $b, c \geq 0$.

Example 2.4. $G(t_1, \ldots, t_6) := t_1^p - t_2^p - \dfrac{at_5^qt_6 + bt_5t_6^r}{t_2}$, where $p, q, r > 0$ and $a, b \geq 0$.

Example 2.5. $G(t_1, \ldots, t_6) := t_1^{p+2} - \dfrac{2t_3^{p+1}t_4^{p+1}}{t_2^p + t_3^p + t_4^p + t_5 + t_6}$, where $p > 0$.

Remark 2.2. Example 2.5 shows that G is increasing in the variables t_5 and t_6. So the classes $\mathcal{G}$ and $\mathcal{F}$ are essentially different.

3 Main results

Before giving our main result, we recall this definition.

Definition 3.1. *Let X be a non empty set and let 2^X be the set of non empty subsets of X. Let $F : X \to 2^X$ be a set-valued map on X. $x \in X$ is a fixed point of F if $x \in Fx$ and is a strict fixed point of F if $Fx = \{x\}$.*

We recall that a strict fixed point is also denoted by "absolutely fixed point" in [5] and "stationary point" in [1] .

It is natural to introduce the following definition.

Definition 3.2. *Let X be a non empty set and $F : X \to 2^X$ be a set-valued map on X. Let $f : X \to X$ be a self-mapping of X. A point $u \in X$ is said to be a strict coincidence point for the pair $\{f, F\}$ if u is a coincidence point for f and F which statisfies $Fx = \{fx\}$.*

Theorem 3.1. *Let (X, d) be a metric space, $S, T : X \to CC(X)$ be two multivalued maps and f, g be two selfmappings of X. Suppose that*

i) The below condition holds

$$G(\delta(Sx, Ty), d(fx, gy), H(fx, Sx), H(gy, Ty), D(fx, Ty), D(gy, Sx)) < 0 \tag{3.1}$$

for all x, y in X with $fx \neq gy$ and some $G \in \mathcal{G}$.

ii) $S(X) \subset g(X)$ and $T(X) \subset f(X)$.

iii) One of the sets $\{H(fx, Sx) : x \in X\}$ or $\{H(gx, Tx) : x \in X\}$ is closed. Then, either $\{f, S\}$ or $\{g, T\}$ has a strict coincidence point in X.

Proof. We assume without loss of generality that the set $\{H(fx, Sx) : x \in X\}$ is closed in the set of nonnegative real numbers. Let α be the lower bound of this set. By assumption there exists a point $u \in X$ such that

$$\alpha = H(fu, Su) = \inf\{H(fx, Sx) : x \in X\}.$$

It is easy to see that there is a point $y \in Su$ with $d(fu, y) = H(fu, Su)$. Since $S(X) \subset g(X)$, then there exists a point $v \in X$ with $y = gv$. Consequently, $d(fu, gv) = H(fu, Su)$ for some $gv \in Su$. Similarly, there exists $w \in X$ such that $d(gv, fw) = H(gv, Tv)$ where $fw \in Tv$. We now assert that $H(fu, Su).H(gv, Tv) = 0$, otherwise $H(fu, Su).H(gv, Tv) > 0$, and then $H(fu, Su) > 0$. Therefore $d(fu, gv) > 0$ and $fu \neq gv$. Then by (3.1) we have

$$G(\delta(Su, Tv), d(fu, gv), H(fu, Su), H(gv, Tv), D(fu, Tv), D(gv, Su)) < 0,$$

$$G(\delta(Su, Tv), H(fu, Su), H(fu, Su), H(gv, Tv), D(fu, gv), 0) < 0.$$

Since $H(gv, Tv) \leq \delta(Su, Tv)$ and G is increasing in the first variable, then we obtain

$$G(H(gv, Tv), H(fu, Su), H(fu, Su), H(gv, Tv), D(fu, gv), 0) < 0.$$

Then, by (Ga), we obtain $H(gv, Tv) < H(fu, Su)$.

Similarly since $d(fw, gv) = H(gv, Tv) > 0$ then $fw \neq gv$ and by (3.1) we have

$$G(\delta(Sw, Tv), d(fw, gv), H(fw, Sw), H(gv, Tv), D(fw, Tv), D(gv, Sw)) < 0,$$

$$G(\delta(Sw, Tv), H(gv, Tv), H(fw, Sw), H(gv, Tv), 0, D(gv, Sw)) < 0.$$

Since $H(fw, Sw) \leq \delta(Tv, Sw)$, we obtain

$$G(H(fw, Sw), H(gv, Tv), H(fw, Sw), H(gv, Tv), 0, D(gv, Sw)) < 0.$$

Then, by (Gb), we have $H(fw,Sw) < H(gv,Tv)$. Hence $H(fw,Sw) < H(fu,Su)$ which is a contradiction and therefore

$$H(fu,Su).H(gv,Tv) = 0$$

which implies that $Su = \{fu\}$ or $Tv = \{gv\}$. This ends the proof. □

The following result is the analogous of Theorem 1.3 (see [9]).

Corollary 3.1. *Let (X,d) be a compact metric space. Let $S,T : X \to CL(X)$. Suppose that f and g are selfmappings of X such that*

$$G(\delta(Sx,Ty), d(fx,gy), H(fx,Sx), H(gy,Ty), D(fx,Ty), D(gy,Sx)) < 0 \tag{3.1}$$

for all x,y in X with $fx \neq gy$, where $G \in \mathcal{G}$.

Let $S(X) \subset g(X)$ and $T(X) \subset f(X)$. If either f and S or g and T are continuous, then either $\{f,S\}$ or $\{g,T\}$ has a strict coincidence point in X.

Proof. If X is compact and the mappings f and S are continuous then the function $x \mapsto H(fx,Sx)$ is continous. Then its range is a compact subset of the real numbers. Therefore the set $\{H(fx,Sx) : x \in X\}$ is closed. So the proof follows from Theorem 3.1. □

If f and g are the identity mapping on X, Theorem 3.1 reduces to the following

Corollary 3.2. *Let (X,d) be a metric space and S and T mappings of X into $CC(X)$ satisfying the inequality:*

$$G(\delta(Sx,Tx), d(x,y), H(x,Sx), H(y,Ty), D(x,Ty), D(y,Tx)) < 0, \tag{3.2}$$

for all $x,y \in X$ with $x \neq y$, where $G \in \mathcal{G}$.

If one of the sets $\{H(x,Sx) : x \in X\}$ or $\{H(x,Tx) : x \in X\}$ is closed, then S or T has a strict fixed point u in X.

Corollary 3.3. *Let (X,d) be a compact metric space and S and T be mappings of X into $CC(X)$ satisfying the inequality:*

$$G(\delta(Sx,Tx), d(x,y), H(x,Sx), H(y,Ty), D(x,Ty), D(y,Sx)) < 0 \tag{3.3}$$

for all $x,y \in X$ with $x \neq y$, where $G \in \mathcal{G}$.

If S or T is continuous, then S or T has a strict fixed point in X.

Corollary 3.4. *([3]) Let (X, d) be a compact metric space and let S and T be mappings of X into $CC(X)$ satisfying the inequality:*

$$\delta(Sx, Ty) < \max\{d(fx, gy), H(fx, Sx), H(gy, Ty), \frac{1}{2}[D(fx, Ty) + D(gy, Sx)],$$
$$\frac{H(fx, Sx).H(gy, Ty)}{d(fx, gy)}, \frac{D(fx, Ty).D(gy, Sx)}{d(fx, gy)}, \} \tag{3.4}$$

for all $x, y \in X$ with $x \neq y$.

If S or T is continuous, then S or T has a strict fixed point in X.

The proof follows by Corollary 3.3 and Example 1.

Theorem 3.2. *Let (X, d) be a metric space and let S and T be mappings of X into $CC(X)$. Assume that f and g are selfmappings of X satisfying the inequality:*

$$G(H(Sx, Ty), d(fx, gy), D(fx, Sx), D(gy, Ty), D(fx, Ty), D(gy, Sx)) < 0, \tag{3.5}$$

for all $x, y \in X$ with $fx \neq gy$, where $G \in \mathcal{G}$.

We suppose that $S(X) \subset g(X)$ and $T(X) \subset f(X)$.

If one of the sets $\{D(fx, Sx) : x \in X\}$ or $\{D(fx, Tx) : x \in X\}$ is closed, then either $\{f, S\}$ or $\{g, T\}$ has a coincidence point.

Proof. We may assume that the set $\{D(fx, Sx) : x \in X\}$ is closed. Since the lower bound of this set is in its closure, then the function $D(fx, Sx)$ attains its minimum at some $u \in X$. As in Theorem 3.1., there exists v, w such that

$$d(fu, gv) = D(fu, Su), \quad \text{and} \quad d(gv, fw) = D(gv, Tv),$$

where $gv \in Su$ and $fw \in Tv$. Assume that $D(fu, Su).D(gv, Tv) > 0$. The same arguments as those of the proof of Theorem 3.1 show that

$$D(fw, Sw) < D(gv, Tv) < D(fu, Su),$$

which contradicts the minimality of $D(fu, Su)$. Hence $D(fu, Su).D(gv, Tv) = 0$, which implies that $fu \in Su$ or $gv \in Tv$. This completes the proof. □

Corollary 3.5. *Let (X, d) be a compact metric space and let S and T be mappings of X into $CL(X)$. Assume that f and g are selfmappings of X satisfying the inequality:*

$$G(H(Sx, Ty), d(fx, gy), D(fx, Sx), D(gy, Ty), D(fx, Ty), D(gy, Sx)) < 0, \tag{3.6}$$

for all $x, y \in X$ with $fx \neq gy$, where $G \in \mathcal{G}$.

Let $S(X) \subset g(X)$ and $T(X) \subset f(X)$. If either f and S or g and T are continuous, then either $\{f, S\}$ or $\{g, T\}$ has a coincidence point.

If f and g are the identity mapping on X, Theorem 3.2 reduces to the following.

Corollary 3.6. *Let (X, d) be a metric space and let S and T be mappings of X into $CC(X)$ satisfying the inequality:*

$$G(H(Sx, Ty), d(x, y), D(x, Sx), D(y, Ty), D(x, Ty), D(y, Sx)) < 0, \tag{3.7}$$

for all $x, y \in X$ with $fx \neq gy$, where $G \in \mathcal{G}$.

If one of the sets $\{D(x, Sx) : x \in X\}$ or $\{D(x, Tx) : x \in X\}$ is closed, then S or T have a fixed point in X.

Corollary 3.7. *([3]) Let (X, d) be a compact metric space and let S and T be mappings of X into $CL(X)$. Suppose that f and g are selfmappings of X satisfying the inequality:*

$$H(Sx, Ty) < \max\{d(fx, gy), D(fx, Sx), D(gy, Ty), \frac{1}{2}[D(fx, Ty)+D(gy, Sx)],$$
$$\frac{D(fx, Sx).D(gy, Ty)}{d(fx, gy)}, \frac{D(fx, Ty).D(gy, Sx)}{d(fx, gy)}, \} \tag{3.8}$$

for all $x, y \in X$ with $fx \neq gy$.

Let $S(X) \subset g(X)$ and $T(X) \subset f(X)$. If either f and S or g and T are continuous, then either $\{f, S\}$ or $\{g, T\}$ has a coincidence point.

The proof follows from Corollary 3.5 and Example 1.

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