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CONVERGENCE THEOREMS FOR TOTAL ASYMPTOTICALLY NONEXPANSIVE MAPPINGS

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Abstract

In this paper, a demiclosed principle for total asymptotically non-expansive mappings is established. An implicit iterative method for the class of total asymptotically nonexpansive mappings is considered. Weak and strong convergence theorems are established in a real Hilbert space. As applications of main results, an equilibrium problem is considered based on implicit iterative process.

1. Introduction and Preliminaries

Throughout this paper, we assume that H is a real Hilbert space, whose inner product and norm are denoted by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$. We also assume that $\psi:[0,\infty)\to[0,\infty)$ is strictly increasing continuous function with $\psi(0)=0$. $\to$ and $\rightharpoonup$ are denoted by strong convergence and weak convergence, respectively. Let C be a nonempty closed and convex subset of H and $T:C\to C$ a mapping. In this paper, we use $F(T)$ to denote the fixed point set of the mapping T. Recall that T is said to be nonexpansive if

$$\|Tx-Ty\|\le\|x-y\|,\quad \forall x,y\in C. \tag{1.1}$$

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T is said to be asymptotically nonexpansive if there exists a positive sequence $h_n \subset [1, \infty)$ with $\lim_{n\to\infty} h_n = 1$ such that

$$\|T^n x - T^n y\| \leq h_n \|x - y\|, \quad \forall x, y \in C, n \geq 1. \tag{1.2}$$

The class of asymptotically nonexpansive mappings was introduced by Goebel and Kirk [14] as a generalization of the class of nonexpansive mappings. They proved that if C is a nonempty closed convex and bounded subset of a real uniformly convex Banach space and T is an asymptotically nonexpansive mapping on C, then T has a fixed point.

T is said to be asymptotically nonexpansive in the intermediate sense if it is continuous and the following inequality holds:

$$\limsup_{n\to\infty} \sup_{x,y\in C} (\|T^n x - T^n y\| - \|x - y\|) \leq 0. \tag{1.3}$$

Observe that if we define

$$\sigma_n = \sup_{x,y\in C} (\|T^n x - T^n y\| - \|x - y\|) \quad \text{and } \nu_n = \max\{0, \sigma_n\},$$

then $\nu_n \to 0$ as $n \to \infty$. It follows that (1.3) is reduced to

$$\|T^n x - T^n y\| \leq \|x - y\| + \nu_n, \quad \forall x, y \in C, n \geq 1. \tag{1.4}$$

The class of mappings which are asymptotically nonexpansive in the intermediate sense was introduced by Bruck, Kuczumow and Reich [2]. It is known [16] that if C is a nonempty closed convex bounded subset of a uniformly convex Banach space E and T is asymptotically nonexpansive in the intermediate sense, then T has a fixed point. It is worth mentioning that the class of mappings which are asymptotically nonexpansive in the intermediate sense contains properly the class of asymptotically nonexpansive mappings.

Recently, Alber, Chidume and Zegeye [1] introduced the concept of total asymptotically nonexpansive mappings. Recall that T is said to be total asymptotically nonexpansive if

$$\|T^n x - T^n y\| \leq \|x - y\| + \mu_n \psi(\|x - y\|) + \nu_n, \quad \forall x, y \in C, \tag{1.5}$$

where $\{\mu_n\}$ and $\{\nu_n\}$ are nonnegative real sequences such that $\mu_n \to 0$ and $\nu_n \to 0$ as $n \to \infty$. From the definition, we see that the class of total asymptotically nonexpansive mappings include the class of asymptotically nonexpansive mappings as a special case; see also [13] for more details.

Recently, weak convergence problems of implicit (or non-implicit) iterative processes to a common fixed point for a finite family of nonexpansive mappings and asymptotically nonexpansive mappings have been considered by a number of authors (see, for example, [1],[8],[9],[12],[15],[19],[21],[23],[24],[27]-[32],[34]-[36],[38]).

In 2001, Xu and Ori [34] introduced the following implicit iteration process for a finite family of nonexpansive mappings $\{T_1, T_2, \dots, T_N\}$, with $\{\alpha_n\}$ a real sequence in $(0, 1)$, and an initial point $x_0 \in C$:

$$
\begin{aligned}
&x_1 = \alpha_1 x_0 + (1-\alpha_1)T_1 x_1,\\
&x_2 = \alpha_2 x_1 + (1-\alpha_2)T_2 x_2,\\
&\vdots\\
&x_N = \alpha_N x_{N-1} + (1-\alpha_N)T_N x_N,\\
&x_{N+1} = \alpha_{N+1} x_N + (1-\alpha_{N+1})T_1 x_{N+1},\\
&\vdots
\end{aligned}
$$

which can be re-written in the following compact form:

$$x_n = \alpha_n x_{n-1} + (1-\alpha_n)T_n x_n, \quad \forall n \geq 1, \tag{1.6}$$

where $T_n = T_{n(modN)}$ (here the mod N function takes values in $\{1, 2, \dots, N\}$). They obtained the following results in a real Hilbert space.

Theorem XO. *Let H be a real Hilbert space, C be a nonempty closed convex subset of H, and $T : C \to C$ be a finite family of nonexpansive self-mappings on C such that $F = \cap_{i=1}^N F(T_i) \neq \emptyset$. Let $\{x_n\}$ be defined by (1.6). If $\{\alpha_n\}$ is chosen so that $\alpha_n \to 0$, as $n \to \infty$, then $\{x_n\}$ converges weakly to a common fixed point of the family of $\{T_i\}_{i=1}^N$.*

In 2006, Chang et al. [9] improved the results of Xu and Ori [34] from the class of nonexpansive mappings to the class of asymptotically nonexpansive mappings which is defined on a nonempty closed and convex subset C of H such that $C + C \subset C$. To be more precise, they considered the following implicit iterative process for a finite family of asymptotically nonexpansive mappings $\{T_1, T_2, \dots, T_N\}$, with $\{\alpha_n\}$ a real sequence in $(0, 1)$, $\{u_n\}$ a bounded sequence in C, and an initial point $x_0 \in C$:

$$\begin{aligned}
&x_1 = \alpha_1 x_0 + (1-\alpha_1)T_1 x_1 + u_1,\\
&x_2 = \alpha_2 x_1 + (1-\alpha_2)T_2 x_2 + u_2,\\
&\vdots\\
&x_N = \alpha_N x_{N-1} + (1-\alpha_N)T_N x_N + u_N,\\
&x_{N+1} = \alpha_{N+1} x_N + (1-\alpha_{N+1})T_1 x_{N+1} + u_{N+1},\\
&\vdots\\
&x_{2N} = \alpha_{2N} x_{2N-1} + (1-\alpha_{2N})T_N^2 x_{2N} + u_{2N},\\
&x_{2N+1} = \alpha_{2N+1} x_{2N} + (1-\alpha_{2N+1})T_1^3 x_{2N+1} + u_{2N+1},\\
&\vdots
\end{aligned}$$

Since for each $n \geq 1$, it can be written as $n = (k-1)N + i$, where $i = i(n) \in \{1, 2, \ldots, N\}$, $k = k(n) \geq 1$ is a positive integer and $k(n) \to \infty$ as $n \to \infty$. Hence the above table can be rewritten in the following compact form:

$$x_n = \alpha_n x_{n-1} + (1-\alpha_n)T_{i(n)}^{k(n)} x_n + u_n, \quad \forall n \geq 1. \tag{1.7}$$

They obtained weak convergence theorems of the implicit iterative scheme (1.7) for a finite family of asymptotically nonexpansive mappings $\{T_1, T_2, \ldots, T_N\}$; see [9] for more details.

The purpose of this paper is to establish weak and strong convergence theorems of the implicit iteration process (1.7) for a finite family of uniformly Lipschitz total asymptotically nonexpansive mappings in a real Hilbert space.

Next, we recall some well-known concepts.

Recall that a space X is said to satisfy Opial's condition [18] if for each sequence $\{x_n\}$ in X, the condition that the sequence $x_n \to x$ weakly implies that

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|$$

for all $y \in E$ and $y \neq x$.

Recall that a mapping $T : C \to C$ is semicompact if any sequence $\{x_n\}$ in C satisfying $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$ has a convergent subsequence.

Recall that the mapping T is said to be demiclosed at the origin if for each sequence $\{x_n\}$ in C, the condition $x_n \to x_0$ weakly and $Tx_n \to 0$ strongly implies $Tx_0 = 0$.

Next, we show that the process (1.7) is well-defined if the control sequence satisfies $0 < \frac{L-1}{L} < \alpha_n < 1$, where $L = \max_{1 \le i \le N}\{L_i\}$. Indeed, define a mapping

$$W_n x = \alpha_n x_{n-1} + (1-\alpha_n)T_{i(n)}^{k(n)}x + u_n, \quad \forall n \ge 1, \quad \forall x \in C.$$

It follows that

$$\|W_n x - W_n y\| \le (1-\alpha_n)L\|x - y\|, \quad \forall x, y \in C.$$

Since $(1-\alpha_n)L < 1$, it follows that W_n is a contractive mapping and hence has a unique fixed point x_n in C. This is, the process (1.7) is well-defined.

In order to prove our main results, we also need the following lemmas.

Lemma 1.1 ([30]). *Let $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ be three nonnegative sequences satisfying the following condition:*

$$a_{n+1} \le (1+b_n)a_n + c_n, \quad \forall n \ge n_0,$$

where n_0 is some nonnegative integer, $\sum_{n=0}^{\infty} c_n < \infty$ and $\sum_{n=0}^{\infty} b_n < \infty$. Then $\lim_{n\to\infty} a_n$ exists.

Lemma 1.2 ([27]). *Let H be a real Hilbert space and $0 < p \le t_n \le q < 1$ for all $n \ge 0$. Suppose that $\{x_n\}$ and $\{y_n\}$ are sequences of H such that*

$$\limsup_{n\to\infty} \|x_n\| \le r, \quad \limsup_{n\to\infty} \|y_n\| \le r$$

and

$$\lim_{n\to\infty} \|t_n x_n + (1-t_n)y_n\| = r$$

hold for some $r \ge 0$. Then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.

Lemma 1.3. *Let C be a nonempty closed convex and bounded subset of a real Hilbert space H and T be a L-Lipschitz continuous and total asymptotically nonexpansive mapping with the function ψ and sequences $\{\mu_n\}$ and $\{\nu_n\}$ such that $\mu_n \to 0$ and $\nu_n \to 0$ as $n \to \infty$. Then $I - T$ is demiclosed at zero.*

Proof. Let $\{x_n\}$ be a sequence in C such that $x_n \rightharpoonup x^*$ and $x_n - Tx_n \to 0$ as $n \to \infty$. Next, we show that $x^* \in C$ and $x^* = Tx^*$. Since C is closed and convex, we see that $x^* \in C$. It is sufficient to show that $x^* = Tx^*$. Choose $\alpha \in (0, \frac{1}{1+L})$ and define $y_{\alpha,m} = (1-\alpha)x^* + \alpha T^m x^*$ for arbitrary but fixed $m \ge 1$. Note that

$$\begin{aligned}\|x_n - T^m x_n\| &\le \|x_n - Tx_n\| + \|Tx_n - T^2 x_n\| + \cdots + \|T^{m-1}x_n - T^m x_n\| \\ &\le mL\|x_n - Tx_n\|.\end{aligned}$$

It follows that

$$\lim_{n\to\infty} \|x_n - T^m x_n\| = 0. \tag{1.8}$$

Note that

$$\begin{aligned}
&\langle x^* - y_{\alpha,m}, y_{\alpha,m} - T^m y_{\alpha,m}\rangle \\
&= \langle x^* - x_n, y_{\alpha,m} - T^m y_{\alpha,m}\rangle + \langle x_n - y_{\alpha,m}, y_{\alpha,m} - T^m y_{\alpha,m}\rangle = \\
&= \langle x^* - x_n, y_{\alpha,m} - T^m y_{\alpha,m}\rangle + \langle x_n - y_{\alpha,m}, T^m x_n - T^m y_{\alpha,m}\rangle - \\
&\quad - \langle x_n - y_{\alpha,m}, x_n - y_{\alpha,m}\rangle + \langle x_n - y_{\alpha,m}, x_n - T^m x_n\rangle \\
&\le \langle x^* - x_n, y_{\alpha,m} - T^m y_{\alpha,m}\rangle + \\
&\quad + \|x_n - y_{\alpha,m}\|(\|x_n - y_{\alpha,m}\| + \mu_n \psi(\|x_n - y_{\alpha,m}\|) + \nu_m) - \\
&\quad - \|x_n - y_{\alpha,m}\|^2 + \|x_n - y_{\alpha,m}\|\|x_n - T^m x_n\| \le \\
&\le \langle x^* - x_n, y_{\alpha,m} - T^m y_{\alpha,m}\rangle + \mu_m M\psi(M) + \nu_m M + \\
&\quad + \|x_n - y_{\alpha,m}\|\|x_n - T^m x_n\|,
\end{aligned} \tag{1.9}$$

where $M = \sup_{n\ge 0}\{\|x_n - y_{\alpha,m}\|\}$. Since $x_n \rightharpoonup x^*$ and (1.8), we arrive at

$$\langle x^* - y_{\alpha,m}, y_{\alpha,m} - T^m y_{\alpha,m}\rangle \le \mu_m M\psi(M) + \nu_m M. \tag{1.10}$$

On the other hand, we have

$$\begin{aligned}
\langle x^* - y_{\alpha,m}, (x^* - T^m x^*) - (y_{\alpha,m} - T^m y_{\alpha,m})\rangle &\le (1+L)\|x^* - y_{\alpha,m}\|^2 \\
&= (1+L)\alpha^2\|x^* - T^m x^*\|^2.
\end{aligned} \tag{1.11}$$

Note that

$$\begin{aligned}
\|x^* - T^m x^*\|^2 &= \langle x^* - T^m x^*, x^* - T^m x^*\rangle = \\
&= \frac{1}{\alpha}\langle x^* - y_{\alpha,m}, x^* - T^m x^*\rangle = \\
&= \frac{1}{\alpha}\langle x^* - y_{\alpha,m}, (x^* - T^m x^*) - (y_{\alpha,m} - T^m y_{\alpha,m})\rangle + \\
&\quad + \frac{1}{\alpha}\langle x^* - y_{\alpha,m}, y_{\alpha,m} - T^m y_{\alpha,m}\rangle.
\end{aligned} \tag{1.12}$$

Substituting (1.10) and (1.11) into (1.12), we arrive at

$$\|x^* - T^m x^*\|^2 \le (1+L)\alpha\|x^* - T^m x^*\|^2 + \frac{\mu_m M\psi(M) + \nu_m M}{\alpha}.$$

This implies that

$$\alpha[1 - (1+L)\alpha]\|x^* - T^m x^*\|^2 \le \mu_m M\psi(M) + \nu_m M, \quad \forall m \ge 1. \tag{1.13}$$

Letting $m \to \infty$ in (1.13), we see that $T^m x^* \to x^*$. Since T is uniformly L-Lipschitz, we obtain that $x^* = Tx^*$. This completes the proof.

2. Main results

Theorem 2.1. *Let H be a real Hilbert space and C be a nonempty closed convex and bounded subset of H such that $C + C \subset C$. Let $T_i : C \to C$ be a uniformly L_i-Lipschitz total asymptotically nonexpansive mapping with the function ψ_i and sequences $\{\mu_n^{(i)}\}$, $\{\nu_n^{(i)}\}$ for each $i \in \{1, 2, \ldots, N\}$. Assume that $\sum_{n=1}^{\infty} \mu_n^{(i)} < \infty$ and $\sum_{n=1}^{\infty} \nu_n^{(i)} < \infty$ for each $i \in \{1, 2, \ldots, N\}$. Let $\{u_n\}$ be a bounded sequence in C such that $\sum_{n=1}^{\infty} \|u_n\| < \infty$ and $\{\alpha_n\}$ a sequence in $[\frac{L-1}{L}, a]$, where $L = \max_{1\le i\le N}\{L_i\} > 1$ and a is some constant in $(0, 1)$. Assume that $\mathcal{F} = \cap_{i=1}^{N} F(T_i) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by (1.7). Then the sequence $\{x_n\}$ converges weakly to some point $x^* \in \mathcal{F}$.*

Proof. Define the following sequences

$$\mu_n = \max\{\mu_n^{(1)}, \mu_n^{(2)}, \ldots, \mu_n^{(N)}\}$$

and

$$\nu_n = \max\{\nu_n^{(1)}, \nu_n^{(2)}, \ldots, \nu_n^{(N)}\}.$$

It is easy to see that $\sum_{n=1}^{\infty} \mu_n < \infty$ and $\sum_{n=1}^{\infty} \nu_n < \infty$. Fixing $p \in \mathcal{F}$, we have

$$\begin{aligned} \|x_n - p\| &\le \alpha_n \|x_{n-1} - p\| + (1-\alpha_n)\|T_{i(n)}^{k(n)} x_n - p\| + \|u_n\| \le \\ &\le \alpha_n \|x_{n-1} - p\| + (1-\alpha_n)\big(\|x_n - p\| + \mu_{k(n)}\psi_{i(n)}(\|x_n - p\|) + \nu_{k(n)}\big) + \|u_n\| \le \\ &\le \|x_{n-1} - p\| + \mu_{k(n)}\psi_{i(n)}((diam\ C)) + \nu_{k(n)} + \|u_n\| leq \\ &\le \|x_{n-1} - p\| + \mu_{k(n)}\psi_r((diam\ C)) + \nu_{k(n)} + \|u_n\|, \end{aligned} \tag{2.1}$$

where

$$\psi_r((diam\ C)) = \max\{\psi_1((diam\ C)), \psi_2((diam\ C)), \ldots, \psi_N((diam\ C))\}.$$

In view of Lemma 1.1, we obtain that the limit of the sequence $\{\|x_n - p\|\}$ exits. Next, we assume that

$$d = \lim_{n\to\infty} \|x_n - p\|, \tag{2.2}$$

where $d > 0$ is some constant. It follows that

$$\limsup_{n\to\infty} \|x_{n-1} - p - u_n\| \le \limsup_{n\to\infty} \|x_{n-1} - p\| + \limsup_{n\to\infty} \|u_n\| = d. \tag{2.3}$$

Note that

$$\begin{aligned}\|T_{i(n)}^{k(n)}x_n - p + u_n\| &\le \|T_{i(n)}^{k(n)}x_n - p\| + \|u_n\| \le \\ &\le \|x_n - p\| + \mu_{k(n)}\psi_{i(n)}(\|x_n - p\|) + \nu_{k(n)} + \|u_n\| \le \\ &\le \|x_n - p\| + \mu_{k(n)}\psi_r((diam\ C)) + \nu_{k(n)} + \|u_n\|.\end{aligned}$$

It follows that

$$\limsup_{n\to\infty} \|T_{i(n)}^{k(n)}x_n - p + u_n\| \le d. \tag{2.4}$$

On the other hand, we have

$$d = \lim_{n\to\infty} \|x_n - p\| = \lim_{n\to\infty} \|\alpha_n(x_{n-1} - p - u_n) + (1-\alpha_n)(T_{i(n)}^{k(n)}x_n - p + u_n)\|. \tag{2.5}$$

Combining (2.3), (2.4) with (2.4) and from Lemma 1.2, we obtain that

$$\lim_{n\to\infty} \|x_{n-1} - T_{i(n)}^{k(n)}x_n\| = 0. \tag{2.6}$$

Note that

$$\|x_n - x_{n-1}\| \le (1-\alpha_n)\|x_{n-1} - T_{i(n)}^{k(n)}x_n\| + \|u_n\|.$$

From (2.6), we arrive at

$$\lim_{n\to\infty} \|x_n - x_{n-1}\| = 0. \tag{2.7}$$

On the other hand, we have

$$\|T_{i(n)}^{k(n)}x_n - x_n\| \le \|T_{i(n)}^{k(n)}x_n - x_{n-1}\| + \|x_{n-1} - x_n\|,$$

which combined with (2.6) gives that

$$\lim_{n\to\infty} \|T_{i(n)}^{k(n)}x_n - x_n\| = 0. \tag{2.8}$$

From (2.7), we also have

$$\lim_{n\to\infty} \|x_n - x_{n+l}\| = 0, \quad \forall l = 1, 2, \ldots, N. \tag{2.9}$$

For any positive $n > N$, it can be rewritten as $n = (k(n)-1)N + i(n)$, $i(n) \in \{1, 2, \ldots, N\}$. Note that

$$\begin{aligned}\|x_{n-1} - T_n x_n\| &\le \|x_{n-1} - T_{i(n)}^{k(n)}x_n\| + \|T_{i(n)}^{k(n)}x_n - T_n x_n\| \le \\ &\le \|x_{n-1} - T_{i(n)}^{k(n)}x_n\| + L\|T_{i(n)}^{k(n)-1}x_n - x_n\| \le \\ &\le \|x_{n-1} - T_{i(n)}^{k(n)}x_n\| + L\{\|T_{i(n)}^{k(n)-1}x_n - T_{i(n-N)}^{k(n)-1}x_{n-N}\| + \\ &\quad + \|T_{i(n-N)}^{k(n)-1}x_{n-N} - x_{(n-N)-1}\| + \|x_{(n-N)-1} - x_n\|\}.\end{aligned} \tag{2.10}$$

On the other hand, we have $n-N=((k(n)-1)-1)N+i(n)=((k(n)-1)-1)N+i(n-N)$, i.e.,

$$k(n-N)=k(n)-1 \quad \text{and} \quad i(n-N)=i(n).$$

It follows that

$$\|T_{i(n)}^{k(n)-1}x_n - T_{i(n-N)}^{k(n)-1}x_{n-N}\| \le L\|x_n - x_{n-N}\| \tag{2.11}$$

and

$$\|T_{i(n-N)}^{k(n)-1}x_{n-N} - x_{(n-N)-1}\| = \|T_{i(n-N)}^{k(n-N)}x_{n-N} - x_{(n-N)-1}\|. \tag{2.12}$$

Combining (2.6), (2.9) with (2.10), we see that

$$\lim_{n\to\infty}\|x_{n-1} - T_n x_n\| = 0. \tag{2.13}$$

On the other hand, we have that

$$\|x_n - T_n x_n\| \le \|x_n - x_{n-1}\| + \|x_{n-1} - T_n x_n\|.$$

From (2.7) and (2.13), we obtain that

$$\lim_{n\to\infty}\|x_n - T_n x_n\| = 0. \tag{2.14}$$

Consequently, for any $j=1,2,\ldots,N$, we see that

$$\begin{aligned}\|x_n - T_{n+j}x_n\| &\le \|x_n - x_{n+j}\| + \|x_{n+j} - T_{n+j}x_{n+j}\| + \|T_{n+j}x_{n+j} - T_{n+j}x_n\| \le \\ &\le (1+L)\|x_n - x_{n+j}\| + \|x_{n+j} - T_{n+j}x_{n+j}\|.\end{aligned}$$

From (2.9) and (2.14), we arrive at

$$\lim_{n\to\infty}\|x_n - T_{n+j}x_n\| = 0.$$

Therefore, for $\forall i \in \{1,2,\ldots,N\}$, there exists some $e \in \{1,2,\ldots,N\}$ such that $n+e = i(mod\ N)$. It follows that

$$\lim_{n\to\infty}\|x_n - T_i x_n\| = \lim_{n\to\infty}\|x_n - T_{n+e}x_n\| = 0. \tag{2.15}$$

Since $\{x_n\}$ is bounded, we see that there exists a subsequence $\{x_{n_i}\} \subset \{x_n\}$ such that $x_{n_i} \rightharpoonup x^*$. From Lemma 1.3, we can obtain that $x^* \in \mathcal{F}$. Next we prove that $\{x_n\}$ converges weakly to x^*. Suppose the contrary. Then we see that there exists some subsequence $\{x_{n_j}\} \subset \{x_n\}$ such that $\{x_{n_j}\}$ converges weakly to $\bar{x} \in C$ and $\bar{x} \neq x^*$. From Lemma 1.3, we also have $\bar{x} \in \mathcal{F}$. Put $w = \lim_{n\to\infty} \|x_n - x^*\|$. Since H is an Opial's space, we see that

$$\begin{aligned} w = \lim_{n_i\to\infty} \|x_{n_i} - x^*\| &< \lim_{n_i\to\infty} \|x_{n_i} - \bar{x}\| = \\ = \lim_{n_j\to\infty} \|x_{n_j} - \bar{x}\| &< \lim_{n_j\to\infty} \|x_{n_j} - x^*\| = \\ = \lim_{n_i\to\infty} \|x_{n_i} - x^*\| &= w. \end{aligned}$$

This derives a contradiction. It follows that $\bar{x} = x^*$. This completes the proof.

Remark 2.2. Theorem 2.1 improves Theorem 1 of Chang et al. [9] from asymptotically nonexpansive mappings to total asymptotically nonexpansive mappings.

Corollary 2.3. *Let H be a real Hilbert space and C be a nonempty closed convex and bounded subset of H. Let $T_i : C \to C$ be a uniformly L_i-Lipschitz total asymptotically nonexpansive mapping with the function ψ_i and sequences $\{\mu_n^{(i)}\}$, $\{\nu_n^{(i)}\}$ for each $i \in \{1, 2, \ldots, N\}$. Assume that $\sum_{n=1}^{\infty} \mu_n^{(i)} < \infty$ and $\sum_{n=1}^{\infty} \nu_n^{(i)} < \infty$ for each $i \in \{1, 2, \ldots, N\}$. Let $\{\alpha_n\}$ be a sequence in $[\frac{L-1}{L}, a]$, where $L = \max_{1\le i\le N}\{L_i\} > 1$ and a is some constant in $(0, 1)$. Assume that $\mathcal{F} = \cap_{i=1}^N F(T_i) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by the following manner:*

$$x_0 \in C, \quad x_n = \alpha_n x_{n-1} + (1 - \alpha_n) T_{i(n)}^{k(n)} x_n, \quad \forall n \geq 1. \tag{2.16}$$

Then the sequence $\{x_n\}$ converges weakly to some point $x^* \in \mathcal{F}$.

Next, we prove a strong convergence theorem under the condition of semicompactness.

Theorem 2.4. *Let H be a real Hilbert space and C a nonempty closed convex and bounded subset of H such that $C + C \subset C$. Let $T_i : C \to C$ be a uniformly L_i-Lipschitz total asymptotically nonexpansive mapping with the function ψ_i and sequences $\{\mu_n^{(i)}\}$, $\{\nu_n^{(i)}\}$ for each $i \in \{1, 2, \ldots, N\}$. Assume that $\sum_{n=1}^{\infty} \mu_n^{(i)} < \infty$ and $\sum_{n=1}^{\infty} \nu_n^{(i)} < \infty$ for each $i \in \{1, 2, \ldots, N\}$. Let $\{u_n\}$ be a bounded sequence in C such that $\sum_{n=1}^{\infty} \|u_n\| < \infty$ and $\{\alpha_n\}$ a sequence in $[\frac{L-1}{L}, a]$, where $L = \max_{1\le i\le N}\{L_i\} > 1$ and a is some constant in $(0, 1)$. Assume that $\mathcal{F} = \cap_{i=1}^N F(T_i) \neq \emptyset$ and at least there exists a mapping T_r which is semicompact. Let $\{x_n\}$ be a sequence generated by (1.7). Then the sequence $\{x_n\}$ converges weakly to some point $x^* \in \mathcal{F}$.*

Proof. Without loss of generality, we may assume that T_1 is semicompact. It follows from (2.15) that

$$\lim_{n\to\infty} \|x_n - T_1 x_n\| = 0.$$

By the semicompactness of T_1, we have there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \to x \in C$ strongly. From (2.15), we have

$$\lim_{n_i \to \infty} \|x_{n_i} - T_l x_{n_i}\| = \|x - T_l x\| = 0,$$

for all $l = 1, 2, \cdots, N$. This implies that $x \in \mathcal{F}$. From Theorem 2.1, we know that $\lim_{n\to\infty} \|x_n - p\|$ exists for each $p \in \mathcal{F}$. This shows that $\lim_{n\to\infty} \|x_n - x\|$ exists. From $x_{n_i} \to x$, we have

$$\lim_{n\to\infty} \|x_n - x^*\| = 0.$$

This completes the proof of Theorem 2.4.

Corollary 2.5. *Let H be a real Hilbert space and C be a nonempty closed convex bounded subset of H. Let $T_i : C \to C$ be a uniformly L_i-Lipschitz total asymptotically nonexpansive mapping with the function ψ_i and sequences $\{\mu_n^{(i)}\}$, $\{\nu_n^{(i)}\}$ for each $i \in \{1, 2, \dots, N\}$. Assume that $\sum_{n=1}^{\infty} \mu_n^{(i)} < \infty$ and $\sum_{n=1}^{\infty} \nu_n^{(i)} < \infty$ for each $i \in \{1, 2, \dots, N\}$. Let $\{\alpha_n\}$ be a sequence in $[\frac{L-1}{L}, a]$, where $L = \max_{1\le i\le N}\{L_i\} > 1$ and a is some constant in $(0, 1)$. Assume that $\mathcal{F} = \cap_{i=1}^{N} F(T_i) \neq \emptyset$ and at least there exists a mapping T_r which is semicompact. Let $\{x_n\}$ be a sequence generated by (2.16). Then the sequence $\{x_n\}$ converges weakly to some point $x^* \in \mathcal{F}$.*

3. Applications

Let $A : C \to H$ be a mapping. Recall that the classical variational inequality problem is to find $x \in C$ such that

$$\langle Ax, y - x\rangle \geq 0, \quad \forall y \in C. \tag{3.1}$$

It is known that $x \in C$ is a solution to the variational inequality problem (3.1) if and only if x is a fixed point of the mapping $P_C(I - \rho A)$, where P_C is the metric projection from H onto C, $\rho > 0$ is a constant and I is the identity mapping. This implies that the variational inequality problem (3.1) is equivalent to a fixed point problem. This alternative formula is very important form the numerical analysis point of view. Recently, many authors studied the variational inequality (3.1) by iterative methods.

Let F be a bifunction of $C \times C$ into $\mathbb{R}$, where $\mathbb{R}$ is the set of real numbers. We consider the following equilibrium problem:

$$\text{Find } x \in C \text{ such that } F(x, y) \geq 0, \quad \forall y \in C. \tag{3.2}$$

In this paper, the set of such $x \in C$ is denoted by $EP(F)$, i.e.,

$$EP(F) = \{x \in C : F(x, y) \geq 0, \quad \forall y \in C\}.$$

Numerous problems in physics, optimization and economics reduce to find a solution of (1.1); see ([3],[5],[10],[11]). Approximating solutions of the problem (3.2) based on iterative methods was studied by many authors, see, for example, ([4]-[7],[17],[20],[22],[25],[26],[33],[37]) and the reference therein. Putting $F(x, y) = \langle Ax, y - x\rangle$, $\forall x, y \in C$, we see that $z \in FP(F)$ if and only if $\langle Az, y - z\rangle \geq 0$, $\forall y \in C$. That is, z is a solution to the variational inequality (3.1). Numerous problems in physics, optimization, and economics reduce to find a solution of the problem (3.2). To study the problem (3.2), we may assume that the bifunction $F : C \times C \to \mathbb{R}$ satisfies the following conditions:

(A1) $F(x, x) = 0$ for all $x \in C$;

(A2) F is monotone, i.e., $F(x, y) + F(y, x) \leq 0$ for all $x, y \in C$;

(A3) for each $x, y, z \in C$, $\lim_{t\downarrow 0} F(tz + (1 - t)x, y) \leq F(x, y)$;

(A4) for each $x \in C$, $y \mapsto F(x, y)$ is convex and weakly lower semi-continuous.

Next, we consider the convergence of implicit iterative process (1.7) for the equilibrium problem (3.2). To prove the main results in this section, we need the following lemma which can be found in [3] and [4].

Lemma 3.1. *Let C be a nonempty closed convex subset of H ad let F : $C \times C \to \mathbb{R}$ be a bifunction satisfying (A1)-(A4). Then, for any $r > 0$ and $x \in H$, there exists $z \in C$ such that $F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0$, $\forall y \in C$. Further, define a mapping*

$$T_r x = \{z \in C : F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \quad \forall y \in C\}$$

for all $r > 0$ and $x \in H$. Then, the following hold:

(1) *T_r is single-valued;*

(2) *T_r is firmly nonexpansive, i.e., for any $x, y \in H$,*

$$\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y\rangle;$$

(3) *$F(T_r) = EP(F)$;*

(4) *$EP(F)$ is closed and convex.*

Theorem 3.2. *Let C be a nonempty closed and convex subset of a real Hilbert space H. Let F_1 be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying $(A1)$-$(A4)$ such that $EP(F_1)$ is nonempty. Let $\{x_n\}$ be a sequence generated in the following manner:*

$$\begin{cases} x_0 \in H, \text{ chosen arbitrily} \\ F_1(y_n, y) + \frac{1}{r_n}\langle y - y_n, y_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ x_n = \alpha_n x_{n-1} + (1-\alpha_n)y_n + u_n, \quad \forall n \geq 1. \end{cases} \tag{3.3}$$

where $\{r_n\} \subset (0, \infty)$, $\{\alpha_n\} \subset (0, 1)$ and $\{u_n\}$ is a bounded sequence. Assume that the following conditions are satisfied

(1) $a \leq \alpha_n \leq b$, *where* $0 < a < b < 1$;

(2) $\liminf_{n\to\infty} r_n > 0$;

(3) $\sum_{n=1}^{\infty} \|u_n\| < \infty$.

Then the sequence $\{x_n\}$ generated in (3.3) *converges weakly to some point in $EP(F)$.*

Proof. Putting $y_n = T_{r_n}x_n$ for each $n \geq 1$, we from Lemma 3.1 see that T_{r_n} is firmly nonexpansive. Whenever needed, we shall equivalently write the implicit iteration (3.3) as

$$x_0 \in H, \quad x_n = \alpha_n x_{n-1} + (1-\alpha_n)T_{r_n}x_n + u_n, \quad \forall n \geq 1, \tag{3.4}$$

Fixing $p \in EP(F)$, we see that

$$\begin{aligned} \|x_n - p\| &\leq \alpha_n\|x_{n-1} - p\| + (1-\alpha_n)\|T_{r_n}x_n - p\| + \|u_n\| \\ &\leq \alpha_n\|x_{n-1} - p\| + (1-\alpha_n)\|x_n - p\| + \|u_n\|. \end{aligned}$$

This in turn implies that

$$\|x_n - p\| \leq \|x_{n-1} - p\| + \frac{\|u_n\|}{a}.$$

From Lemma 1.1, we obtain that $\lim_{n\to\infty}\|x_n - p\|$ exits. Next, we assume that $\lim_{n\to\infty}\|x_n - p\| = r > 0$. Note that

$$\limsup_{n\to\infty}\|x_{n-1} - p + u_n\| \leq r \tag{3.5}$$

and

$$\limsup_{n\to\infty}\|T_{r_n}x_n - p + u_n\| \leq \limsup_{n\to\infty}(\|x_n - p\| + \|u_n\|) \leq r. \tag{3.6}$$

On the other hand, we have

$$\lim_{n\to\infty}\|x_n-p\| = \lim_{n\to\infty}\|\alpha_n(x_{n-1}-p+u_n)+(1-\alpha_n)(T_{r_n}x_n-p+u_n)\| = r. \quad (3.7)$$

Combining (3.5), (3.6) with (3.7), from Lemma 1.2, we obtain that

$$\lim_{n\to\infty}\|x_{n-1} - T_{r_n}x_n\| = 0. \quad (3.8)$$

From (3.1), we arrive at

$$x_n - x_{n-1} = (1-\alpha_n)(T_{r_n}x_n - x_{n-1}) + u_n.$$

From the conditions (3) and (3.8), we see that

$$\lim_{n\to\infty}\|x_{n-1} - x_n\| = 0. \quad (3.9)$$

Note that

$$\|x_n - T_{r_n}x_n\| \le \|x_n - x_{n-1}\| + \|x_{n-1} - T_{r_n}x_n\|.$$

In view of (3.8) and (3.9), we obtain that

$$\lim_{n\to\infty}\|x_n - T_{r_n}x_n\| = 0. \quad (3.9)$$

Since the sequence $\{x_n\}$ is a bounded, we see that there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup q$.

Next, we show that $q \in EP(T)$. Since $y_n = T_{r_n}x_n$, we have

$$F(y_n, y) + \frac{1}{r_n}\langle y - y_n, y_n - x_n\rangle \ge 0, \quad \forall y \in C.$$

It follows from (A2) that

$$\langle y - y_n, \frac{y_n - x_n}{r_n}\rangle \ge F(y, y_n)$$

and hence

$$\langle y - y_{n_i}, \frac{y_{n_i} - x_{n_i}}{r_{n_i}}\rangle \ge F(y, y_{n_i}).$$

Since $\frac{y_{n_i}-x_{n_i}}{r_{n_i}} \to 0$, $y_{n_i} \rightharpoonup q$ and (A4), we have $F(y,q) \le 0$ for all $y \in C$. For t with $0 < t \le 1$ and $y \in C$, let $y_t = ty + (1-t)q$. Since $y \in C$ and $q \in C$, we have $y_t \in C$ and hence $F(y_t, q) \le 0$. So, from (A1) and (A4), we have

$$0 = F(y_t, y_t) \le tF(y_t, y) + (1-t)F(y_t, q) \le tF(y_t, y).$$

That is, $F(y_t, y) \geq 0$. It follows from (A3) that $F(q, y) \geq 0$ for all $y \in C$ and hence $q \in EP(F)$. This proves that $q \in EP(T)$.

Finally, we show that the sequence $\{x_n\}$ converges weakly to q. Suppose the contrary holds. It follows that there exists some subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \rightharpoonup \bar{q}$ and $q \neq \bar{q}$. By the same method as given above, we can prove that $\bar{q} \in EP(T)$. Put

$$\lim_{n\to\infty} \|x_n - q\| = d_1 \quad \text{and} \quad \lim_{n\to\infty} \|x_n - \bar{q}\| = d_2,$$

where d_1 and d_2 are two nonnegative numbers. In view of Opial's condition, we see that

$$d_1 = \liminf_{i\to\infty} \|x_{n_i} - q\| < \liminf_{j\to\infty} \|x_{n_i} - \bar{q}\| = \liminf_{j\to\infty} \|x_{n_j} - \bar{q}\| < \liminf_{j\to\infty} \|x_{n_j} - q\| = d_1.$$

This is a contradiction. Hence $\bar{q} = q$. This shows that the sequence $\{x_n\}$ converges weakly to q. The proof is completed.

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