BIG PICARD THEOREMS FOR HOLOMORPHIC MAPPINGS INTO THE COMPLEMENT OF 2n+1 MOVING HYPERSURFACES IN $\mathbb{C}P^n$

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Abstract

In this paper, we generalize the Big Picard Theorem to the case of holomorphic mappings of several complex variables into the complement of $2n+1$ moving hypersurfaces in general position in the n-dimensional complex projective space.

1 Introduction

The classical theorems of Picard are stated as follows:

Theorem 1.1 (Little Picard Theorem). *Let f be a holomorphic mapping of $\mathbb{C}$ into $\mathbb{C}P^1$. If f omits three distinct points in $\mathbb{C}P^1$, then f is constant.*

Theorem 1.2 (Big Picard Theorem). *Let f be a holomorphic mapping of a punctured disc $\triangle_R^* \subset \mathbb{C}$ into $\mathbb{C}P^1$. If f omits three distinct points in $\mathbb{C}P^1$, then f can be extended to a holomorphic mapping of $\triangle_R$ into $\mathbb{C}P^1$.*

In [4], [5], Fujimoto generalized the above results to the case of holomorphic mappings into the complement of $2n+1$ hyperplanes in general position in $\mathbb{C}P^n$. He proved the following theorems.

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Theorem 1.3. *Let f be a holomorphic mapping of $\mathbb{C}^m$ into $\mathbb{C}P^n$. If f omits $2n+1$ hyperplanes in $\mathbb{C}P^n$ in general position, then f is constant.*

Theorem 1.4. *Let S be a regular thin analytic subset of a domain D in $\mathbb{C}^m$. Then every holomorphic mapping f of $D \setminus S$ into the complement X of $2n+1$ hyperplanes $H_1, \ldots, H_{2n+1}$ in general position in $\mathbb{C}P^n$ can be extended to a holomorphic mapping of D into $\mathbb{C}P^n$.*

Furthermore, these results were generalized to the case where the holomorphic f intersects the hyperplane H_j with multiplicity at least m_j ($j \in \{1, \ldots, q\}, q \geq 2n+1$) where $m_1, \ldots, m_q$ are positive integers and may be ∞, with $\sum_{j=1}^{2n+1} \frac{1}{m_j} < \frac{q-n-1}{n}$ (see Nochka [9], and Tu [13] for the both of cases: fixed and moving hyperplanes).

For the case of hypersurfaces, in [10] Noguchi and Winkelmann obtained the following result.

Theorem 1.5. *Let $X \subset \mathbb{C}P^n$ be an irreducible subvariety, and $\{D_j\}_{j=1}^{q}$ ($q \geq 2dimX + 1$) be distinct hypersurfaces cuts of X that are in general position as hypersurfaces of X. Then $X \setminus \cup_{j=1}^{q} D_j$ is complete hyperbolic and hyperbolically imbedded into X.*

On the other hand, we have the following result of Kierman: Let X be an m-dimensional complex manifold and A a closed complex subspace consisting of hypersurfaces with normal crossing singularities. If a complex space Y is hyperbolically imbedded in a complex space Z, then every holomorphic mapping f of $X \setminus A$ into Y extends to a holomorphic mapping of X into Z (see [7], p. 285 or [6] for more further results). Therefore, we immediately obtain Picard's theorems for holomorphic mappings into the complement of $2n+1$ hypersurfaces in general position in $\mathbb{C}P^n$. This means that the extension problem is already known for the case of fixed hypersurfaces. However, in the case of moving hypersurfaces this problem is not yet studied. On the other hand, recently, the Nevanlinna theory was studied for the case of moving hypersurfaces (see Dethloff-Tan [2], and Eremenko-Sodin [3]). This motivates us to study the extension problem of holomorphic mappings involving moving hypersurfaces. In particular, we shall prove the Big Picard Theorem for holomorphic mappings into the complement of $2n+1$ moving hypersurfaces in $\mathbb{C}P^n$.

Denote by $\mathcal{H}_D$ the ring of all holomorphic functions on D. Let Q be a homogeneous polynomial in $\mathcal{H}_D[x_0, \ldots, x_n]$ of degree $d \geq 1$. Denote by $Q(z)$ the homogeneous polynomial over $\mathbb{C}$ obtained by substituting a specific point $z \in D$ into the coefficients of Q. We define a moving hypersurface in $\mathbb{C}P^n$ to be any homogeneous polynomial $Q \in \mathcal{H}_D[x_0, \ldots, x_n]$ such that the coefficients of Q have no common zero point. Let f be a holomorphic mapping of an open subset $\Omega \subset D$ into $\mathbb{C}P^n$. We say that f omits the moving hypersurface Q if for any $z_0 \in \Omega$ and for a reduced presentation $\tilde{f} = (f_0, \cdots, f_n)$ of f in a neighborhood U of z_0 in Ω, $Q(f_0, \cdots, f_n) \neq 0$ on U. We say that moving hypersurfaces $\{Q_j\}_{j=1}^q$ $(q \geq n+1)$ in $\mathbb{C}P^n$ are in pointwise general position on D if for any $z \in D$ and for any $1 \leq j_0 < \cdots < j_n \leq q$, the system of equations

$$\begin{cases} Q_{j_i}(z)(w_0, \ldots, w_n) = 0 \\ \qquad 0 \leq i \leq n \end{cases}$$

has only the trivial solution $w = (0, \ldots, 0)$ in $\mathbb{C}^{n+1}$.

In connection with the Little Picard Theorem, we remak that there exist nonconstant holomorphic mappings of $\mathbb{C}$ into $\mathbb{C}P^1$ omits three moving points (moving hypersurfaces) in $\mathbb{C}P^1$ in pointwise general position. This can be seen easily as follows: consider the nonconstant meromorphic mapping $f := (e^z : 1)$ and three moving points $Q_1 := (1 : 0)$, $Q_2 := (0 : 1)$, $Q_3 := (1 : e^z)$.

In connection with the Big Picard Theorem, we shall prove the following theorems.

Theorem 1.6. *Let S be an analytic subset of a domain $D \subset \mathbb{C}^m$ with codimension one, whose singularities are normal crossings. Let f be a holomorphic mapping of $D \setminus S$ into $\mathbb{C}P^n$. Assume that f omits $2n+1$ moving hypersurfaces $\{Q_j\}_{j=1}^{2n+1}$ in $\mathbb{C}P^n$ in pointwise general position on D. Then f can be extended to a holomorphic mapping of D into $\mathbb{C}P^n$.*

Theorem 1.7. *Let D be a domain in $\mathbb{C}^m$ and let $S \subset D$ be either a closed analytic subset with codimension at least two or a closed subset with (2m-2)-dimensional Hausdorff measure equal to zero. Let f be a holomorphic mapping of $D \setminus S$ into $\mathbb{C}P^n$. Assume that f omits $2n+1$ moving hypersurfaces $\{Q_j\}_{j=1}^{2n+1}$ in $\mathbb{C}P^n$ in pointwise general position on D. Then f can be extended to a holomorphic mapping of D into $\mathbb{C}P^n$.*

2 Proof of our results

In order to prove Theorems 1.6-1.7, we need some preparations.

Definition 2.1. *Let D be a domain in $\mathbb{C}^m$ and $\triangle$ be the unit disc in $\mathbb{C}$.*

i) A family $\mathcal{F}$ of holomorphic mappings of D into $\mathbb{C}P^n$ is said to be normal if $\mathcal{F}$ is relatively compact in $Hol(D, \mathbb{C}P^n)$ in the compact-open topology.

ii) We say that a holomorphic mapping f of $D \subset \mathbb{C}^m$ into $\mathbb{C}P^n$ is normal if the family $\{f \circ \psi : \psi \in Holl(\triangle, D)\}$ is normal.

Lemma 2.1. *Let S be an analytic subset of a domain D in $\mathbb{C}^m$ with codimension one, whose singularities are normal crossings. Let f be a holomorphic* *mapping of $D \setminus S$ into $\mathbb{C}P^n$. If f is normal then it can be extended to a holomorphic mapping of D into $\mathbb{C}P^n$.*

For the proof, we refer to [6], Theorem 2.3.

Lemma 2.2 ([12], Corollary 2.8)**.** *Let $\mathcal{F}$ be a family of holomorphic mappings of a domain $D \subset \mathbb{C}^m$ into $\mathbb{C}P^n$. Then $\mathcal{F}$ is not normal if and only if there exist sequences $\{f_j\} \subset \mathcal{F}$, $\{p_j\} \subset D$ with $p_j \to p_0 \in D$, $\{r_j\} \subset (0, +\infty)$ with $r_j \to 0$ such that $\lim_{j\to\infty} r_j/d(p_j, \mathbb{C}^m \setminus D) = 0$ (where d is the Euclidean distance) and the sequence $g_j(\xi) := f_j(p_j + r_j\xi)$ ($\xi \in \mathbb{C}^m$ with $p_j + r_j\xi \in D$) converges uniformly on compact subsets of $\mathbb{C}^m$ to a nonconstant holomorphic mapping g of $\mathbb{C}^m$ into $\mathbb{C}P^n$.*

Lemma 2.3 ([1], Proposition 2.9)**.** *Let Ω be a hyperbolic domain in $\mathbb{C}^m$ and M be a compact complete Hermitian manifold with a length function E_M. Let f be a holomorphic mapping of Ω into M. Then f is normal if and only if there exists a positive constant c such that for all $z \in \Omega$ and all $\xi \in T_z\Omega$,*

$$|E_M(f(z), df(z)(\xi)| \le cF_K^{\Omega}(z, \xi),$$

where $df(z)$ is the tangent mapping from $T_z\Omega$ to $T_{f(z)}M$ induced by f and F_K^{Ω} denotes by the infinitesimal Kobayashi metric on Ω.

Proof of Theorem 1.6. Let w_0 be an arbitrary point in S. Take bounded neighborhoods U and V of w_0 in D such that $\overline{U} \subset V \subset D$.
Case 1: $f_{|U\setminus S}$ is not normal.
Then, the family $\{f \circ \psi : \psi \in Hol(\triangle, U \setminus S)\}$ is not normal. By Lemma 2.2, there exist sequences $\{f_j\} \subset \mathcal{F}$, $\{p_j\} \subset \triangle$ with $p_j \to p \in \triangle$, $\{r_j\} \subset (0, +\infty)$ with $r_j \to 0$ such that $\lim_{j\to\infty} r_j/d(p_j, \mathbb{C} \setminus \triangle) = 0$ and the sequence $g_j(t) := f_j(p_j + r_jt)$ ($t \in \mathbb{C}$ with $p_j + r_jt \in \triangle$) converges uniformly on compact subsets of $\mathbb{C}$ to a nonconstant holomorphic mapping g of $\mathbb{C}$ into $\mathbb{C}P^n$. We write $f_j = f \circ \psi_j$ where $\psi_j \in Hol(\triangle, U \setminus S)$. Since $\overline{U \setminus S}$ is compact, without loss of generality, we may assume that $\{\psi_j(p_j)\}$ converges to z_0 in $\overline{U \setminus S}$ ($\subset V$).

Let R be an arbitrary positive constant. Then, there exists a positive integer j_0 such that $p_j + r_jt \in \triangle$ for all $j > j_0$ and $t \in \triangle_R := \{z \in \mathbb{C} : |z| < R\}$. Let $\tilde{g} = (g_0, \cdots, g_n)$, $\tilde{g}_j = (g_{j0}, \cdots, g_{jn})$ be reduced presentations of g, g_j on $\triangle_R$ respectively such that $\{g_{ji}\}_{j>j_0}$ converges uniformly on compact subsets of $\triangle_R$ to g_i ($i \in \{0, \ldots, n\}$).

Let K be an arbitrary compact subset of $\triangle_R$.
For any $\epsilon_1 > 0$, there exists a positive integer j_1 such that

$$|g_{ji}(t) - g_i(t)| < \epsilon_1, \text{ for all } t \in K, j > j_1, i \in \{0, \ldots, n\}. \tag{2.1}$$

Set $d_k := \deg Q_k$ and $\mathcal{T}_{d_k} := \big\{(i_0,\dots,i_n) \in \mathbb{N}_0^{n+1} : i_0 + \cdots + i_n = d\big\}$.
Assume that

$$Q_k = \sum_{I \in \mathcal{T}_{d_k}} a_{kI} x^I \quad (k = 1,\dots,2n+1)$$

where $a_{jI} \in \mathcal{H}_D$, $x^I = x_0^{i_0}\cdots x_n^{i_n}$ for $x = (x_0,\dots,x_n)$ and $I = (i_0,\dots,i_n)$.

Since $\triangle$ and V are hyperbolic, by Barth Theorem (see (3.2.1) p. 60 in [7]), the Kobayashi distances $d^K_\triangle$, d^K_V define the topologies of $\triangle$, V respectively.

For any $\epsilon_2 > 0$, there exists a positive constant $\delta > 0$ such that

$$|a_{kI}(z) - a_{kI}(z_0)| < \epsilon_2 \tag{2.2}$$

for all $z \in W_\delta(z_0) := \{z \in V : d^K_V(z_0, z) < \delta\}$ and for all $k \in \{1,\dots,2n+1\}$, $I \in \mathcal{T}_{d_k}$.

Set $B_p := \{\xi \in \triangle : d^K_\triangle(p,\xi) < \frac{\delta}{3}\}$. Since $p_j \to p$, $r_j \to 0$, $\psi_j(p_j) \to z_0$, we easily obtain that there exists a positive integer j_2 such that $|\psi_j(p_j) - z_0| < \frac{\delta}{3}$, $p_j \in B_p$ and $p_j + r_j t \in B_p$ for all $t \in K$, $j > j_2$.
Therefore, for all $t \in K, j > j_2$ we have

$$\begin{aligned}
d^K_V\big(\psi_j(p_j + r_j t), z_0\big) &\le d^K_V\big(\psi_j(p_j + r_j t), \psi_j(p_j)\big) + d^K_V\big(\psi_j(p_j), z_0)\big)\\
&\le d^K_{U\setminus S}\big(\psi_j(p_j + r_j t), \psi_j(p_j)\big) + \frac{\delta}{3}\\
&\le d^K_\triangle(p_j + r_j t, p_j) + \frac{\delta}{3}\\
&\le d^K_\triangle(p_j + r_j t, p) + d^K_\triangle(p_j, p) + \frac{\delta}{3} < \delta.
\end{aligned}$$

This means that $\psi_j(p_j + r_j t) \in W_\delta(z_0)$ for all $t \in K, j > j_2$.
Then, by (2.2) we have

$$|a_{kI}(\psi_j(p_j + r_j t)) - a_{kI}(z_0)| < \epsilon_2, \tag{2.3}$$

for all $k \in \{1,\dots,2n+1\}$, $I \in \mathcal{T}_{d_k}, t \in K$.
Therefore, there exists a positive constant c_1 such that

$$|a_{kI}(\psi_j(p_j + r_j t))| < c_2, \text{ for all } k \in \{1,\dots,2n+1\},\ I \in \mathcal{T}_{d_k}, t \in K. \tag{2.4}$$

Since $\{g_{ji}\}_{j>j_0}$ converges uniformly on compact subsets of $\triangle_R$ to g_i $(i \in \{0,\dots,n\})$ and since K is compact, there exists a positive constant c_2 such that

$$|g_{ji}(t)| < c_2 \text{ and } |g_i(t)| < c_2, \text{ for all } j \ge j_0, i \in \{0,\dots,0\}, t \in K.$$

Combining with (2.1), (2.3) and (2.4), we easily obtain that for any $\epsilon > 0$, there exists a positive integer $j_3 > j_0$ such that

$$|Q_k(\psi_j(p_j + r_j t))(\tilde{g}_j(t)) - Q_k(z_0)(\tilde{g}(t))| < \epsilon,$$

for all $t \in K,\ j > j_3$ and $k \in \{1, \ldots, 2n+1\}$.
This means that $\{Q_k(\psi_j(p_j+r_j t))(\tilde{g}_j(t))\}_{j>j_3}$ converges uniformly on compact subsets of $\triangle_R$ to $Q_k(z_0)(\tilde{g}(t))$, for all $k \in \{1, \ldots, 2n+1\}$. Since f omits hypersurfaces $\{Q_k\}_{k=1}^{2n+1}$, we get that $Q_k(\psi_j(p_j + r_j t))(\tilde{g}_j(t)) \neq 0$ on $\triangle_R$ for all $j > j_3$. Then by Hurwitz theorem, the holomorphic function $Q_k(z_0)(\tilde{g})$ on $\triangle_R$ vanishes either nowhere or every where.
Letting $R \to \infty$, for each $k \in \{1, \ldots, 2n+1\}$ we have
i) $Q_k(z_0)(\tilde{g}) \neq 0$ on $\mathbb{C}$, i.e. $g(\mathbb{C}) \cap Q_k(z_0) = \emptyset$, or
ii) $Q_k(z_0)(\tilde{g}) \equiv 0$ on $\mathbb{C}$, i.e. $g(\mathbb{C}) \subset Q_k(z_0)$
(we identify the polynomial $Q_k(z_0) \in \mathbb{C}[x_0, \cdots, x_n]$ with the hypersurface in $\mathbb{C}P^n$ defined by $Q_k(z_0)$).
Denote by I the set of all indies $k \in \{1, \cdots, 2n+1\}$ with $g(\mathbb{C}) \subset Q_k(z_0)$. Set $X := \cap_{k\in I} Q_k(z_0)$ if $I \neq \emptyset$ and $X := \mathbb{C}P^n$ if $I = \emptyset$. Since $\mathbb{C}$ is irreducible, there exists an irreducible component Z of X such that $g(\mathbb{C}) \subset Z \setminus (\cup_{i\notin I} Q_k(z_0))$. Since $z_0 \in \overline{U \setminus S} \subset D$, we have that $\{Q_k(z_0)\}_{k=1}^{2n+1}$ are in general position in $\mathbb{C}P^n$. This implies that $\{Q_k(z_0) \cap Z\}_{k\notin I}$ are in general position in Z. Furthermore, it is easy to see that $\#(\{1, \cdots, 2n+1\} \setminus I) \geq 2\mathrm{dim}Z + 1$. By Theorem 1.5, we get that $Z \setminus (\cup_{i\notin I} Q_i(z_0))$ is hyperbolic. Hence, g is constant. This is a contradiction.
Case 2: If $f_{|U\setminus S}$ is normal. Then, since singularities of S are normal crossings and by Lemma 2.1 we get that f extends to a holomorphic mapping of U into $\mathbb{C}P^n$.

We have completed the proof of Theorem 1.6. □

Proof of Theorem 1.7. Let w_0 be an arbitrary point in S. Take bounded neighborhoods $U,\ V$ of w_0 such that $\overline{U} \subset V \subset D$. By an argument similar to the proof of Theorem 1.6, we get that $f_{|U\setminus S}$ is normal. It is clear that U is hyperbolic and $d^K_{U\setminus S} = d^K_U$ on $U \setminus S$ (see (3.2.19) p.65 and (3.2.22) p.67 in [7]). Denote by $d_{\mathbb{C}P^n}$ the Fubini-Study distance on $\mathbb{C}P^n$. By Lemma 2.3, there exists a positive constant c such that

$$d_{\mathbb{C}P^n}(f(w), f(v)) \leq c d^K_{U\setminus S}(w, v) = c d^K_U(w, v), \tag{2.5}$$

for all $w, v \in U \setminus S$. Let p be an arbitrary point in $S \cap U$ and $\{p_j\}$ a sequence of $U \setminus S$ such that $p_j \to p$ ($\in U$). Then by (2.5), we have

$$d_{\mathbb{C}P^n}(f(p_i), f(p_j)) \leq c d^K_{U\setminus S}(p_i, p_j) = c d^K_U(p_i, p_j).$$

On the other hand, $\{p_j\}$ is a Cauchy sequence in U. Hence, $\{f(p_j)\}$ is a Cauchy sequence in $\mathbb{C}P^n$. Then $\{f(p_j)\}$ converges to a point in $\mathbb{C}P^n$. Furthermore, for two arbitrary sequences $\{p_j\}$ and $\{p'_j\}$ of $U \setminus S$ with $\lim p_j = \lim p'_j = p$, by (2.5) we have $\lim f(p_j) = \lim f(p'_j)$. Then, by Riemann extension theorem, $f_{|U\setminus S}$ can be extended to a holomorphic mapping of U into $\mathbb{C}P^n$. We have completed the proof of Theorem 1.7. $\square$

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