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IMPLICIT AND EXPLICIT ITERATIVE PROCESS WITH ERRORS FOR COMMON FIXED POINTS OF A FINITE FAMILY OF STRICTLY PSEUDOCONTRACTIVE MAPPINGS

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Abstract

In this paper, a necessary and sufficient conditions for the strong convergence to a common fixed point of a finite family of strictly pseudocontractive mappings of Browder-Petryshyn type are proved in an arbitrary real Banach spaces using a implicit iteration scheme with errors. The results presented in this paper not only correct some mistakes appeared in the paper by Y. Su and S. Li [Composite implicit iteration process for common fixed points of a finite family of strictly pseudocontractive maps, J. Math. Anal. Appl., 320(2006), 882-891] but also improve and extend some recent results by M. O. Osilike [M. O. Osilike, Implicit iteration process for common fixed points of a finite family of strictly pseudocontractive maps, J. Math. Anal. Appl., 294(2004), 73-81],and F.Gu [The new composite implicit iteration process with errors for common fixed points of a finite of strictly pseudocontractive mappings, J. Math. Anal. Appl., 329 (2007), 766-776]. Moreover, in this paper the methods of proof of main results are also different from that of Osilike, Su and Li.

Key Words: Strictly pseudocontractive mappings; Implicit iteration process with errors; Common fixed points

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1 Introduction and preliminaries

In this paper we assume that E is a real Banach space and let J denote the *normalized duality mapping* from E into 2^{E^*} given by $J(x) = \{f \in E^* : \langle x, f\rangle = ||x||^2, ||x|| = ||f||\}$, where E^* denotes the dual space of E and $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing. If E^* is strictly convex, then J is single-valved. In the sequel, we shall denote the single-valved duality mapping by j.

Definition 1.1. Let K be a closed subset of real Banach space E and $T : K \to K$ be a mapping. T is said to be *semi-compact*, if for any bounded sequence $\{x_n\}$ in K such that $||x_n - Tx_n|| \to 0$ $(n \to \infty)$, then there exists a subsequence $\{x_{n_i}\} \subset \{x_n\}$ such that $x_{n_i} \to x^* \in K$.

Definition 1.2. A mapping T with domain $D(T)$ and range $R(T)$ in E is called *nonexpansive* if

$$||Tx - Ty|| \leq ||x - y||, \ \ \forall x, y \in D(T) \tag{1.1}$$

Definition 1.3. A mapping T with domain $D(T)$ and range $R(T)$ in E is called *strictly pseudocontractive* in the terminology of Brower and Petryshyn [1], if for all $x, y \in D(T)$, there exists $k \in (0, 1)$ and $j(x - y) \in J(x - y)$ such that

$$\langle Tx - Ty, j(x - y)\rangle \leq ||x - y||^2 - k||x - y - (Tx - Ty)||^2 \tag{1.2}$$

If I denotes the identity operator, then (1. 2) can be written in the form

$$\langle (I - T)x - (I - T)y, j(x - y)\rangle \geq k||(I - T)x - (I - T)y)||^2 \tag{1.3}$$

It is easy to know that every strictly pseudocontractive mapping is L-Lipschitzian and continuous. Indeed, it follows from (1.3) that

$$k||(x - y) - (Tx - Ty)||^2 \leq ||(x - y) - (Tx - Ty)|| \cdot ||j(x - y)||,$$

$$k(||Tx - Ty|| - ||x - y||) \leq k||(x - y) - (Tx - Ty)|| \leq ||x - y||,$$

i.e.,

$$||Tx - Ty|| \leq L||x - y||, \quad \text{where} \quad L = \frac{k + 1}{k}.$$

The class of strictly pseudocontractive mappings has been studied by several authors (see, for example, [1, 3-6, 8-12]).

Let K be a nonempty convex subset of E, and let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive self-maps of K. In [13], Xu and Ori introduced the following implicit iteration process. For any $x_0 \in K$ and $\{\alpha_n\}_{n=1}^{\infty} \subset (0,1)$, the sequence $\{x_n\}_{n=1}^{\infty}$ is generated as follows:

$$\begin{cases} x_1 = (1-\alpha_1)x_0 + \alpha_1 T_1 x_1, \\ x_2 = (1-\alpha_2)x_1 + \alpha_2 T_2 x_2, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdot , \\ x_N = (1-\alpha_N)x_{N-1} + \alpha_N T_N x_N, \\ x_{N+1} = (1-\alpha_{N+1})x_N + \alpha_{N+1} T_1 x_{N+1}, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdot , \end{cases}$$

which can be written in the following compact form as follows:

$$x_n = (1-\alpha_n)x_{n-1} + \alpha_n T_n x_n, \ \ \forall n \geq 1, \tag{1.4}$$

where $T_n = T_{n(modN)}$.

Using this iteration process, they proved the following convergence theorem for nonexpansive mappings in Hilbert spaces.

Theorem XO [13] *Let H be a Hilbert space and let K be a nonempty closed convex subset of H. Let $\{T_i\}_{i=1}^N : K \to K$ be N nonexpansive mappings such that $F = \cap_{i=1}^N F(T_i) \neq \emptyset$ (the set of common fixed points of $\{T_i\}_{i=1}^N$). Let $x_0 \in K$ and $\{\alpha_n\}$ be a sequence in (0, 1) with $\lim_{n\to\infty}(1-\alpha_n) = 0$. Then the sequence $\{x_n\}$ defined by (1.4) converges weakly to a common fixed point of $\{T_i\}_{i=1}^N$.*

In [7], M. O. Osilike extended their results from the nonexpansive mappings to strictly pseudocontractive mappings. by this iteration process, he proved the following convergence theorems in Hilbert and Banach spaces.

Theorem MO1[7] *Let H be a Hilbert space and let K be a nonempty closed convex subset of H. Let $\{T_i\}_{i=1}^N : K \to K$ be N strictly pseudocontractive mappings such that $F = \cap_{i=1}^N F(T_i) \neq \emptyset$ (the set of common fixed points of $\{T_i\}_{i=1}^N$). Let $x_0 \in K$ and $\{\alpha_n\}$ be a sequence in (0, 1) with $\lim_{n\to\infty}(1-\alpha_n) = 0$. Then the sequence $\{x_n\}$ defined by (1.4) converges weakly to a common fixed point of $\{T_i\}_{i=1}^N$.*

Theorem MO2[7] *Let E be a real Banach space and let K be a nonempty closed convex subset of E. Let $\{T_i\}_{i=1}^N : K \to K$ be N strictly pseudocontractive mappings such that $F = \cap_{i=1}^N F(T_i) \neq \emptyset$ (the set of common fixed points of $\{T_i\}_{i=1}^N$). Let $x_0 \in K$ and $\{\alpha_n\}$ be a sequence in (0, 1) satisfying the conditions:*

(i) $0 < \alpha_n < 1$,

(ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(iii) $\sum_{n=1}^{\infty} \alpha_n^2 < \infty$.

Then the sequence $\{x_n\}$ defined by (1.4) converges strongly to a common fixed point of the mappings $\{T_i\}_{i=1}^N$ if and only if $\liminf_{n\to\infty} d(x_n, F) = 0$.

Recently, Su and Li introduced the following implicit iteration process. For any $x_0 \in K$, the sequence $\{x_n\}_{n=1}^{\infty}$ is generated as follows:

$$\begin{cases} x_n = \alpha_n x_{n-1} + (1-\alpha_n)T_n y_n, \\ y_n = \beta_n x_{n-1} + (1-\beta_n)T_n x_n, \quad \forall n \geqslant 1, \end{cases} \tag{1.5}$$

where $T_n = T_{n(modN)}$, $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$ be two real sequences in [0,1].

Using this iteration process, they proved the following theorem in real Banach space.

Theorem SL[12]. *Let E be a real Banach space and let K be a nonempty closed convex subset of E. Let $\{T_i\}_{i=1}^{N}$ be N strictly pseudocontractive self-maps of K such that $F = \bigcap_{i=1}^{N} F(T_i) \neq \emptyset$, where $F(T_i) = \{x \in K : T_i x = x\}$ and let $\{\alpha_n\}_{n=1}^{\infty}, \{\beta_n\}_{n=1}^{\infty} \subset [0,1]$ be two real sequences satisfying the conditions:*

(i) $\sum_{n=1}^{\infty}(1-\alpha_n) = +\infty$;

(ii) $\sum_{n=1}^{\infty}(1-\alpha_n)^2 < +\infty$;

(iii) $\sum_{n=}^{\infty}(1-\beta_n) < +\infty$;

(iv) $(1-\alpha_n)(1-\beta_n)L^2 < 1, \ \forall n \geq 1,$ *where $L \geq 1$ is common Lipschitz constant of $\{T_i\}_{i=1}^{N}$.*

Let $x_0 \in K$ and let $\{x_n\}_{n=1}^{\infty}$ be defined by (1.5), then

(1) $\lim_{n\to\infty} ||x_n - p||$ *exists for all $p \in F$;*

(2) $\liminf_{n\to\infty} ||x_n - T_n x_n|| = 0.$

Remark 1.1. It should be pointed the Theorem SL generalize and improve the results of Osilike [7] in 2004, but the proof of [12, Theorem 2.1] has some problems.

Motivated and inspired by the above works, in this paper, we introduce a composite implicit iteration process as follows:

$$\begin{cases} x_n &= (1-\alpha_n-\gamma_n)x_{n-1} + \alpha_n T_n y_n + \gamma_n u_n, \ n \geq 1, \\ y_n &= (1-\beta_n-\delta_n)x_{n-1} + \beta_n T_n x_n + \delta_n v_n, \ n \geq 1, \end{cases} \tag{1.6}$$

where $T_n = T_{n(modN)}$, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\delta_n\}$ are four real sequences in [0, 1] satisfying $\alpha_n + \gamma_n \leq 1$ and $\beta_n + \delta_n \leq 1$ for all $n \geq 1$, $\{u_n\}$ and $\{v_n\}$ are two bounded sequences in K and x_0 is a given point.

Observe that if K is a nonempty closed convex subset of E and $T_i : K \to K$ is a k_i-strictly pseudocontractive mapping, then it is a L_i Lpschitzian mapping with $L_i = 1 + \frac{1}{k_i}$. If $\alpha_n \beta_n L^2 < 1$, where $L = \max_{1\leq i\leq N}\{L_i\}$, then for given $x_{n-1} \in K$, $\gamma_n u_n \in K$ and $\delta_n v_n \in K$, the mapping $S_n : K \to K$ defined by:

$$S_n(x) = (1-\alpha_n-\gamma_n)x_{n-1} + \alpha_n T_n\{(1-\beta_n-\delta_n)x_{n-1} + \beta_n T_n x + \delta_n v_n\} + \gamma_n u_n,$$

for all $n \geq 1$, is a contractive mapping. In fact, we have

$$\begin{aligned} ||S_n x - S_n y|| &= \alpha_n ||T_n\{(1-\beta_n-\delta_n)x_{n-1} + \beta_n T_n x + \delta_n v_n\} \\ &\quad - T_n\{(1-\beta_n-\delta_n)x_{n-1} + \beta_n T_n y + \delta_n v_n\}|| \\ &\leq \alpha_n L_n ||\beta_n (T_n x - T_n y)|| \\ &\leq \alpha_n \beta_n L_n^2 ||x-y)||, \ \forall\, x,\ y\ \in K. \end{aligned}$$

Since $\alpha_n\beta_n L^2 < 1$, hence $S_n : K \to K$ is a contractive mapping. By Banach contractive mapping principle there exists a unique fixed point $x_n \in K$ such that

$$\begin{cases} x_n &= (1-\alpha_n-\gamma_n)x_{n-1} + \alpha_n T_n y_n + \gamma_n u_n,\ n \geq 1, \\ y_n &= (1-\beta_n-\delta_n)x_{n-1} + \beta_n T_n x_n + \delta_n v_n,\ n \geq 1, \end{cases}$$

Therefore if $\alpha_n\beta_n L^2 < 1$, $\forall\ n \geq 1$, then the iterative sequence (1.6) can be employed for the approximation of common fixed points of an finite family of strictly pseudocontractive mappings

Especially, if $\{\alpha_n\}$, $\{\gamma_n\}$ be two sequences in [0, 1] satisfying $\alpha_n + \gamma_n \leq 1$ for all $n \geq 1$, $\{u_n\}$ be a bounded sequence in K and x_0 is a given point in K, then the sequence $\{x_n\}$ defined by

$$x_n = (1-\alpha_n-\gamma_n)x_{n-1} + \alpha_n T_n x_{n-1} + \gamma_n u_n,\ \ \forall n \geq 1 \tag{1.7}$$

Remark 1.2. As $\gamma_n = \delta_n = 0$ for all $n \geq 1$, the iteration scheme (1.6) reduces (1.5).

The purpose of this paper is to study the convergence of implicit iterative sequence $\{x_n\}$ defined by (1.6) and (1.7) to a common fixed point for a finite family of strictly pseudocontractive mappings of Browder-Petryshyn type in an arbitrary real Banach spaces. The results presented in this paper generalized and extend the corresponding results of F. Gu [3], M. O. Osilike [7] and Su-Li [12], even in the case of $\beta_n = \delta_n = 0$, $\forall n \geq 1$ or $N = 1$ are also new. Moreover, in this paper the methods of proof of main results are also different from that of Osilike [7] and Su and Li [12]. At the same time, we also revised the mistake in [12].

In order to prove the main results of this paper, we need the following Lemmas:

Lemma 1.1[2]. *Let E be a real Banach space and let J be the normalized duality mapping. Then for any given $x, y \in E$, we have*

$$||x+y||^2 \leq ||x||^2 + 2\langle y, j(x+y)\rangle,\ \ \forall j(x+y) \in J(x+y)$$

Lemma 1.2[8]. *Let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ be three nonnegative real sequences satisfying the following condition:*

$$a_{n+1} \leq (1+b_n)a_n + c_n,\ \ \forall n \geq n_0,$$

where n_0 *is some nonnegative integer,* $\sum_{n=0}^{\infty} c_n < \infty$ *and* $\sum_{n=0}^{\infty} b_n < \infty$*. Then*

(1) *the limit* $\lim_{n\to\infty} a_n$ *exists.*

(2) *In addition, if there exists a subsequence* $\{a_{n_i}\} \subset \{a_n\}$ *such that* $a_{n_i} \to 0$*, then* $a_n \to 0$ $(n \to \infty)$.

2 Main results

We are now in a position to prove our main results in this paper.

Theorem 2.1. *Let* E *be a real Banach space and* K *be a nonempty closed convex subset of* E*. Let* $\{T_1, T_2, \cdots, T_N\} : K \to K$ *be* N *strictly pseudocontractive mappings with* $F = \cap_{i=1}^{N} F(T_i) \neq \emptyset$ *(the set of common fixed points of* $\{T_1, T_2, \cdots, T_N\}$*). Let* $\{\alpha_n\}$*,* $\{\beta_n\}$*,* $\{\gamma_n\}$*,* $\{\delta_n\}$ *are four real sequences in [0, 1] satisfying* $\alpha_n + \gamma_n \leq 1$ *and* $\beta_n + \delta_n \leq 1$ *for all* $n \geq 1$*,* $\{u_n\}$ *and* $\{v_n\}$ *are two bounded sequences in* K *satisfying the following conditions:*

(i) $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(ii) $\sum_{n=1}^{\infty} \alpha_n^2 < \infty$;

(iii) $\sum_{n=1}^{\infty} \alpha_n \beta_n < \infty$;

(iv) $\sum_{n=1}^{\infty} \alpha_n \delta_n < \infty$;

(v) $\sum_{n=1}^{\infty} \gamma_n < \infty$;

(vi) $\alpha_n \beta_n L^2 < 1$, where $L = \max_{1\leq i\leq N}\{L_i\}$.

Suppose further that $x_0 \in K$ *be any given point and* $\{x_n\}$ *is the implicit iteration sequence defined by (1.6), then the following conclusions hold:*

(1) $\lim_{n\to\infty} ||x_n - p||$ *exists for all* $p \in F$;

(2) $\liminf_{n\to\infty} ||x_n - T_n x_n|| = 0$.

Proof. Since each $T_i : K \to K$, $i \in I = \{1, 2, \cdots, N\}$ be strictly pseudocontractive, then we have $\forall x, y \in K$, there exists constants $k_i \in (0, 1)$ and $L_i \geq 1$ such that

$$\langle T_i x - T_i y, j(x - y)\rangle \leq ||x - y||^2 - k_i||x - T_i x - (y - T_i y)||^2, \ \ \forall i \in I$$

and

$$||T_i x - T_i y|| \leq L_i||x - y||, \ \ \forall i \in I.$$

Let $k = \min_{1\leq i\leq N}\{k_i\}$ and $L = \max_{1\leq i\leq N}\{L_i\}$, then

$$\langle T_i x - T_i y, j(x - y)\rangle \leq ||x - y||^2 - k||x - T_i x - (y - T_i y)||^2, \ \ \forall i \in I \quad (2.1)$$

and

$$||T_i x - T_i y|| \leq L||x - y||, \ \ \forall i \in I. \quad (2.2)$$

Let $p \in F$, it follows from (1.5), (2.1), (2.2) and Lemma1.1 that

$$\begin{aligned}
||x_n - p||^2 &= ||(1-\alpha_n-\gamma_n)(x_{n-1}-p) + \alpha_n(T_n y_n - p) + \gamma_n(u_n - p)||^2 \\
&\leq (1-\alpha_n-\gamma_n)^2||x_{n-1}-p||^2 + 2\alpha_n\langle T_n y_n - p, j(x_n-p)\rangle \\
&\quad + 2\gamma_n\langle u_n - p, j(x_n - p)\rangle \\
&= (1-\alpha_n-\gamma_n)^2||x_{n-1}-p||^2 + 2\alpha_n\langle T_n y_n - T_n x_n, j(x_n-p)\rangle \\
&\quad + 2\alpha_n\langle T_n x_n - p, j(x_n-p)\rangle + 2\gamma_n\langle u_n - p, j(x_n-p)\rangle \\
&\leq (1-\alpha_n)^2||x_{n-1}-p||^2 + 2\alpha_n||T_n y_n - T_n x_n||\cdot||x_n-p|| + 2\alpha_n||x_n-p||^2 \\
&\quad - 2\alpha_n k||x_n - T_n x_n||^2 + 2\gamma_n||u_n-p||\cdot||x_n-p|| \\
&\leq (1-\alpha_n)^2||x_{n-1}-p||^2 + 2\alpha_n L||y_n - x_n||\cdot||x_n-p|| + 2\alpha_n||x_n-p||^2 \\
&\quad - 2\alpha_n k||x_n - T_n x_n||^2 + 2\gamma_n||u_n-p||\cdot||x_n-p||. \qquad (2.3)
\end{aligned}$$

From (1.6) and (2.2), we also have that

$$\begin{aligned}
||y_n - x_n|| &= ||\beta_n(T_n x_n - x_{n-1}) + \delta_n(v_n - x_{n-1}) + \alpha_n(x_{n-1} - T_n y_n) + \gamma_n(x_{n-1} - u_n)|| \\
&\leq \beta_n||T_n x_n - x_{n-1}|| + \delta_n||v_n - x_{n-1}|| + \alpha_n||x_{n-1} - T_n y_n|| + \gamma_n||x_{n-1} - u_n|| \\
&\leq \beta_n||T_n x_n - p|| + \beta_n||x_{n-1}-p|| + \delta_n||v_n - p|| + \delta_n||x_{n-1}-p|| \\
&\quad + \alpha_n||x_{n-1}-p|| + \alpha_n||T_n y_n - p|| + \gamma_n||x_{n-1}-p|| + \gamma_n||u_n-p|| \\
&\leq \beta_n L||x_n-p|| + \alpha_n||x_{n-1}-p|| + \beta_n||x_{n-1}-p|| + \gamma_n||x_{n-1}-p|| \\
&\quad + \delta_n||x_{n-1}-p|| + \alpha_n L||y_n-p|| + \gamma_n||u_n-p|| + \delta_n||v_n-p|| \\
&\leq \beta_n L||x_n-p|| + (\alpha_n+\beta_n+\gamma_n+\delta_n)||x_{n-1}-p|| \\
&\quad + \alpha_n L||y_n-p|| + \gamma_n||u_n-p|| + \delta_n||v_n-p|| \qquad (2.4)
\end{aligned}$$

and

$$\begin{aligned}
||y_n - p|| &= ||(1-\beta_n-\delta_n)(x_{n-1}-p) + \beta_n(T_n x_n - p) + \delta_n(v_n - p|| \\
&\leq (1-\beta_n-\delta_n)||x_{n-1}-p|| + \beta_n||T_n x_n - p|| + \delta_n||v_n-p|| \\
&\leq ||x_{n-1}-p|| + \beta_n L||x_n-p|| + \delta_n||v_n-p|| \qquad (2.5)
\end{aligned}$$

Setting $M_1 = \max\{\sup\{||u_n-p||^2 : n \geq 1\}, \ \sup\{||v_n-p||^2 : \ n \geq 1\}\}$, substituting (2.4),(2.5) into (2.3), and noticing that $2||x_{n-1}-p||\cdot||x_n-p|| \leq ||x_{n-1}-p||^2 + ||x_n-p||^2$, $2||u_n-p||\cdot||x_n-p|| \leq ||u_n-p||^2 + ||x_n-p||^2$ and

$2||v_n - p|| \cdot ||x_n - p|| \leq ||v_n - p||^2 + ||x_n - p||^2$ we obtain that

$$\begin{aligned}
||x_n - p||^2 &\leq (1-\alpha_n)^2||x_{n-1} - p||^2 + 2\alpha_n L(\beta_n L + \alpha_n\beta_n L^2)||x_n - p||^2 \\
&+ 2\alpha_n L(\alpha_n + \beta_n + \gamma_n + \delta_n + \alpha_n L)||x_{n-1} - p|| \cdot ||x_n - p|| \\
&+ 2\alpha_n\gamma_n L||u_n - p|| \cdot ||x_n - p|| + 2\alpha_n L(\delta_n + \alpha_n\delta_n L)||v_n - p|| \cdot ||x_n - p|| \\
&+ 2\alpha_n||x_n - p||^2 - 2\alpha_n k||x_n - T_n x_n||^2 + 2\gamma_n||u_n - p|| \cdot ||x_n - p|| \\
&\leq (1-\alpha_n)^2||x_{n-1} - p||^2 + 2\alpha_n\beta_n L^2(1+\alpha_n L)||x_n - p||^2 \\
&+ \alpha_n L[\alpha_n(1+L) + \beta_n + \gamma_n + \delta_n](||x_{n-1} - p||^2 + ||x_n - p||^2) \\
&+ \alpha_n\gamma_n L(||u_n - p||^2 + ||x_n - p||^2) + \alpha_n\delta_n L(1+\alpha_n L)(||v_n - p||^2 + ||x_n - p||^2) \\
&+ 2\alpha_n||x_n - p||^2 - 2\alpha_n k||x_n - T_n x_n||^2 + \gamma_n(||u_n - p||^2 + ||x_n - p||^2) \\
&= \{(1-\alpha_n)^2 + \alpha_n L[\alpha_n(1+L) + \beta_n + \gamma_n + \delta_n]\}||x_{n-1} - p||^2 \\
&+ \{2\alpha_n\beta_n L^2(1+\alpha_n L) + \alpha_n L[\alpha_n(1+L) + \beta_n + \gamma_n + \delta_n] \\
&+ \alpha_n\gamma_n L + \alpha_n\delta_n L(1+\alpha_n L) + 2\alpha_n + \gamma_n\}||x_n - p||^2 + \gamma_n(1+\alpha_n L)M_1 \\
&+ \alpha_n\delta_n L(1+\alpha_n L)M_1 - 2\alpha_n k||x_n - T_n x_n||^2 \\
&\leq \{(1-\alpha_n)^2 + \alpha_n L[\alpha_n(1+L) + \beta_n + \gamma_n + \delta_n]\}||x_{n-1} - p||^2 \\
&+ \{2\alpha_n\beta_n L^2(1+L) + \alpha_n L[\alpha_n(1+L) + \beta_n + \gamma_n + \delta_n] \\
&+ \alpha_n\gamma_n L + \alpha_n\delta_n L(1+L) + 2\alpha_n + \gamma_n\}||x_n - p||^2 + \gamma_n(1+L)M_1 \\
&+ \alpha_n\delta_n L(1+L)M_1 - 2\alpha_n k||x_n - T_n x_n||^2 \\
&= \tau_n||x_{n-1} - p||^2 + \sigma_n||x_n - p||^2 + \gamma_n(1+L)M_1 \\
&+ \alpha_n\delta_n L(1+L)M_1 - 2\alpha_n k||x_n - T_n x_n||^2 \qquad (2.6)
\end{aligned}$$

where

$$\tau_n = (1-\alpha_n)^2 + \alpha_n L[\alpha_n(1+L) + \beta_n + \gamma_n + \delta_n]$$

and

$$\begin{aligned}
\sigma_n &= 2\alpha_n\beta_n L^2(1+L) + \alpha_n L[\alpha_n(1+L) + \beta_n + \gamma_n + \delta_n] \\
&+ \alpha_n\gamma_n L + \alpha_n\delta_n L(1+L) + 2\alpha_n + \gamma_n.
\end{aligned}$$

Transposing and simplifying above inequality (2.6), we have

$$\begin{aligned}
||x_n - p||^2 &\leq \left(\frac{\tau_n}{1-\sigma_n}\right)||x_{n-1} - p||^2 + \frac{(\gamma_n + \alpha_n\delta_n L)(1+L)M_1}{1-\sigma_n} \\
&- \left(\frac{2\alpha_n k}{1-\sigma_n}\right)||x_n - T_n x_n||^2 \\
&= \left(1 + \frac{\mu_n}{1-\sigma_n}\right)||x_{n-1} - p||^2 + \frac{(\gamma_n + \alpha_n\delta_n L)(1+L)M_1}{1-\sigma_n} \\
&- \left(\frac{2\alpha_n k}{1-\sigma_n}\right)||x_n - T_n x_n||^2, \qquad (2.7)
\end{aligned}$$

where

$$\begin{aligned}\mu_n &= \tau_n + \sigma_n - 1\\ &= \alpha_n^2 + 2\alpha_n L[\alpha_n(1+L) + \beta_n + \gamma_n + \delta_n]\\ &\quad + 2\alpha_n\beta_n L^2(1+L) + \alpha_n\gamma_n L + \alpha_n\delta_n L(1+L) + \gamma_n\end{aligned}$$

It follows from the conditions (ii)-(v) that

$$\begin{aligned}\sigma_n &= 2\alpha_n\beta_n L^2(1+L) + \alpha_n L[\alpha_n(1+L) + \beta_n + \gamma_n + \delta_n]\\ &\quad + \alpha_n\gamma_n L + \alpha_n\delta_n L(1+L) + 2\alpha_n + \gamma_n \to 0\,(n\to\infty),\end{aligned}$$

therefore there exists a natural number n_0 such that $1-\sigma_n \geq \frac{1}{2}$ for any $n \geq n_0$. Hence, from (2.7) we have

$$\begin{aligned}||x_n - p||^2 &\leq (1+2\mu_n)||x_{n-1}-p||^2 + 2(\gamma_n + \alpha_n\delta_n L)(1+L)M_1\\ &\quad - 2\alpha_n k||x_n - T_n x_n||^2\\ &= (1+b_n)||x_{n-1}-p||^2 + c_n - 2\alpha_n k||x_n - T_n x_n||^2,\ \ \forall n \geq n_0\end{aligned} \tag{2.8}$$

where $b_n = 2\mu_n$ and $c_n = 2(\gamma_n + \alpha_n\delta_n L)(1+L)M_1$. From the conditions (ii)-(v) it is easy to see that $\sum_{n=1}^{\infty} b_n < \infty$ and $\sum_{n=1}^{\infty} c_n < \infty$. Thus using (2.8) and Lemma 1.2 we have limit $\lim_{n\to\infty}||x_n - p||^2$ exists, and so limit $\lim_{n\to\infty}||x_n - p||$ exists (since $||x_n - p|| \geq 0$).

Since $\lim_{n\to\infty}||x_n - p||$ exists, then $\{x_n\}$ is bounded, hence there exists constant $M_2 > 0$ such that $||x_n - p||^2 \leq M_2$, $\forall n \geq 1$. It also follows from (2.8) that

$$\begin{aligned}2\alpha_n k||x_n - T_n x_n||^2 &\leq ||x_{n-1}-p||^2 - ||x_n - p||^2 + b_n||x_{n-1}-p||^2 + c_n\\ &\leq ||x_{n-1}-p||^2 - ||x_n - p||^2 + b_n M_2 + c_n,\ \ \forall n \geq n_0.\end{aligned}$$

Thus

$$2k\sum_{j=n_0+1}^{\infty}\alpha_j||x_j - T_j x_j||^2 \leq ||x_{n_0} - p||^2 + M_2\sum_{j=n_0+1}^{\infty} b_j + \sum_{j=n_0+1}^{\infty} c_j,$$

and hence

$$2k\sum_{n=1}^{\infty}\alpha_n||x_n - T_n x_n||^2 \leq ||x_{n_0} - p||^2 + M_2\sum_{n=1}^{\infty} b_n + \sum_{n=1}^{\infty} c_n. \tag{2.9}$$

By virtue of the $\sum_{n=1}^{\infty} b_n < \infty$ and $\sum_{n=1}^{\infty} c_n < \infty$, it follows from (2.9) that

$$\sum_{n=1}^{\infty}\alpha_n||x_n - T_n x_n||^2 < \infty.$$

Since $\sum_{n=1}^{\infty} \alpha_n = \infty$, then we must have

$$\liminf_{n\to\infty} ||x_n - T_n x_n|| = 0.$$

This completes the proof of Theorem 2.1. □

Remark 2.1. Theorem 2.1 is a generalization of Theorem SL, that is, if $\gamma_n = \delta_n = 0$ for all $n \geq 1$, then one can get Theorem SL from Theorem 2.1.

Remark 2.2. Noticing that, the inequality (2.12) is error in Su and Li [12]. Moreover, it can not be obtained about the Theorem SL [12] because of the error. In here, we give a correction for proof of the Theorem SL use a new method.

Corollary 2.2. *Let E be a real Banach space and K be a nonempty closed convex subset of E. Let $\{T_1, T_2, \cdots, T_N\} : K \to K$ be N strictly pseudocontractive mappings with $F = \cap_{i=1}^{N} F(T_i) \neq \emptyset$ (the set of common fixed points of $\{T_1, T_2, \cdots, T_N\}$). Let $\{\alpha_n\}$ and $\{\gamma_n\}$ are two real sequences in [0, 1] satisfying $\alpha_n + \gamma_n \leq 1$ for all $n \geq 1$, $\{u_n\}$ be a bounded sequence in K satisfying the following conditions:*

(i) $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(ii) $\sum_{n=1}^{\infty} \alpha_n^2 < \infty$;
(iii) $\sum_{n=1}^{\infty} \gamma_n < \infty$.

Suppose further that $x_0 \in K$ be any given point and $\{x_n\}$ is the explicit iteration sequence defined by (1.7), then the following conclusions hold:

(i) $\lim_{n\to\infty} ||x_n - p||$ *exists for all* $p \in F$;
(ii) $\liminf_{n\to\infty} ||x_n - T_n x_n|| = 0$.

Proof. Taking $\beta_n = \delta_n = 0$, $\forall n \geq 1$ in Theorem 2.1, then the conclusion of Corollary 2.2 can be obtained from Theorem 2.1 immediately. This completes the proof of Corollary 2.2.□

Theorem 2.3. *Let E be a real Banach space and K be a nonempty closed convex subset of E. Let $\{T_1, T_2, \cdots, T_N\} : K \to K$ be N strictly pseudocontractive mappings with $F = \cap_{i=1}^{N} F(T_i) \neq \emptyset$ (the set of common fixed points of $\{T_1, T_2, \cdots, T_N\}$). Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\delta_n\}$ are four real sequences in [0, 1] satisfying $\alpha_n + \gamma_n \leq 1$ and $\beta_n + \delta_n \leq 1$ for all $n \geq 1$, $\{u_n\}$ and $\{v_n\}$ are two bounded sequences in K satisfying the following conditions:*

(i) $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(ii) $\sum_{n=1}^{\infty} \alpha_n^2 < \infty$;
(iii) $\sum_{n=1}^{\infty} \alpha_n \beta_n < \infty$;
(iv) $\sum_{n=1}^{\infty} \alpha_n \delta_n < \infty$;
(v) $\sum_{n=1}^{\infty} \gamma_n < \infty$;
(vi) $\alpha_n \beta_n L^2 < 1$, where $L = \max_{1 \leq i \leq N}\{L_i\}$.

Suppose further that $x_0 \in K$ be any given point and $\{x_n\}$ is the implicit iteration sequence defined by (1.6), then the sequence $\{x_n\}$ convergence strongly to a common fixed point of the mappings family $\{T_i\}_{i=1}^N$ if and only if

$$\liminf_{n\to\infty} d(x_n, F) = 0. \tag{2.10}$$

Proof. The necessity of condition (2.10) is obvious.

Next we prove the sufficiency of Theorem 2.3. For any given $p \in F$, it follows from (2.8) in Theorem 2.1 that

$$||x_n - p||^2 \le (1 + b_n)||x_{n-1} - p||^2 + c_n, \ \ \forall n \ge n_0, \tag{2.11}$$

where sequences $\{b_n\}$ and $\{c_n\}$ satisfying $\sum_{n=1}^{\infty} b_n < \infty$ and $\sum_{n=1}^{\infty} c_n < \infty$. Hence, we have

$$[d(x_n, F)]^2 \le (1 + b_n)[d(x_{n-1}, F)]^2 + c_n, \ \ \forall n \ge n_0. \tag{2.12}$$

It follows from (2.12) and Lemma 1.2 that the limit $\lim_{n\to\infty}[d(x_n, F)]^2$ exists, further, limit $\lim_{n\to\infty} d(x_n, F)$ exists. By the condition (2.10), we have

$$\lim_{n\to\infty} d(x_n, F) = 0.$$

Next we prove that the sequence $\{x_n\}$ is a Cauchy sequence in K. In fact, since $\sum_{n=1}^{\infty} b_n < \infty$, $1 + t \le \exp\{t\}$ for all $t > 0$, and (2.11), therefore we have

$$||x_n - p||^2 \le \exp\{b_n\}||x_{n-1} - p||^2 + c_n, n \ge n_0. \tag{2.13}$$

Hence, for any positive integers n, m, $n \ge n_0$, from (2.13) we have

$$\begin{aligned}
||x_{n+m} - p||^2 &\le \exp\{b_{n+m}\}||x_{n+m-1} - p||^2 + c_{n+m} \\
&\le \exp\{b_{n+m}\}[\exp\{b_{n+m-1}\}||x_{n+m-2} - p||^2 + c_{n+m-1}] + c_{n+m} \\
&= \exp\{b_{n+m} + b_{n+m-1}\}||x_{n+m-2} - p||^2 + \exp\{b_{n+m}\}c_{n+m-1} + c_{n+m} \\
&\le \cdots\cdots \\
&\le \exp\left\{\sum_{i=n+1}^{n+m} b_i\right\}||x_n - p||^2 + \exp\left\{\sum_{i=n+2}^{n+m} b_i\right\}\sum_{i=n+1}^{n+m} c_i \\
&\le W||x_n - p||^2 + W\sum_{i=n+1}^{\infty} c_i.
\end{aligned}$$

where $W = \exp\{\sum_{n=1}^{\infty} b_n\} < \infty$.

Since $\lim_{n\to\infty} d(x_n, F) = 0$ and $\sum_{n=1}^{\infty} c_n < \infty$, for any given $\epsilon > 0$, there exists a positive integer $n_1 \geq n_0$ such that

$$[d(x_n, F)]^2 < \frac{\epsilon^2}{8(W+1)}, \quad \sum_{i=n+1}^{\infty} c_i < \frac{\epsilon^2}{4W}, \quad \forall n \geq n_1.$$

Therefore there exists $p_1 \in F$ such that

$$||x_n - p_1||^2 < \frac{\epsilon^2}{4(W+1)}, \quad \forall n \geq n_1$$

Consequently, for any $n \geq n_1$ and for all $m \geq 1$ we have

$$\begin{aligned}
||x_{n+m} - x_n||^2 &\leq (||x_{n+m} - p_1|| + ||x_n - p_1||)^2 \\
&\leq 2(||x_{n+m} - p_1||^2 + ||x_n - p_1||^2) \\
&\leq 2(1+W)||x_n - p_1||^2 + 2W \sum_{i=n+1}^{\infty} c_i \\
&< 2 \cdot \frac{\epsilon^2}{4(W+1)}(1+W) + 2W \cdot \frac{\epsilon^2}{4W} \\
&= \epsilon^2.
\end{aligned}$$

i.e.,

$$||x_{n+m} - x_n|| < \epsilon.$$

This implies that $\{x_n\}$ is a Cauchy sequence in K. By the completeness of K, we can assume that $x_n \to x^* \in K$. Observe that if $T : K \to K$ is strictly pseudocontractive and $\{p_n\}_{n=1}^{\infty}$ is a sequence in $F(T)$ which converges strongly to some p, them

$$\begin{aligned}
||p - Tp|| &\leq ||p - p_n|| + ||p_n - Tp|| \\
&= ||p - p_n|| + ||Tp_n - Tp|| \\
&\leq (1+L)||p - p_n|| \to 0 \ (n \to \infty).
\end{aligned}$$

Thus $p \in F(T)$, so that $F(T)$ is closed. It follows that $F(T_i)$ is closed for all $i \in I$, so that F is closed. Since

$$\lim_{n\to\infty} d(x_n, F) = 0,$$

we must have that $x^* \in F$. This completes the proof of Theorem 2.3. □

Corollary 2.4. *Let E be a real Banach space and K be a nonempty closed* *convex subset of E. Let $\{T_1, T_2, \cdots, T_N\} : K \to K$ be N strictly pseudocontractive mappings with $F = \cap_{i=1}^N F(T_i) \neq \emptyset$ (the set of common fixed points of $\{T_1, T_2, \cdots, T_N\}$). Let $\{\alpha_n\}$, and $\{\gamma_n\}$ be two real sequences in [0, 1] satisfying $\alpha_n + \gamma_n \leq 1$ for all $n \geq 1$, $\{u_n\}$ be a bounded sequence in K satisfying the following conditions:*

(i) $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(ii) $\sum_{n=1}^{\infty} \alpha_n^2 < \infty$;
(iii) $\sum_{n=1}^{\infty} \gamma_n < \infty$.

Suppose further that $x_0 \in K$ be any given point and $\{x_n\}$ is the explicit iteration sequence defined by (1.7), then the sequence $\{x_n\}$ convergence strongly to a common fixed point of the mappings family $\{T_i\}_{i=1}^N$ if and only if the condition (2.10) is satisfied.

Proof. Taking $\beta_n = \delta_n = 0$, $\forall n \geq 1$ in Theorem 2.3, then the conclusion of Corollary 2.4 can be obtained from Theorem 2.3 immediately. This completes the proof of Corollary 2.4.□

In the case of $N = 1$, (1.6) become the implicit iteration process as follows:

$$\begin{cases} x_n &= (1-\alpha_n-\gamma_n)x_{n-1} + \alpha_n T y_n + \gamma_n u_n, \ n \geq 1, \\ y_n &= (1-\beta_n-\delta_n)x_{n-1} + \beta_n T x_n + \delta_n v_n, \ n \geq 1, \end{cases} \tag{2.14}$$

The conclusion of Theorems 2.1 and 2.3 are still valid for the iteration process (2.14). Furthermore, we have the following result:

Theorem 2.5. *Let E be a real Banach space and K be a nonempty closed convex subset of E. Let $T : K \to K$ be a semi-compact strictly pseudocontractive mappings with $F(T) = \{x \in K : Tx = x\} \neq \emptyset$. Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\delta_n\}$ are four real sequences in [0, 1] satisfying $\alpha_n + \gamma_n \leq 1$ and $\beta_n + \delta_n \leq 1$ for all $n \geq 1$, $\{u_n\}$ and $\{v_n\}$ are two bounded sequences in K satisfying the following conditions:*

(i) $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(ii) $\sum_{n=1}^{\infty} \alpha_n^2 < \infty$;
(iii) $\sum_{n=1}^{\infty} \alpha_n \beta_n < \infty$;
(iv) $\sum_{n=1}^{\infty} \alpha_n \delta_n < \infty$;
(v) $\sum_{n=1}^{\infty} \gamma_n < \infty$;
(vi) $\alpha_n \beta_n L^2 < 1$.

Suppose further that $x_0 \in K$ be any given point and $\{x_n\}$ is the implicit iteration sequence defined by (2.14), then the sequence $\{x_n\}$ convergence strongly to a fixed point of T

Proof. By the Theorem 2.1 we known that

$$\liminf_{n\to\infty} ||x_n - Tx_n|| = 0,$$

then there exists a subsequence $\{n_k\}$ of $\{n\}$ such that

$$\lim_{k\to\infty} ||x_{n_k} - Tx_{n_k}|| = 0. \tag{2.15}$$

By the semi-compactness of T, there must exists a subsequence $\{x_{n_{k_i}}\}$ of $\{x_{n_k}\}$ such that

$$\lim_{i\to\infty} x_{n_{k_i}} = p_0.$$

It follows from (2.15) that $p_0 = Tp_0$, hence $p_0 \in F(T)$. Since $\lim_{n\to\infty} ||x_n - p_0||$ exists, then

$$\lim_{n\to\infty} x_n = p_0.$$

This completes the proof of Theorem 2.5.□

Corollary 2.6. *Let E be a real Banach space and K be a nonempty closed convex subset of E. Let $T : K \to K$ be a semi-compact strictly pseudocontractive mappings with $F(T) = \{x \in K : Tx = x\} \neq \emptyset$. Let $\{\alpha_n\}$ and $\{\gamma_n\}$ be two real sequences in [0, 1] satisfying $\alpha_n + \gamma_n \leq 1$ for all $n \geq 1$, $\{u_n\}$ be a bounded sequence in K satisfying the following conditions:*
(i) $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(ii) $\sum_{n=1}^{\infty} \alpha_n^2 < \infty$;
(iii) $\sum_{n=1}^{\infty} \gamma_n < \infty$.

Suppose further that $x_0 \in K$ be any given point and $\{x_n\}$ is the explicit iteration sequence defined by

$$x_n = (1 - \alpha_n - \gamma_n)x_{n-1} + \alpha_n Tx_{n-1} + \gamma_n u_n,\ n \geq 1. \tag{2.16}$$

Then the sequence $\{x_n\}$ convergence strongly to a fixed point of T

Proof. Taking $\beta_n = \delta_n = 0$, $\forall n \geq 1$ in Theorem 2.5, then the conclusion of Corollary 2.6 can be obtained from theorem 2.5 immediately. This completes the proof of Corollary 2.6.

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