An. Şt. Univ. Ovidius Constanţa Vol. 9(2), 2001, 31–44

# EFFECTIVE DETERMINATION OF ALL THE HAHN-BANACH EXTENSIONS OF SOME LINEAR AND CONTINUOUS FUNCTIONALS

**Constantin Costara and Dumitru Popa**

*Dedicated to Professor Mirela Ştefănescu on the occasion of her 60th birthday*

The Hahn-Banach theorem is the one of the fundamental results in functional analysis. Recall one of its forms, perhaps the most well known:

Let $X$ be a normed space over $\mathbb{K} = \mathbb{R}$ or ( $\mathbb{C}$ ), $G \subset X$ a linear subspace, $f : G \to \mathbb{K}$ be a linear and continuous functional. Then there exists $\overline{f} : X \to \mathbb{K}$ a linear and continuous functional which extends $f$ with the same norm as $f$.

In the sequel we will call a such extension a *Hahn-Banach extension.*

As it is well known and we will see in the sequel, the Hahn-Banach extension is not unique in general.

Also it is well known that the Hahn-Banach theorem has profound applications, so what we consider that some reasonably simple examples of effective determination of all the Hahn-Banach extensions of some linear and continuous functional can be of some interest, see [5] for the bellow Example 1, Propositions 1, 3 ( which in [5] are unsolved), [4] for the Proposition 4 ( also unsolved in [4] ) and [2] where the all below examples will be are included.

We advertise from the beginning the reader that in the bellow examples the essential difficulty will be in the calculation of the norm of the given functional $f$ relative to the linear subspace $G$.

The next 1, 2, 3 examples are elementary.

**Example 1.** In the space $\mathbb{R}^2$ let us consider the subspace $G = \{(x, y) \in \mathbb{R}^2 \mid 2x - y = 0\}$, $f : G \to \mathbb{R}$, $f(x, y) = x$. Then $g : \mathbb{R}^2 \to \mathbb{R}$, $g(x, y) = \frac{x}{5} + \frac{2y}{5}$, $\forall (x, y) \in \mathbb{R}^2$ is the unique Hahn-Banach extension of $f$.

Received: June, 2001

**Proof.** If $g : \mathbb{R}^2 \to \mathbb{R}$ is a linear functional, then $g(x,y) = xg(1,0) + yg(0,1) = ax + by$, $\forall (x,y) \in \mathbb{R}^2$ and as is easy $||g|| = \sqrt{a^2+b^2}$. Let us observe that $G = \{(x,2x) \mid x \in \mathbb{R}\} = Sp\{(1,2)\}$. In addition$|f(x,y)| = |x| = \frac{1}{\sqrt{5}}||(x,y)||$, $\forall (x,y) \in G$ i.e. $||f|| = \frac{1}{\sqrt{5}}$. Let be $g : \mathbb{R}^2 \to \mathbb{R}$ a Hahn-Banach extension of $f$. A such extension there exist by the Hahn-Banach theorem. Then there exists $a, b \in \mathbb{R}$ such that $g(x,y) = ax + by$, $\forall (x,y) \in \mathbb{R}^2$ and $||g|| = \sqrt{a^2+b^2}$. As $g \mid_G = f$ it follows that, $g(x,y) = x$, $\forall (x,y) \in G$ i.e. $ax + by = x$, $\forall (x,y) \in G$ or $ax + 2bx = x$, $\forall x \in \mathbb{R}$, $a + 2b = 1$. But, by the hypothesis $g$ has the same norm as $f$ i.e. $||g|| = ||f||$ or $\sqrt{a^2+b^2} = \frac{1}{\sqrt{5}}$ i.e.$\begin{cases} a+2b=1 \\ a^2+b^2=\frac{1}{5} \end{cases}$. Solving this system we obtain $b = \frac{2}{5}$ şi $a = \frac{1}{5}$ i.e. the Hahn-Banach extension is unique and given by the formula $g(x,y) = \frac{x}{5} + \frac{2y}{5}$, $\forall (x,y) \in \mathbb{R}^2$.

**Example 2.** Let be $1 \le p \le \infty$, $\alpha \in \mathbb{K}$ fixed, $G \subseteq \mathbb{K}^2$, $G = \{(x_1, 0) \mid x_1 \in \mathbb{K}\}$. Let us consider the linear and continuous functional $e : \left(G, \|\cdot\|_p\right) \to \mathbb{K}$, $e(x) = \alpha x_1$, the norm $\|\cdot\|_p$ being the usual. Then the all Hahn-Banach extension of $e \in G^*$ are:

1) $\varphi(x_1,x_2) = \alpha x_1 + v x_2$, $\forall\ (x_1,x_2) \in \mathbb{K}^2$, with $|v| \le |\alpha|$, for $p = 1$ and we have an infinity of such extension if $\alpha$ is non null.

ii) $\varphi(x_1,x_2) = \alpha x_1$, $\forall\ (x_1,x_2) \in \mathbb{K}^2$, for $1 < p \le \infty$.

**Proof.** It is easy to see that the norm of $\| e \| = |\alpha|$, for each $p$. Let now $\varphi : \mathbb{K}^2 \to \mathbb{K}$ be a Hahn-Banach extension of $e$. Then there exist $u, v \in K$ such that $\varphi(x_1,x_2) = ux_1 + vx_2$, $\forall (x_1,x_2) \in \mathbb{K}^2$. If on $\mathbb{K}^2$ we will consider the norm $\|\cdot\|_p$, $1 \le p \le \infty$, then $\|\varphi\| = \begin{cases} \max\{|u|, |v|\}, \text{ if } p = 1 \\ (|u|^q + |v|^q)^{\frac{1}{q}}, \text{ if } 1 < p < \infty \\ |u| + |v|, \text{ if } p = \infty \end{cases}$, as it follows from the Holder inequality, where $\frac{1}{p} + \frac{1}{q} = 1$. As $\varphi \mid_G = e$, then $u = \alpha$. So we have the cases:

i) If $p = 1$, then $\varphi(x_1,x_2) = \alpha x_1 + v x_2$, $\forall\ (x_1,x_2) \in \mathbb{K}^2$, with $|v| \le |\alpha|$.

ii) If $1 < p < \infty$, then $v = 0$, so $\varphi(x_1,x_2) = \alpha x_1$, $\forall\ (x_1,x_2) \in \mathbb{K}^2$.

iii) If $p = \infty$, again $v = 0$, so $\varphi(x_1,x_2) = \alpha x_1$, $\forall\ (x_1,x_2) \in \mathbb{K}^2$

**Example 3.** Let $G = \{(x_n)_{n\in\mathbb{N}} \in l_1 \mid x_1 - 3x_2 = 0\}$, $f : G \to \mathbb{R}$, $f((x_n)_{n\in\mathbb{N}}) = x_1$. Then $g : l_1 \to \mathbb{R}$, $g_1(x) = \frac{3}{4}x_1 + \frac{3}{4}x_2$, $\forall x = (x_n)_{n\in\mathbb{N}} \in l_1$ is the unique Hahn-Banach extension of $f$.

**Proof.** We have $|f(x)| = |x_1| = \frac{3}{4}(|x_1| + |x_2|) \le \frac{3}{4}||x||$, $\forall x \in G$, so $||f|| \le \frac{3}{4}$. But $|f(-3,1,0,...)| = 3 \le ||f|| \cdot ||(-3,1,0,...)|| = 4\,||f||$. Let

$g : l_1 \to \mathbb{R}$ be a Hahn-Banach extension. As $l_1^* = l_\infty$ i.e. there exist $\xi = (\xi_n)_{n\in\mathbb{N}} \in l_\infty$ such that $g(x) = \sum_{n=1}^{\infty}\xi_n x_n$, $\forall x = (x_n)_{n\in\mathbb{N}} \in l_1$ and $||g|| = ||\xi|| = \sup_{n\in\mathbb{N}} |\xi_n|$. But $g\mid_G = f$, hence $g(x) = x_1$, $\forall (x_n)_{n\in\mathbb{N}} \in l_1$ with $x_1 - 3x_2 = 0$, or $\xi_1 x_1 + \frac{\xi_2}{3}x_1 + \xi_3 x_3 + ... = x_1$, $\forall |x_1| + |x_3| + |x_4| + ... < \infty$. From this we deduce that $\xi_1 + \frac{\xi_2}{3} = 1$ and $\xi_n = 0$, $\forall n \geq 3$ i.e. $3\xi_1 + \xi_2 = 3$, $\xi_n = 0$, $\forall n \geq 3$. So the Hahn-Banach extension $g$ has the expression $g(x) = \xi_1 x_1 + \xi_2 x_2 = \xi_1 x_1 + (3 - 3\xi_1)x_2$, $\forall x = (x_n)_{n\in\mathbb{N}} \in l_1$. But $||g|| = ||f|| = \frac{3}{4}$ i.e. $\max(|\xi_1|, |3 - 3\xi_1|) = \frac{3}{4}$, which has the solution $\xi_1 = \frac{3}{4}$, i.e. $f$ has an unique Hahn-Banach extension to $l_1$, namely $g(x) = \frac{3}{4}x_1 + \frac{3}{4}x_2$, $\forall x = (x_n)_{n\in\mathbb{N}} \in l_1$.

**Proposition 1.** *Any linear and continuous functional on $c_0$ has a unique Hahn-Banach extension to $l_\infty$.*

**Proof.** Let be $f \in c_0^*$. As it is well known there exists $(a_n)_{n\in\mathbb{N}} \in l_1$ such that $f(x) = \sum_{n=1}^{\infty} a_n x_n$, $\forall x = (x_n)_{n\in\mathbb{N}} \in c_0$ and $\|f\| = \sum_{n=1}^{\infty} |a_n|$. Let be $g : l_\infty \to \mathbb{K}$, $g(x) = \sum_{n=1}^{\infty} a_n x_n$ , $\forall x = (x_n)_{n\in\mathbb{N}} \in l_\infty$ . Then $g$ is linear and $|g(x)| \leq \sum_{n=1}^{\infty} |a_n| \cdot |x_n| \leq \|x\| \cdot \sum_{n=1}^{\infty} |a_n|)$, $\forall x \in l_\infty$ i.e. $g$ is continuous. Using a well known procedure, we obtain that $\|g\| = \sum_{n=1}^{\infty} |a_n| = \|f\|$ i.e. $g$ is a Hahn-Banach extension of $f$ to $l_\infty$. Let now

$\overline{f} : l_\infty \to \mathbb{K}$ an another Hahn-Banach extension of $f$ i.e. $\overline{f}\mid_{c_0} = f$ şi $\|\overline{f}\| = \|f\|$. We prove that $\overline{f} = g$. Let be $x \in l_\infty$, $x = (x_n)_{n\in\mathbb{N}}$ with $\|x\| \leq 1$. For $n \in \mathbb{N}$ denote by $y_n = (sgna_1, ..., sgna_n, x_{n+1,...}) \in l_\infty$ and obviously $\|y_n\| \leq 1$. As $y_n - x = (sgn\ a_1 - x_1, ..., sgn\ a_n - x_n, 0, ...) \in c_0$ and $\overline{f} = g = f$ on $c_0$ it follows that $\overline{f}(y_n - x) = g(y_n - x)$. Let us denote $h = \overline{f} - g$. Then the above relation shows that $h(y_n - x) = 0$, or $h(y_n) = h(x)$. We have $\overline{f}(y_n) = h(y_n) + g(y_n) = h(x) + g(y_n) = h(x) + \sum_{n=1}^{n} |a_k| + \sum_{k=n+1}^{\infty} a_k \cdot x_k$, from where $\left|\overline{f}(y_n) - h(x) - \sum_{k=1}^{n} |a_k|\right| = \left|\sum_{k=n+1}^{\infty} a_k \cdot x_k\right| \leq \sum_{k=n+1}^{\infty} |a_k| \cdot |x_k| \leq \sum_{k=n+1}^{\infty} |a_k| \to 0$ , since the series $\sum_{n=1}^{\infty} |a_n|$ is convergent and $\|x\| \leq 1$, $|\ x_n \ | \leq 1$, $\forall\ n \in \mathbb{N}$. So $\overline{f}(y_n) - h(x) - \sum_{k=1}^{n} |a_k| \to 0$ and as $\sum_{k=1}^{n} |a_n| \to \sum\limits_{n=1}^{\infty} |a_n| = \|f\|$, we obtain that $\overline{f}(y_n) \to h(x) + \|f\|$.

Let us resume, we prove that: $\forall x \in l_\infty$ with $\|x\| \leq 1$, $\forall n \in \mathbb{N}$, $\exists y_n \in l_\infty$ with $\|y_n\| \leq 1$ such that: $\overline{f}(y_n) \to h(x) + \|f\|$, (1).

Let be now $x \in l_\infty$ with $\|x\| \leq 1$. There exist $\lambda \in \mathbb{K}$, $|\lambda| \leq 1$ such that $|h(x)| = \lambda \cdot h(x) = h(\lambda x)$. As $\|\lambda x\| = |\lambda| \cdot \|x\| \leq 1$, from the relation (1) it follows that there exist $y_n \in l_\infty$ with $\|y_n\| \leq 1$ such that: $\overline{f}(y_n) \to h(\lambda x) + \|f\|$, or $\overline{f}(y_n) \to |h(x)| + \|f\|$ i.e. $|\ \overline{f}(y_n)\ | \to |h(x)| + \|f\|$ (2). But: $\left|\overline{f}(y_n)\right| \leq \|\overline{f}\| \cdot \|y_n\| \leq \|\overline{f}\| = \|f\|$, $\forall n \in \mathbb{N}$, from where passing to the limit for $n \to \infty$ from (2) we obtain: $|h(x)| + \|f\| \leq \|f\|$, $|h(x)| \leq 0$, $h(x) = 0$, $\forall x \in l_\infty$ with $\|x\| \leq 1$, hence by homogeneity: $h = 0$, $\overline{f} - g = 0$, $\overline{f} = g$.

**Comment**. There exist other situations in which the above proposition is true? The next proposition shows that the answer is yes!

Let us make some usual notations.

For a given sequence $(X_n)_{n\in\mathbb{N}}$ of normed spaces we will denote $c_0(X_n \mid n\in\mathbb{N}) = \{(x_n)_{n\in\mathbb{N}} \in \prod\limits_{n\in\mathbb{N}} X_n \mid \lim\limits_{n\to\infty} \| x_n \| = 0\}$, with the norm $\| (x_n)_{n\in\mathbb{N}} \| = \sup\limits_{n\in\mathbb{N}} \| x_n \|$;

$l_\infty(X_n \mid n\in\mathbb{N}) = \{(x_n)_{n\in\mathbb{N}} \in \prod\limits_{n\in\mathbb{N}} X_n \mid \sup\limits_{n\in\mathbb{N}} \| x_n \| < \infty\}$, with the norm $\| (x_n)_{n\in\mathbb{N}} \| = \sup\limits_{n\in\mathbb{N}} \| x_n \|$;

$l_1(X_n \mid n\in\mathbb{N}) = \{(x_n)_{n\in\mathbb{N}} \in \prod\limits_{n\in\mathbb{N}} X_n \mid \sum\limits_{n=1}^{\infty} \| x_n \| < \infty\}$, with the norm $\| (x_n)_{n\in\mathbb{N}} \| = \sum\limits_{n=1}^{\infty} \| x_n \|$;

As it is well known $(c_0(X_n \mid n\in\mathbb{N}))^* = l_1(X_n^* \mid n\in\mathbb{N})$, by: If $f \in (c_0(X))^*$ , then there exist $(x_n^*)_{n\in\mathbb{N}} \in l_1(X^*)$ such that: $f(x) = \sum_{n=1}^{\infty} x_n^*(x_n)$, $\forall x = (x_n)_{n\in\mathbb{N}} \in c_0(X)$ and $\|f\| = \sum_{n=1}^{\infty} \|x_n^*\|$, and conversely; see [3] for a proof.

**Proposition 2.** *Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of normed spaces. Then each linear and continuous functional on $c_0(X_n \mid n\in\mathbb{N})$ has an unique Hahn-Banach extension to $l_\infty(X_n \mid n\in\mathbb{N})$.*

**Proof.** Let be $f \in (c_0(X_n \mid n\in\mathbb{N}))^*$. Then there exists $(x_n^*)_{n\in\mathbb{N}} \in l_1(X_n^* \mid n\in\mathbb{N})$ such that $f(x) = \sum_{n=1}^{\infty} x_n^*(x_n)$,$\forall x = (x_n)_{n\in\mathbb{N}} \in c_0(X)$ and $\|f\| = \sum_{n=1}^{\infty} \|x_n^*\|$. We define $g : l_\infty(X_n \mid n\in\mathbb{N}) \to \mathbb{K}$, $g(x) = \sum_{n=1}^{\infty} x_n^*(x_n)$, $\forall x = (x_n)_{n\in\mathbb{N}} \in l_\infty(X)$.
Obviously $g$ is linear and $| g(x) | \le \sum_{n=1}^{\infty} |x_n^*(x_n)| \le \sum_{n=1}^{\infty} \|x_n^*\| \cdot \|x_n\| \le (\sup_{n\in\mathbb{N}} \|x_n\|) \cdot \sum_{n=1}^{\infty} \|x_n^*\| = \|x\| \cdot \sum_{n=1}^{\infty} \|x_n^*\| = \|x\| \cdot \|f\|$, $\forall x \in l_\infty(X)$, so $\|g\| \le \|f\|$. Let be $n \in \mathbb{N}$ and $\|x_1\| \le 1$, ..., $\|x_n\| \le 1$. The sequence $x = (a_1 \cdot x_1, ..., a_n \cdot x_n, 0, ...) \in l_\infty(X_n \mid n\in\mathbb{N})$, where $a_k = sgn x_k^*(x_k)$ and $g(x) = \sum_{k=1}^{n} a_k x_k^*(x_k) = \sum_{k=1}^{n} |x_k^*(x_k)| \le \|g\| \cdot \|x\| = \|g\|$ i.e.$\sum_{k=1}^{n} |x_k^*(x_k)| \le \|g\|$ so taking the supremum over $\|x_1\| \le 1, ..., \|x_n\| \le 1$ it follows that:$\sum_{k=1}^{n} \|x_k^*\| \le \|g\|$, $\forall n \in \mathbb{N}$, so passing to the limit for $n \to \infty$, $\sum_{n=1}^{\infty} \|x^*\| \le \|g\|$ i.e. $\|f\| \le \|g\|$. So $g$ is an extension of with $\|g\| = \|f\| = \sum_{n=1}^{\infty} \|x^*\|$. Let be now $\overline{f} : l_\infty(X_n \mid n\in\mathbb{N}) \to \mathbb{K}$ a linear and continuous functional which extend $f$ with the same norm i.e. $\overline{f} \mid_{c_0(X_n|n\in\mathbb{N})} = f$ and $\|\overline{f}\| = \|f\|$. We will prove that $\overline{f} = g$. We denote $h = \overline{f} - g$ and let us consider $x \in l_\infty(X_n \mid n\in\mathbb{N})$, $\|x\| \le 1$.

There exists $\lambda \in \mathbb{K}$ such that $|\lambda| \leq 1$ and $|h(x)| = h(\lambda x) = h(t)$, $t = \lambda x \in l_\infty(X_n \mid n \in \mathbb{N})$, $\|t\| = |\lambda| \cdot \|x\| \leq 1$. Let be $t = (t_n)_{n\in\mathbb{N}} \in l_\infty(X_n \mid n \in \mathbb{N})$, $n \in \mathbb{N}$ and $a_1 \in X_1$, ..., $a_n \in X_n$ with $\|a_k\| \leq 1$, $\forall k = \overline{1,n}$. The element $y = (\lambda_1 a_1, ..., \lambda_n a_n, t_{n+1}, ...) \in l_\infty(X_n \mid n \in \mathbb{N})$, where $\lambda_k = sgn x_k^*(a_k)$, $k = \overline{1,n}$, and $y - t = (\lambda_1 a_1 - t_1, ..., \lambda_n a_n - t_n, 0, ...) \in c_0(X_n \mid n \in \mathbb{N})$. But $h\mid_{c_0(X_n|n\in\mathbb{N})} = 0$, so $h(y-t) = 0$ hence by the linearity of $h$ on $l_\infty(X_n \mid n \in \mathbb{N})$ it follows that $h(y) = h(t)$ i.e. $|h(x)| = h(y) = \overline{f}(y) - g(y) = \overline{f}(y) - \sum_{k=1}^{n} \lambda_k x_k^*(a_k) - \sum_{k=n+1}^{\infty} x_k^*(t_k)$, or $|h(x)| + \sum_{k=1}^{n} |x_k^*(a_k)| = \overline{f}(y) - \sum_{k=n+1}^{\infty} x_k^*(t_k) = |\overline{f}(y) - \sum_{k=n+1}^{\infty} x_k^*(t_k)| \leq |\overline{f}(y)| + \sum_{k=n+1}^{\infty} |x_k^*(t_k)| \leq \|\overline{f}\| \cdot \|y\| + \sum_{k=n+1}^{\infty} \|x_k^*\| \leq \|f\| + \sum_{k=n+1}^{\infty} \|x_k^*\|$. Hence: $|h(x)| + \sum_{k=1}^{n} |x_k^*(a_k)| \leq \|f\| + \sum_{k=n+1}^{\infty} \|x_k^*\|$, $\forall \|a_1\| \leq 1$,..., $\forall \|a_n\| \leq 1$, so passing to the supremum over $a_1$,..., $a_n$ we obtain $|h(x)| + \sum_{k=1}^{n} \|x_k^*\| \leq \|f\| + \sum_{k=n+1}^{\infty} \|x_k^*\|$, $\forall n \in \mathbb{N}$. For $n \to \infty$ this gives: $|h(x)| + \sum_{n=1}^{\infty} \|x_n^*\| \leq \|f\| + 0$ i.e. $|h(x)| + \|f\| \leq \|f\|$, $h(x) = 0$, $\forall x \in l_\infty(X_n \mid n \in \mathbb{N})$ with $\|x\| \leq 1$ and by the homogeneity of $h$, $h(x) = 0$, $\forall x \in l_\infty(X)$ i.e. $h = 0$, $\overline{f} = g$.

**Proposition 3.** *Let $H$ be a Hilbert space, $G \subseteq H$ a closed linear subspace. Then each linear and continuous functional on $G$ has an unique Hahn-Banach extension.*

*More precisely:*

*1) If $f : G \to \mathbb{K}$ is a linear and continuous functional, then $\overline{f} : H \to \mathbb{K}$, $\overline{f}(x) = f(pr_G(x))$, $\forall x \in H$, is the unique Hahn-Banach extension of $f$, where $pr_G(x)$ is the orthogonal projection of $x$ on $G$.*

*2) If $b \in H$, $f : G \to \mathbb{K}$, $f(x) =< x, b >$, $\forall x \in G$, then the unique Hahn-Banach extension of $f$ is $\overline{f} : H \to \mathbb{K}$ , $\overline{f}(x) =< x, pr_G(b) >$, $\forall x \in X$.*

**Proof. Existence**. Let $f : G \to \mathbb{K}$ linear and continuous. Since $H$ is a Hilbert space we have the decomposition $H = G \oplus G^\perp$. Define now $\overline{f} : H \to \mathbb{K}$, $\overline{f}(x) = f(pr_G(x))$, $\forall x \in H$, where $x = pr_G(x) + pr_{G^\perp}(x)$. Then $\overline{f}$ is linear and $\overline{f}\mid_G = f$, so $\|\overline{f}\| \geq \|f\|$. But $|\overline{f}(x)| = |f(pr_G(x))| \leq \|f\| \|pr_G(x)\| \leq \|f\| \|x\|$, $\forall x \in H$ ( We use the fact that $\| x \|^2 = \| pr_G(x) \|^2 + \| pr_{G^\perp}(x) \| \geq \| pr_G(x) \|^2$ ). From here it follows $\|\overline{f}\| \leq \|f\|$.

**Uniqueness.** Let $g : H \to \mathbb{K}$ be a linear and continuous with $g\mid_G = f$ and $\|g\| = \|f\|$. Using the Riesz Theorem, there exists $b \in H$ such that $g(x) = \langle x, b \rangle$, $\forall x \in H$. Since $b = pr_G(b) + pr_{G^\perp}(b)$, then for each $y \in G$ we have: $f(y) = g(y) = \langle y, pr_G(b) \rangle + \langle y, pr_{G^\perp}(b) \rangle = \langle y, pr_G(b) \rangle$, from where: $\|f\| = \|pr_G(b)\|$. But $\|f\| = \|g\|$, hence $\|f\| = \|pr_G(b)\|$, from where $\|b\|^2 = \|pr_G(b)\|^2$. As $\|b\|^2 = \|pr_G(b)\|^2 + \|pr_{G^\perp}(b)\|^2$, $pr_{G^\perp}(b) = 0$, i.e. $g(x) = \langle x, b \rangle =< x, pr_G(b) >=< pr_G(x), b >=< pr_G(x), pr_G(b) >= \overline{f}(x)$, $\forall x \in X$, i.e. $g = \overline{f}$.

**Comment.** This example shows that in the Hilbert case the problem of the effective determination of the Hahn-Banach extension requires the calculation of the orthogonal projection to a closed linear subspace.

**Example 4.** a) Let $H$ be a Hilbert space, $a \in H$, $a \neq 0$ and $G = \{x \in H \mid < x, a >= 0\}$, $b \in H$, $f : G \to \mathbb{K}$, $f(x) =< x, b >$, $\forall x \in G$.

Then the unique Hahn-Banach extension of $f$ is $\overline{f} : H \to \mathbb{K}$ , $\overline{f}(x) =< x, b > -\frac{<a,b>}{\|a\|^2} < x, a >$, $\forall x \in X$.

b) Let $G = \{f \in L_2[0,1] \mid \int\limits_0^1 xf(x)dx = 0\}$, and $L : G \to \mathbb{K}$, $L(f) = \int\limits_0^1 x^2 f(x)dx$. Then the unique Hahn-Banach extension of $L$ is $x^*(f) = \int\limits_0^1 (x^2 - \frac{3}{4}x)f(x)dx$, $\forall f \in L_2[0,1]$.

c) Let $\mathcal{M}_n(\mathbb{C})$ be the set of all $n \times n$ complex matrix which is a Hilbert space with respect to the scalar product $< A, B >= tr(AB^*)$, where $B^*$ is the adjoint of the operator $B$. Let $G = \{A \in \mathcal{M}_n(\mathbb{C}) \mid tr(A) = 0\}$, $B \in \mathcal{M}_n(\mathbb{C})$ and $f : G \to \mathbb{C}$, $f(A) = tr(AB^*)$, $\forall A \in G$.

Then the unique Hahn-Banach extension of $f$ is $\overline{f} : \mathcal{M}_n(\mathbb{C}) \to \mathbb{C}$, $\overline{f}(A) = tr(AB^*) - \frac{\overline{tr(B)}tr(A)}{n}$, $\forall A \in \mathcal{M}_n(\mathbb{C})$.

d) Let $\mathcal{M}_n(\mathbb{C})$ be the Hilbert space as in c) and $\mathcal{M}_n(\mathbb{C}) \times \mathcal{M}_n(\mathbb{C})$ the hilbertian product. Let $G = \{A \in \mathcal{M}_n(\mathbb{C}) \times \mathcal{M}_n(\mathbb{C}) \mid tr(A) = tr(B)\}$, $C, D \in \mathcal{M}_n(\mathbb{C})$ and $f : G \to \mathbb{C}$, $f(A, B) = tr(AC^*)+tr(BD^*)$, $\forall (A, B) \in G$. Then the unique Hahn-Banach extension of $f$ is $\overline{f} : \mathcal{M}_n(\mathbb{C}) \times \mathcal{M}_\ltimes(\mathbb{C}) \to \mathbb{C}$, $\overline{f}(A, B) = tr(AC^*) + tr(BD^*) - \frac{(\overline{tr(C)} - \overline{tr(D)})(tr(B) - tr(A))}{2n}$, $\forall (A, B) \in \mathcal{M}_n(\mathbb{C}) \times \mathcal{M}_\ltimes(\mathbb{C})$.

**Proof.** a) As it is well known $pr_G(b) = b - \frac{<b,a>}{\|a\|^2}a$ and we can use the proposition 3. 2.

b) We have $pr_G(x^2) = x^2 - \frac{<x^2,x>}{\|x\|^2}x = x^2 - \frac{\int_0^1 x^3 dx}{\int_0^1 x^2 dx}x = x^2 - \frac{3}{4}x$, hence proposition 3.2 or a) implies the result.

c) For the fact that $(\mathcal{M}_n(\mathbb{C}) < . >)$ is a Hilbert space, see [1], Spring 1982 ex. 3, p.125; in passing let us observe that if $A = (a_{ij})_{i,j} \in \mathcal{M}_n(\mathbb{C})$, then $\| A \| = \sqrt{tr(AA^*)} = \sqrt{\sum\limits_{i,j=1}^n | a_{ij} |^2}$, which is usually called the Frobenius norm of a matrix.

We observe that $G = \{A \in \mathcal{M}_n(\mathbb{C}) \mid tr(AI^*) = 0\} = \{A \in \mathcal{M}_n(\mathbb{C}) \mid < A, I_n >= 0\}$, where $I_n$ is the unit matrix. But $pr_G(B) = B - \frac{<B,I_n>}{\|I_n\|^2}I_n = B - \frac{tr(B)}{tr(I_n)}I_n = B - \frac{tr(B)}{n}I_n$. Since $f(A) =< A, B >$, $\forall A \in G$, from the proposition 3.2 or a) we obtain $\overline{f}: \mathcal{M}_n(\mathbb{C}) \to \mathbb{C}$, $\overline{f}(A) =< A, pr_G(B) >= tr(AB^*) - \frac{\overline{tr(B)}tr(A)}{n}$, $\forall A \in \mathcal{M}_n(\mathbb{C})$.

d) Recall that if $H_1, H_2$ are two Hilbert space then, the cartesian product $H_1 \times H_2$ is a Hilbert space relatively to the scalar product $< (x_1, y_1), (x_2, y_2) >=< x_1, x_2 > + < y_1, y_2 >$. Let us observe that $G = \{A \in \mathcal{M}_n(\mathbb{C}) \times \mathcal{M}_n(\mathbb{C}) \mid < (A, B), (I_n, -I_n) >= 0\}$, hence $pr_G(C, D) = (C, D) - \frac{<(C,D),(I_n,-I_n)>}{\|(I_n,-I_n)\|^2}(I_n, -I_n) = (C, D) - \frac{trC - trD}{2n}(I_n, -I_n) = (C - \lambda I_n, D + \lambda I_n)$, where $\lambda = \frac{trC-trD}{2n}$. Also $f(A, B) = tr(AC^*) + tr(BD^*) =< (A, B), (C, D) >$, $\forall (A, B) \in G$. Now using proposition 3.2 or a) $f(A, B) =< (A, B), pr_G(C, D) >$, $\forall (A, B) \in \mathcal{M}_n(\mathbb{C}) \times \mathcal{M}_n(\mathbb{C})$, i.e. the statement.

**Example 5.** Let $G \subset l_1$, $G = \{(x_n)_n \in l_1 \mid x_1 = x_3 = x_5 = ... = 0\}$. Then any linear and continuous non–null functional on $G$ has an infinity of Hahn-Banach extensions.

**Proof.** Let be $f : G \to \mathbb{K}$ linear and continuous functional, with $\|f\| \neq 0$. By the Hahn-Banach theorem, there exists a Hahn-Banach extension $g : \ell_1 \to \mathbb{K}$. Hence there exists $\xi = (\xi_n)_{n\in\mathbb{N}} \in \ell_\infty$ such that $g(x) = \sum_{n=1}^{\infty} x_n \xi_n$, $\forall x = (x_n)_{n\in\mathbb{N}} \in \ell_1$, $\|g\| = \sup_{n\in\mathbb{N}} |\xi_n| = \|f\|$, $g\mid_G = f$. So $f(x) = \sum_{k=1}^{\infty} x_{2k}\xi_{2k}$, $\forall x \in G$ and $\|f\| = \|g\| \geq \sup_{k\in\mathbb{N}} |\xi_{2k}|$. But $|f(x)| \leq (\sup_{k\in\mathbb{N}} |\xi_{2k}|) \sum_{k=1}^{\infty} |x_{2k}| \leq (\sup_{k\in\mathbb{N}} |\xi_{2k}|) \|x\|_1$, $\forall x \in G$, from where $\|f\| \leq \sup_{k\in\mathbb{N}} |\xi_{2k}|$. We obtain that $f(x) = \sum_{k=1}^{\infty} x_{2k}\xi_{2k}$, $\forall x \in G$ and $\|f\| = \sup_{k\in\mathbb{N}} |\xi_{2k}|$. As $\|f\| \neq 0$, it follows that we can find an infinity of sequences $\tau = (\tau_{2k+1})_{k\geq 0} \in l_\infty$ such that $\sup_{k\geq 0} |\tau_{2k+1}| = \sup_{k\geq 1} |\xi_{2k}|$ and let us consider the sequence $a(\tau) = (\tau_1, \xi_2, \tau_3, \xi_4, ...) \in \ell_\infty$. We have$\|a(\tau)\|_\infty = \|f\|$. With the help of $\tau$ we construct $g_\tau : \ell_1 \to \mathbb{K}$, $g_\tau(x) = \sum_{n=0}^{\infty} \tau_{2n+1}x_{2n+1} + \sum_{n=1}^{\infty} \xi_{2n}x_{2n}$. Then $g_\tau \in \ell_1^*$ and $\|g_\tau\| =\| a(\tau) \|= \|f\|$, and for each $x \in G$ we have $g_\tau(x) = \sum_{n=1}^{\infty} x_{2n}\xi_{2n} = f(x)$, i.e. $g_\tau \mid_G = f$. As $g_{\tau_1} \neq g_{\tau_2}$, for $\tau_1 \neq \tau_2$ and $\tau$ can be chosen in an infinity of way, the statement follows.

**Comment**. The above examples show that it is natural the question: How many Hahn-Banach extensions we have for a given linear and continuous functional? The next proposition gives the answer.

**Proposition 4.** *Let $X$ be a normed space, $G \subseteq X$ be a linear subspace, $f : G \to \mathbb{K}$ a linear and continuous functional for which there exists $g, h : X \to \mathbb{K}$ two distinct Hahn-Banach extension of $f$. Then there exists an infinity of such Hahn-Banach extensions of $f$.*

*More precisely: the set of all Hahn-Banach extensions of $f$ is a convex set.*

**Proof.** For each $t \in [0,1]$ we define $f_t : X \to \mathbb{K}$, $f_t(x) = tg(x) + (1-t)h(x)$. Clearly $f_t$ is a linear and continuous functional. Also for $x \in G$ we have $f_t(x) = t_f(x) + (1-t)h(t) = t_f(x) + (1-t)f(x) = f(x)$ i.e. $f_t$ extends $f$. If we will prove that $\|f_t\| = \|f\|$, $\forall t \in [0,1]$, the proposition is proved since by $g \neq h$, we have $f_{t_1} \neq f_{t_2}$ for $t_1 \neq t_2$. We have for each $x \in X$ $|f_t(x)| \leq t|g(x)| + (1-t)|h(x)| \leq (t\|g\| + (1-t)\|h\|)\|x\| = \|f\| \cdot \|x\|$, since $\|g\| = \|h\| = \|f\|$, i.e. $\|f_t\| \leq \|f\|$, $\forall t \in [0,1]$. Since $f_t$ extends $f$ we have $\|f_t\| \geq \|f\|$, i.e. $\|f_t\| = \|f\|$, $\forall t \in [0,1]$.

Let us observe that in all the above examples we use essentially the fact that we know an explicit structure of the normed space on which we work.

The next result give a way to construct the extension of some linear and continuous functional defined on a finite codimensional subspace.

**Proposition 5.** *Let $X$ be a normed space and $X^*$ his dual, $x_1^*, ..., x_n^*, f \in X^*$, $G = \{x \in X \mid x_1^*(x) = 0, ..., x_n^*(x) = 0\}$, and $g : G \to \mathbb{K}$, $g(x) = f(x)$, $\forall x \in X$. If $h : X \to \mathbb{K}$ is a linear and continuous functional which extends $g$, then there exists $\alpha_1, ..., \alpha_n \in \mathbb{K}$ such that $h = f + \alpha_1 x_1^* + \cdots + \alpha_n x_n^*$.*

**Proof.** Since $h$ extend $f$ we have $h(x) = g(x) = f(x)$, $\forall x \in G$, i.e. $G = \bigcap_{i=1}^{n} \ker x_i^* \subset \ker(h - f)$. But using now a well known result from the linear algebra there exists $\alpha_1, ..., \alpha_n \in \mathbb{K}$ such that $h - f = \alpha_1 x_1^* + \cdots + \alpha_n x_n^*$.

**Comment.** This proposition shows that, in reasonable situations, the problem of the explicit expressions for a Hahn-Banach extension for a linear and continuous functional is solved, so the difficulty of effective determination of the all Hahn-Banach extensions has been moved to the problem of the calculation of the norm of the given functional to the given subspace, which, as we will see, is a difficult question; see also the comment for the case of the Hilbert space.

In the next concrete examples the scalar field will be the set of the real numbers.

The following example cannot be obtained as in the above Examples 1-3, 4, 5, since the structure of the dual space of $L(l_2)$ is unknown.

**Example 6.** Let $G = \{U \in L(l_2) \mid < Ue_1, e_1 > -2 < Ue_1, e_2 >= 0\}$, and $f : G \to \mathbb{R}$, $f(U) =< Ue_1, e_1 >$. Then $\overline{f} : L(l_2) \to \mathbb{R}$, $\overline{f}(U) =< Ue_1, \frac{4}{5}e_1 + \frac{2}{5}e_2 >$, $\forall U \in L(l_2)$ is the unique Hahn-Banach extension of $f$.

**Proof.** For $U \in G$ we have $\frac{5}{4} \mid< Ue_1, e_1 >\mid^2 = \mid< Ue_1, e_1 >\mid^2 + \mid< Ue_1, e_2 >\mid^2 \leq \parallel Ue_1 \parallel^2$, conforming with the Bessel inequality. Hence $\mid f(U) \mid \leq \sqrt{\frac{4}{5}} \parallel Ue_1 \parallel \leq \sqrt{\frac{4}{5}} \parallel U \parallel$, $\forall U \in G$, i.e. $\parallel f \parallel \leq \sqrt{\frac{4}{5}}$. Let $U \in L(l_2)$ be defined by $U(x_1, x_2, ...) = (2x_1, x_1, 0, ...) = x_1(2e_1 + e_2)$. Then $U \in G$, from where $\mid f(U) \mid \leq \parallel f \parallel \parallel U \parallel$. But $f(U) =< Ue_1, e_1 >=< 2e_1 + e_2, e_1 >= 2$ and $\parallel U(x_1, x_2, ...) \parallel = \sqrt{5} \mid x_1 \mid \leq \sqrt{5} \parallel x \parallel$, $\forall x \in l_2$, $\parallel U \parallel \leq \sqrt{5}$. We obtain that $2 \leq \sqrt{5} \parallel f \parallel$, i.e. $\parallel f \parallel = \sqrt{\frac{4}{5}}$. Let be now $\overline{f} : L(l_2) \to \mathbb{R}$ a Hahn-Banach extension of $f$. From the proposition 5 there exists $\alpha \in \mathbb{R}$ such that $h(U) =< Ue_1, e_1 > +\alpha(< Ue_1, e_1 > -2 < Ue_1, e_2 >) =< Ue_1, (1 + \alpha)e_1 - 2\alpha e_2 >$.

But as is easy to see if $x, y \in l_2$, and $g : L(l_2) \to \mathbb{R}$, $g(U) =< Ux, y >$, then $\parallel g \parallel = \parallel x \parallel \parallel y \parallel$. Hence $\parallel h \parallel = \parallel (1+\alpha)e_1 - 2\alpha e_2 \parallel = \sqrt{(1 + \alpha)^2 + 4\alpha^2}$. Using that $\parallel h \parallel = \parallel f \parallel = \sqrt{\frac{4}{5}}$, it follows $(1 + \alpha)^2 + 4\alpha^2 = \frac{4}{5}$, $25\alpha^2 + 10\alpha + 1 = 0$, $\alpha = -\frac{1}{5}$, i.e. the statement.

**Example 7.** Let $(a_n)_{n\in\mathbb{N}} \in l_1$ with $a_1 \neq 0$, $G = \{(x_n)_{n\in\mathbb{N}} \in c_0 \mid \sum_{n=1}^{\infty} a_n x_n = 0\}$, $f : G \to \mathbb{R}$, $f((x_n)_{n\in\mathbb{N}}) = x_1$. Then the all Hahn-Banach extension of $f$ are the following:

1) $h((x_n)_{n\in\mathbb{N}}) = x_1$, if $\mid a_1 \mid < \sum_{n=2}^{\infty} \mid a_n \mid$.

2) $h((x_n)_{n\in\mathbb{N}}) = x_1 + \alpha \sum_{n=1}^{\infty} a_n x_n$, where $-1 \leq \alpha a_1 \leq 0$, if $\mid a_1 \mid = \sum_{n=2}^{\infty} \mid a_n \mid$.

3) $h((x_n)_{n\in\mathbb{N}}) = -\sum_{n=2}^{\infty} \frac{a_n}{a_1} x_n$, if $\mid a_1 \mid > \sum_{n=2}^{\infty} \mid a_n \mid$.

**Proof.** We prove that $\parallel f \parallel = \min(1, \lambda)$, where $\lambda = \frac{\sum_{n=2}^{\infty} \mid a_n \mid}{\mid a_1 \mid}$. We have $\mid f(x) \mid \leq \parallel x \parallel$, $\forall x \in G$, i.e. $\parallel f \parallel \leq 1$. For $x \in G$ we have $-a_1 x_1 = \sum_{n=2}^{\infty} a_n x_n$, from where $\mid a_1 \mid\mid x_1 \mid \leq \sum_{n=2}^{\infty} \mid a_n \mid\mid x_n \mid \leq (\sum_{n=2}^{\infty} \mid a_n \mid)(\sup_{n\geq 2} \mid x_n \mid) \leq (\sum_{n=2}^{\infty} \mid a_n \mid) \parallel x \parallel$, i.e. $\mid x_1 \mid \leq \lambda \parallel x \parallel$, or $\mid f(x) \mid \leq \lambda \parallel x \parallel$, $\forall x \in G$, hence $\parallel f \parallel \leq \min(1, \lambda)$. Let be $n \in \mathbb{N}$. We choose $\alpha, \beta \in \mathbb{R}$ such that $(\alpha, \beta sgn a_2, \beta sgn a_3, ..., \beta sgn a_n, 0, ...) \in G$. Then $\alpha a_1 + \beta(a_2 sgn a_2 + ... +$

$a_n sgna_n) = 0$, i.e. $-\alpha a_1 = \beta(\mid a_2 \mid +...+ \mid a_n \mid)$. We have $\mid \alpha \mid = \mid f(\alpha, \beta sgna_2, \beta sgna_3, ..., \beta sgna_n, 0, ...) \mid \leq \parallel f \parallel \max(\mid \alpha \mid, \mid \beta \mid)$, or $\min(1, \frac{|\alpha|}{|\beta|}) \leq \parallel f \parallel$, that is $\parallel f \parallel \geq \min(1, \frac{|a_2|+...+|a_n|}{|a_1|})$. Passing to the limit for $n \to \infty$ we obtain $\parallel f \parallel \geq \min(1, \frac{\sum_{n=2}^{\infty} |a_n|}{|a_1|}) = \min(1, \lambda)$.

Let now $h : c_0 \to \mathbb{R}$ be a Hahn-Banach extension of $f$. Then, by Proposition 5 there exists $\alpha \in \mathbb{R}$ such that $h(x_1, ..., x_n, ...) = x_1 + \alpha \sum_{n=1}^{\infty} a_n x_n$, $\forall (x_1, ..., x_n, ...) \in c_0$. But $\parallel h \parallel = \mid 1 + \alpha a_1 \mid + \mid \alpha \mid \sum_{n=2}^{\infty} \mid a_n \mid$. As $\parallel h \parallel = \parallel f \parallel$ it follows that $\mid 1 + \alpha a_1 \mid + \mid \alpha \mid \sum_{n=2}^{\infty} \mid a_n \mid = \min(1, \lambda)$, or $\mid 1 + \alpha a_1 \mid + \lambda \mid \alpha a_1 \mid = \min(1, \lambda)$, i.e. denoting $x = \alpha a_1$, $\mid 1 + x \mid + \lambda \mid x \mid = \min(1, \lambda)$.

1) If $\lambda > 1$, we obtain the equation $\mid 1 + x \mid + \lambda \mid x \mid = 1$, which has the real solution $x = 0$, i.e. $\alpha a_1 = 0$, $\alpha = 0$.

2) If $\lambda = 1$, we obtain the equation $\mid 1 + x \mid + \mid x \mid = 1$, which has the real solutions $-1 \leq x \leq 0$, i.e. $-1 \leq \alpha a_1 \leq 0$.

3) If $\lambda < 1$, we obtain the equation $\mid 1 + x \mid + \lambda \mid x \mid = \lambda$, which has the real solution $x = -1$, i.e. $\alpha a_1 = -1$, $\alpha = -\frac{1}{a_1}$.

Now a particular case of Example 7.

**Example 8.** Let be $a \in \mathbb{R}$, $\mid a \mid < 1$, $a \neq 0$, $G = \{(x_n)_{n \in \mathbb{N}} \in c_0 \mid \sum_{n=1}^{\infty} a^n x_n = 0\}$, and $f : G \to \mathbb{R}$, $f((x_n)_{n \in \mathbb{N}}) = x_1$. The the all Hahn-Banach extension of $f$, denoted by $h$, are the following:

1) If $\frac{1}{2} < \mid a \mid < 1$, $h(x_1, x_2, ...) = x_1$, $\forall (x_1, x_2, ...) \in c_0$.

2) If $a = \frac{1}{2}$, $h(x_1, x_2, ...) = x_1 + \alpha \sum_{n=1}^{\infty} a^n x_n$, $\forall (x_1, x_2, ...) \in c_0$, where $-2 \leq \alpha \leq 0$.

3) If $a = -\frac{1}{2}$, $h(x_1, x_2, ...) = x_1 + \alpha \sum_{n=1}^{\infty} a^n x_n$, $\forall (x_1, x_2, ...) \in c_0$, where $0 \leq \alpha \leq 2$.

4) If $\mid a \mid < \frac{1}{2}$, $h(x_1, x_2, ...) = -\sum_{n=1}^{\infty} a^n x_{n+1}$, $\forall (x_1, x_2, ...) \in c_0$

**Example 9.** Let be $(a_n)_{n \in \mathbb{N}} \in l_\infty$ with $a_1 \neq 0$, $G = \{(x_n)_{n \in \mathbb{N}} \in l_1 \mid \sum_{n=1}^{\infty} a_n x_n = 0\}$, $f : G \to \mathbb{R}$, $f((x_n)_{n \in \mathbb{N}}) = x_1$. The $f$ has a unique Hahn-Banach extension of $f$, namely

$h(x_1, ..., x_n, ...) = x_1 - \sum_{n=1}^{\infty} \frac{sgna_1}{\lambda + |a_1|} a_n x_n$, $\forall (x_1, ..., x_n, ...) \in l_1$, where $\lambda =$

$\sup\limits_{n\geq 2} \mid a_n \mid$.

**Proof.** We prove that $\parallel f \parallel = \frac{\lambda}{\lambda+|a_1|}$. For $x \in G$ we have $-a_1x_1 = \sum\limits_{n=2}^{\infty} a_nx_n$, so $\mid a_1 \mid\mid x_1 \mid \leq \lambda \sum\limits_{n=2}^{\infty} \mid x_n \mid \leq \lambda(\parallel x \parallel - \mid x_1 \mid)$, i.e. $\mid x_1 \mid \leq \frac{\lambda}{\lambda+|a_1|} \parallel x \parallel$, hence $\mid f(x) \mid \leq \frac{\lambda}{\lambda+|a_1|} \parallel x \parallel$, $\forall x \in G$, i.e. $\parallel f \parallel \leq \frac{\lambda}{\lambda+|a_1|}$. Let be $n \in \mathbb{N}$. Choose $\alpha \in \mathbb{R}$ such that $(1, 0, 0, ..., \alpha sgna_n, 0, ...) \in G$. Then $a_1+\alpha \mid a_n \mid = 0$, i.e. $-a_1 = \alpha \mid a_n \mid$. We have $1 = \mid f(1, 0, 0, ..., \alpha sgna_n, 0, ...) \mid \leq \parallel f \parallel (1+ \mid \alpha \mid)$, or $\mid a_n \mid \leq \parallel f \parallel (\mid a_n \mid + \mid a_1 \mid)$. Since $n \in \mathbb{N}$ is arbitrary we obtain $\lambda \leq \parallel f \parallel (\lambda + \mid a_1 \mid)$, i.e. $\parallel f \parallel \geq \frac{\lambda}{\lambda+|a_1|}$.

Let $h : c_0 \to \mathbb{R}$ be a Hahn-Banach extension of $f$. From the Proposition 5 there exists $\alpha \in \mathbb{R}$ such that $h(x_1, ..., x_n, ...) = x_1 + \alpha \sum\limits_{n=1}^{\infty} a_nx_n$, $\forall (x_1, ..., x_n, ...) \in l_1$. But $\parallel h \parallel = \max(\mid 1 + \alpha a_1 \mid, \mid \alpha \mid \lambda)$. As $\parallel h \parallel = \parallel f \parallel$, it follows that $\max(\mid 1 + \alpha a_1 \mid, \mid \alpha \mid \lambda) = \frac{\lambda}{\lambda+|a_1|}$, $\max(\mid 1 + \alpha a_1 \mid, \mid \alpha a_1 \mid \frac{\lambda}{|a_1|}) = \frac{\lambda}{\lambda+|a_1|}$, i.e. denoting by $x = \alpha a_1$, $M = \frac{\lambda}{|a_1|}$, $\max(\mid 1 + x \mid, M \mid x \mid) = \frac{M}{M+1}$, which has a unique real solution $x = -\frac{1}{M+1}$, $\alpha = -\frac{1}{a_1(M+1)} = -\frac{|a_1|}{a_1(\lambda+|a_1|)} = -\frac{sgna_1}{\lambda+|a_1|}$, i.e. the statement.

**Example 10.** Let $1 < p < \infty$, $q$ be the conjugate of $p$, i.e. $\frac{1}{p} + \frac{1}{q} = 1$, $(a_n)_{n\in\mathbb{N}} \in l_q$ with $a_1 \neq 0$, $G = \{(x_n)_{n\in\mathbb{N}} \in l_p \mid \sum\limits_{n=1}^{\infty} a_nx_n = 0\}$, $f : G \to \mathbb{R}$, $f((x_n)_{n\in\mathbb{N}}) = x_1$. Then the Hahn-Banach extension of $f$, denoted by $h$, is of the form:

$h((x_n)_{n\in\mathbb{N}}) = x_1 + \alpha \sum\limits_{n=1}^{\infty} a_nx_n$, $\forall (x_n)_{n\in\mathbb{N}} \in l_p$,

where $\alpha \in \mathbb{R}$ is a solution of the equation $(\mid 1 + \alpha a_1 \mid^q + \lambda^q \mid \alpha a_1 \mid^q)^{\frac{1}{q}} = (\frac{\lambda^p}{\lambda^p+1})^{\frac{1}{p}}$, $\lambda = \frac{(\sum\limits_{n=2}^{\infty} |a_n|^q)^{\frac{1}{q}}}{|a_1|}$.

**Proof.** We prove that $\parallel f \parallel = (\frac{M^p}{M^p+|a_1|^p})^{\frac{1}{p}}$, where $M = (\sum\limits_{n=2}^{\infty} \mid a_n \mid^q)^{\frac{1}{q}}$. For $x \in G$, $-a_1x_1 = \sum\limits_{n=2}^{\infty} a_nx_n$, hence $\mid a_1 \mid\mid x_1 \mid \leq \sum\limits_{n=2}^{\infty} \mid a_n \mid\mid x_n \mid \leq (\sum\limits_{n=2}^{\infty} \mid a_n \mid^q)^{\frac{1}{q}} (\sum\limits_{n=2}^{\infty} \mid x_n \mid^p)^{\frac{1}{p}} = M(\parallel x \parallel^p - \mid x_1 \mid^p)^{\frac{1}{p}}$, i.e. $\mid f(x) \mid = \mid x_1 \mid \leq (\frac{M^p}{M^p+|a_1|^p})^{\frac{1}{p}} \parallel x \parallel$, $\parallel f \parallel \leq (\frac{M^p}{M^p+|a_1|^p})^{\frac{1}{p}}$. Let be $n \in \mathbb{N}$. Again choose $\alpha_n \in \mathbb{R}$ such that $(\alpha_n, \mid a_2 \mid^{q-1} sgna_2, \mid a_3 \mid^{q-1} sgna_3, ..., \mid a_n \mid^{q-1} sgna_n, 0, ...) \in G$. Then $-\alpha_n a_1 = \mid a_2 \mid^q + ... + \mid a_n \mid^q$, $\mid \alpha_n \mid = \frac{|a_2|^q+...+|a_n|^q}{|a_1|}$. We have $\mid \alpha_n \mid = \mid f(\alpha_n, \mid$

$a_2 \mid^{q-1} sgn a_2, \mid a_3 \mid^{q-1} sgn a_3, ..., \mid a_n \mid^{q-1} sgn a_n, 0, ...) \mid \leq \parallel f \parallel (\mid \alpha_n \mid^p + \sum_{k=2}^{n} \mid a_k \mid^{(q-1)p})^{\frac{1}{p}}$. As $(q-1)p = q$, $\parallel f \parallel (1 + \frac{\sum_{k=2}^{n} |a_k|^q}{|\alpha_n|^p})^{\frac{1}{p}} \geq 1$. Passing to the limit for $n \to \infty$ and using that $\alpha_n \to \frac{M^q}{|a_1|}$ we obtain $\parallel f \parallel (1 + \frac{M^q |a_1|^p}{M^{qp}})^{\frac{1}{p}} \geq 1$. Since $q(p-1) = p$, $\parallel f \parallel \geq (\frac{M^p}{M^p + |a_1|^p})^{\frac{1}{p}}$.

If $h : l_p \to \mathbb{R}$ is a Hahn-Banach extension of $f$, from the proposition 5 there is $\alpha \in \mathbb{R}$ such that $h(x_1, ..., x_n, ...) = x_1 + \alpha \sum_{n=1}^{\infty} a_n x_n$, $\forall (x_1, ..., x_n, ...) \in l_p$. But $\parallel h \parallel = (\mid 1 + \alpha a_1 \mid^q + \mid \alpha \mid^q \sum_{n=2}^{\infty} \mid a_n \mid^q)^{\frac{1}{q}}$. Since $\parallel h \parallel = \parallel f \parallel$, $(\mid 1 + \alpha a_1 \mid^q + \mid \alpha \mid^q M^q)^{\frac{1}{q}} = (\frac{M^p}{M^p + |a_1|^p})^{\frac{1}{p}}$, i.e. denoting by $\lambda = \frac{M}{|a_1|}$, then $\alpha \in \mathbb{R}$ is the solution of the equation $(\mid 1 + \alpha a_1 \mid^q + \lambda^q \mid \alpha a_1 \mid^q)^{\frac{1}{q}} = (\frac{\lambda^p}{\lambda^p + 1})^{\frac{1}{p}}$.

**Example 11.** Let be $0 < b < 1$, $G = \{(x_n)_{n \in \mathbb{N}} \in c_0 \mid \sum_{n=1}^{\infty} \frac{x_n}{2^n} = 0\}$, $f : G \to \mathbb{R}$, $f((x_n)_{n \in \mathbb{N}}) = \sum_{n=1}^{\infty} b^n x_n$. Then the Hahn-Banach extension of $f$, denoted by $h$, is:

$h((x_n)_{n \in \mathbb{N}}) = \sum_{n=1}^{\infty} b^n x_n x_1 + \alpha \sum_{n=1}^{\infty} \frac{x_n}{2^n}$, $\forall (x_n)_{n \in \mathbb{N}} \in c_0$,

where $\alpha \in \mathbb{R}$ is a solution of the equation $\sum_{n=1}^{\infty} \mid b^n + \frac{\alpha}{2^n} \mid = \frac{b|2b-1|}{1-b}$.

**Proof.** Let us put temporarily $a = \frac{1}{2}$. For $x \in G$, $x_1 = -\sum_{n=2}^{\infty} a^{n-1} x_n$, from where $\mid f((x_n)_{n \in \mathbb{N}}) \mid = \mid \sum_{n=2}^{\infty} b^n x_n - b \sum_{n=2}^{\infty} a^{n-1} x_n \mid \leq$

$b \sum_{n=2}^{\infty} \mid b^{n-1} - a^{n-1} \parallel x_n \mid \leq b \parallel x \parallel \sum_{n=2}^{\infty} \mid b^{n-1} - a^{n-1} \mid = b \parallel x \parallel \mid \sum_{n=2}^{\infty} (b^{n-1} - a^{n-1}) \mid = b \parallel x \parallel \mid \frac{b}{1-b} - \frac{a}{1-a} \mid$, i.e. $\parallel f \parallel \leq b \mid \frac{b}{1-b} - \frac{a}{1-a} \mid$. For a fixed $n \in \mathbb{N}$, choose $x, y \in \mathbb{R}$ such that $(x, y, y, ..., y, 0, ...) \in G$. Then $-x = y(a + ... + a^{n-1})$, from where $\mid xb + y(b^2 + ... + b^n) \mid = \mid f(x, y, y, ..., y, 0, ...) \mid \leq \parallel f \parallel \max(\mid x \mid, \mid y \mid)$, or $\mid \frac{x}{y} b + (b^2 + ... + b^n) \mid \leq \parallel f \parallel \max(1, \frac{|x|}{|y|})$, $\mid -b(a + ... + a^{n-1}) + (b^2 + ... + b^n) \mid \leq \parallel f \parallel \max(1, a + ... + a^{n-1})$. Passing to the limit for $n \to \infty$ we obtain $\mid \frac{ba}{1-a} - \frac{b^2}{b-1} \mid \leq \parallel f \parallel \max(1, \frac{a}{1-a})$, i.e. $\parallel f \parallel \geq b \mid \frac{b}{1-b} - \frac{a}{1-a} \mid \min(1, \frac{1-a}{a})$. As $a = \frac{1}{2}$, $\frac{1-a}{a} = 1$, hence $\parallel f \parallel = b \mid \frac{b}{1-b} - 1 \mid = \frac{b|2b-1|}{1-b}$.

Let $h : c_0 \to \mathbb{R}$ be a Hahn-Banach extension of $f$. Using again the Proposition 5, there exists $\alpha \in \mathbb{R}$ such that $h((x_n)_{n \in \mathbb{N}}) = \sum_{n=1}^{\infty} b^n x_n + \alpha \sum_{n=1}^{\infty} \frac{x_n}{2^n}$,

$\forall (x_n)_{n\in\mathbb{N}} \in c_0$. But $\| h \| = \sum\limits_{n=1}^{\infty} | b^n + \frac{\alpha}{2^n} |$, i.e. $\alpha \in \mathbb{R}$ is a solution of the equation $\sum\limits_{n=1}^{\infty} | b^n + \frac{\alpha}{2^n} | = \frac{b|2b-1|}{1-b}$.

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"Ovidius" University of Constanta,
Department of Mathematics,
8700 Constanta,
Romania

"Ovidius" University of Constanta,
Department of Mathematics,
8700 Constanta,
Romania
e-mail: dpopa@univ-ovidius.ro