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A PAIR OF JORDAN TRIPLE SYSTEMS OF JORDAN PAIR TYPE

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Dedicated to Professor Mirela Ştefănescu on the occasion of her 60th birthday

Abstract

A Jordan pair $V = (V^+, V^-)$ is a pair of modules over a unitary commutative associative ring K, together with a pair (Q_+, Q_-) of quadratic mappings $Q_\sigma : V^\sigma \to \mathrm{Hom}_K(V^{-\sigma}, V^\sigma)$, $\sigma = \pm$, so that the following identities and their liniarizations are fulfilled for $\sigma = \pm$:

JP1 $D_\sigma(x,y)Q_\sigma(x) = Q_\sigma(x)D_{-\sigma}(y,x)$,

JP2 $D_\sigma(Q_\sigma(x)y, y) = D_\sigma(x, Q_{-\sigma}(y)x)$,

JP3 $Q_\sigma(Q_\sigma(x)y) = Q_\sigma(x)Q_{-\sigma}(y)Q_\sigma(x)$.

Here, $D_\sigma(x,y)z = Q_\sigma(x,z)y := Q_\sigma(x+z)y - Q_\sigma(x)y - Q_\sigma(z)y$.

According to an example with quadratic mappings $Q_\sigma : V^\sigma \to \mathrm{End}\left(V^{-\sigma}\right)$, we get a little different approach and we define the concept of a pair of Jordan Triple Systems of Jordan Pair type.

1 Introduction

Let D_n be the dihedral group of degree n, more precisely,

$$D_n =< a, b : \ |a| = n, \ |b| = 2, \ ba = a^{n-1}b > .$$

Usually, the elements of D_n are written in the form a^ib^i, $0 \le i \le n-1$, $0 \le j \le 1$, so that the underlying set of this group is

$$D_n =< a > \cup < a > b, \text{ if } n \text{ is an odd number,}$$

or

$$D_n =< a^2 > \cup < a^2 > a \cup < a^2 > b \cup < a^2 > ab, \text{ if } n \text{ is an even number.}$$

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If we use the notation $p \oplus_m r$ for the sum modulo m in the abelian group $R_m = \{0, 1, \ldots, m-1\}$, then the multiplication in D_n is given by the rule:

$$\left(a^i b^j\right)\left(a^k b^l\right) = a^{i \oplus_n (-1)^j k} b^{l \oplus_2 j}. \tag{1}$$

Indeed, because $ba^k = a^{(n-1)k} b = a^{-k} b = a^{n-k} b$, we obtain

$$\begin{aligned}
\left(a^i b^j\right)\left(a^k b^l\right) &= \begin{cases} a^{i+k} b^l, & \text{if} \quad j = 0, \\ a^{i-k} b^{l+1}, & \text{if} \quad j = 1 \text{ and } i \geq k, \\ a^{n-k+i} b^{l+1}, & \text{if} \quad j = 1 \text{ and } i < k, \end{cases} \\
&= \begin{cases} a^{i \oplus_n k} b^{l+0}, & \text{if} \quad j = 0, \\ a^{i \oplus_n (-k)} b^{l+1}, & \text{if} \quad j = 1 \end{cases} \\
&= a^{i \oplus_n (-1)^j k} b^{l \oplus_2 j}
\end{aligned}$$

Therefore, $D_n \cong R_n \oplus R_2$, where the group structure on $R_n \oplus R_2$ is defined via the composition

$$(i, j)(k, l) = \left(i \oplus_n (-1)^j k, l \oplus_2 j\right).$$

2 Some properties of the group algebra of D_n

Let K be a field and let $K[D_n]$ be the group algebra of D_n over K. Then $K[D_n]$ is a unitary associative algebra of dimension $2n$ over K, noncommutative if $n > 2$. A basis of $K[D_n]$ over K is the set of vectors

$$\left\{a^i b^j | 0 \leq i \leq n-1, \ 0 \leq j \leq 1\right\}$$

with the multiplication (1).

If we look carefully at the multiplication table of $K[D_n]$, we distinguish two cases, depending on n being either an odd or an even number.

When n is an odd number, let T^+ be the K-vector subspace of $K[D_n]$ with the basis $< a >$, and let T^- be the K-vector subspace of $K[D_n]$ with the basis $< a > b$. Then the multiplication table in $K[D_n] = T^+ \oplus T^-$ is

$\cdot$	T^+	T^-
T^+	T^+	T^-
T^-	T^-	T^+

(2)

When n is an even number, let $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3$ and $\mathcal{A}_4$ the K-vector spaces of $K[D_n]$ with the bases $< a^2 >, < a^2 > a, < a^2 > b$ and $< a^2 > ab$, respectively.

Then the multiplication table of the algebra $K[D_n] = \mathcal{A}_1 \oplus \mathcal{A}_2 \oplus \mathcal{A}_3 \oplus \mathcal{A}_4$ is

$\cdot$	$\mathcal{A}_1$	$\mathcal{A}_2$	$\mathcal{A}_3$	$\mathcal{A}_4$
$\mathcal{A}_1$	$\mathcal{A}_1$	$\mathcal{A}_2$	$\mathcal{A}_3$	$\mathcal{A}_4$
$\mathcal{A}_2$	$\mathcal{A}_2$	$\mathcal{A}_1$	$\mathcal{A}_4$	$\mathcal{A}_3$
$\mathcal{A}_3$	$\mathcal{A}_3$	$\mathcal{A}_4$	$\mathcal{A}_1$	$\mathcal{A}_2$
$\mathcal{A}_4$	$\mathcal{A}_4$	$\mathcal{A}_3$	$\mathcal{A}_2$	$\mathcal{A}_1$

(3)

Denote $\{i, j, k\} = \{i, j, k\}$. Then the table (3) shows that $\mathcal{A}_i\mathcal{A}_j = \mathcal{A}_k = \mathcal{A}_j\mathcal{A}_i$, so that the multiplication table can be arranged in the following form:

$\cdot$	$\mathcal{A}_1$	$\mathcal{A}_i$	$\mathcal{A}_j$	$\mathcal{A}_k$
$\mathcal{A}_1$	$\mathcal{A}_1$	$\mathcal{A}_i$	$\mathcal{A}_j$	$\mathcal{A}_k$
$\mathcal{A}_i$	$\mathcal{A}_i$	$\mathcal{A}_1$	$\mathcal{A}_k$	$\mathcal{A}_j$
$\mathcal{A}_j$	$\mathcal{A}_j$	$\mathcal{A}_k$	$\mathcal{A}_1$	$\mathcal{A}_i$
$\mathcal{A}_k$	$\mathcal{A}_k$	$\mathcal{A}_j$	$\mathcal{A}_i$	$\mathcal{A}_1$

(4)

Finally, if $T^+ = \mathcal{A}_1 \oplus \mathcal{A}_i$ and $T^- = \mathcal{A}_j \oplus \mathcal{A}_k$, then the table (4) becomes (2).

Therefore, except for the detailed table (3), the K-algebra $K[D_n]$ decomposes in $K[D_n] = T^+ \oplus T^-$, with multiplication table (2).

3 A pair of Jordan Triple Systems

Let $T = (T^+, T^-)$ be the pair of K-vector spaces defined in the previous section.

Unlikely the Jordan pairs, where we have to do with quadratic mappings $Q_\sigma : T^\sigma \to \mathrm{Hom}\,(T^{-\sigma}, T^\sigma)$, $\sigma = \pm$, the decomposition of $K[D_n] = T^+ \oplus T^-$, via table (2), leads to the quadratic mappings $Q_\sigma : T^\sigma \to \mathrm{End}\,(T^{-\sigma})$. More exactly, for any $x \in T^\sigma$ and $y \in T^{-\sigma}$, $\sigma = \pm$, the product $xyx \in T^{-\sigma}$. Thus the assignments $x \mapsto Q_\sigma(x) : y \mapsto xyx$, give rise to maps $Q_\sigma : T^\sigma \to \mathrm{End}\,(T^{-\sigma})$, $\sigma = \pm$. Does the pair $Q = (Q_+, Q_-)$, defined above, verify the conditions JP1-JP3? For, supposing the characteristic $\neq 2$, these conditions become the following linear identities:

JP1$'$ $\{xy\{xzx\}\} = \{x\{yxz\}x\}$,
JP2$'$ $\{\{xyx\}yz\} = \{x\{yxy\{z\}$,
JP3$'$ $\{\{xyx\}z\{xyx\}\} = \{x\{y\{xzx\}y\}x\}$.
for all $x, z \in T^\sigma$ and $y \in T^{-\sigma}$, where $\{xyz\} := Q_\sigma(x, z)y = D_\sigma(x, y)z = Q(x + z)y - Q(x)y - Q(z)y$.

Studying the succession of the signs in JP1$'$ and JP2$'$ we conclude that the pair $Q = (Q_+, Q_-)$ cannot satisfy these axioms or others of their type.

Concerning JP3, the permitted successions of the signs in the JP3$'$ lead us to consider, in addition to products xyx, with $x \in T^{\sigma}$ and $y \in T^{-\sigma}$, the products xzx, with $x, z \in T^{\sigma}$. So, every K-vector space T^{σ} together with the quadratic applications $P_{\sigma} : T^{\sigma} \to \mathrm{End}(T^{\sigma})$ defined by the assignments $x \to P_{\sigma}(x) : x \to xzx$, is a Jordan triple system; that is, the following identities and their liniarizations hold:

JTS1 $L_{\sigma}(x, y)P_{\sigma}(x) = P_{\sigma}(x)L_{\sigma}(y, x)$,
JTS2 $L_{\sigma}\left(P_{\sigma}(x)y, y\right) = L_{\sigma}\left(x, P_{\sigma}(y)x\right)$,
JTS3 $P_{\sigma}\left(P_{\sigma}(x)y\right) = P_{\sigma}(x)P_{\sigma}(y)P_{\sigma}(x)$,
where $L_{\sigma}(x, y)z = P_{\sigma}(x, z)y = P_{\sigma}(x + z)y - P_{\sigma}(x)y - P_{\sigma}(z)y$.

Finally, the condition JP3 is replaced by the following three connective relations between the mappings P_{σ} and Q_{σ}:

(CR1) $Q_{\sigma}\left(P_{\sigma}(x)y\right) = Q_{\sigma}(x)Q_{\sigma}(y)Q_{\sigma}(x)$,
for all $x, y \in T^{\sigma}$;
(CR2) $P_{\sigma}\left(Q_{-\sigma}(x)y\right) = Q_{-\sigma}(x)Q_{\sigma}(y)Q_{-\sigma}(x)$,
for all $x \in T^{-\sigma}$, $y \in T^{\sigma}$;
(CR3) $Q_{\sigma}\left(Q_{-\sigma}(x)y\right) = P_{\sigma}(x)Q_{-\sigma}(y)P_{\sigma}(x)$,
for all $x \in T^{-\sigma}$, $y \in T^{\sigma}$.

In this way, we are led to the following definition.

Definition 3.1 Let T^{+} and T^{-} be two Jordan triple systems relative to the quadratic mapping $P_{\sigma} : T^{\sigma} \to \mathrm{End}\left(T^{\sigma}\right)$, $\sigma = \pm$. Let $Q_{\sigma} : T^{\sigma} \to \mathrm{End}\left(T^{-\sigma}\right)$, $\sigma = \pm$, be two quadratic applications. We say that $T = (T^{+}, T^{-})$ is a **pair of Jordan triple systems of Jordan pair type** if the connecting axioms CR1-CR3 and their liniarizations are fulfilled for $\sigma = \pm$.

References

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