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NUMERICAL DISCRETE ALGORITHM FOR SOME NONLINEAR PROBLEMS

Cristina Sburlan

Abstract

In this paper we use the eigenfunctions of the Laplacian to approximate the solution of some nonlinear equations which are used to model natural phenomena (as the Navier-Stokes flow equations, for instance). In this respect we propose a numerical algorithm, combining the Uzawa and Arrow-Hurwitz algorithms. The algorithm proposed here shares features from both algorithms, and it has the following advantages upon them: the usage of a single parameter (like in the Uzawa algorithm) and the fact that the approximative equation is linear (like in Arrow-Hurwitz algorithm). We prove the convergence of the approximate solution to the weak solution of the given equation. Next, we apply a Galerkin-type discretization of this algorithm in order to compute the approximate solution.

1 An Arrow-Hurwicz-Uzawa Type Algorithm

In this section we develop a numerical algorithm used to approximate the solution of the stationary nonlinear Navier-Stokes system, combining Uzawa and Arrow-Hurwicz algorithms presented in [6], which represent extensions of the classical Uzawa and Arrow-Hurwicz algorithms that appear in nonlinear optimization problems. We describe the algorithm and prove the convergence

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of the solution obtained by this algorithm to the weak solution of Navier-Stokes system (without a direct reference to the optimization theory, even if the idea of an algorithm of this type comes from the optimization theory).

This numerical method can also be applied to other linear or nonlinear problems modeling natural phenomena, such as diffusion-dispersion problems, Oseen equations, Brinkman equations and so on.

Consider the Navier-Stokes system for the flow of an incompressible fluid:

$$\begin{aligned} \operatorname{div} u(x) &= 0 \\ (u \cdot \nabla)u(x) - \nu \Delta u(x) + \nabla p(x) &= f(x), x \in \Omega \\ u &= 0 \text{ on } \partial\Omega. \end{aligned}$$

where ν is the dynamical viscosity, $\Omega \subset \mathbb{R}^N, 2 \leq N \leq 3$, is a bounded domain with smooth enough boundary to apply the Green formula and the Sobolev-Kondrashov theorem, $f \in L^2(\Omega)$ represents the body forces, the scalar function p represents the pressure and the vector function $u = (u_1, \ldots, u_N)$ represents the velocity of the fluid.

Consider $A \in L(E, E^*)$ (the Stokes operator):

$$(Ay, w) = \sum_{i=1}^{N} \int_{\Omega} \nabla y_i \cdot \nabla w_i dx, \forall\, y, w \in E$$

and define the nonlinear form $b(y, z, w) := \sum_{i,j=1}^{N} \int_{\Omega} y_i D_i z_j w_j dx$.

We write the weak formulation of the problem:

$$\nu(Au, v) + b(u, u, v) = (f - \nabla p, v), \forall v \in E,$$

or, equivalently,

$$\nu < u, v > + b(u, u, v) = (f - \nabla p, v), \forall v \in E, \tag{1}$$

where we have considered the Hilbert spaces $X := \{y \in \left(L^2(\Omega)\right)^N \mid \nabla \cdot y = 0,\ n \cdot y = 0 \text{ on } \partial\Omega\}$ and $E := \{y \in \left(H_0^1(\Omega)\right)^N \mid \nabla \cdot y = 0\}$.

Denote by $< \cdot, \cdot >$ and $\|\cdot\|$ (respectively, by $(\cdot, \cdot)$ and $|\cdot|$) the scalar product and the norm on $V := \left(H_0^1(\Omega)\right)^N$ (respectively on $\left(L^2(\Omega)\right)^N$).

Define the three-linear functional $\hat{b} : V \times V \times V \to \mathbb{R}$,

$$\hat{b}(u, v, w) = \frac{1}{2} \sum_{i,j=1}^{N} \int_{\Omega} u_i (D_i v_j) w_j dx - \frac{1}{2} \sum_{i,j=1}^{N} \int_{\Omega} u_i v_j (D_i w_j) dx, u, v, w \in V,$$

where $u = (u_1, \ldots, u_N)$, $v = (v_1, \ldots, v_N)$, $w = (w_1, \ldots, w_N)$, and $D_i u_j$ represents the partial derivative of u_j with respect to x_i ($x = (x_1, \ldots, x_N) \in \Omega$).

We have (see [6], p. 205) that $\hat{b}(u,u,v) = b(u,u,v), \forall u,v \in E$.

Also, in [6] is proved that $\exists c > 0$ such that $|\hat{b}(u,v,w)| \le c \cdot \|u\| \cdot \|v\| \cdot \|w\|, \forall u,v,w \in V$, and, if $\nu^2 - c\|f\|_{E^*} > 0$ and u is the solution of problem (1), then $\|u\| \le \frac{1}{\nu}\|f\|_{E^*}$. Moreover, in [6] there are described the following numerical algorithms, whose solutions converge to the solution of problem (1).

Uzawa Algorithm: Let $p^0 \in L^2(\Omega)$ be arbitrary given. For a known p^m ($m \in \mathbb{N}$), define $u^{m+1} \in V$ and $p^{m+1} \in L^2(\Omega)$ as the solution of the problem:

$$\nu < u^{m+1}, v > + \hat{b}(u^{m+1}, u^{m+1}, v) = (p^m, \text{div } v) + (f,v), \forall v \in V,$$
$$(p^{m+1} - p^m, q) + \rho(\text{div } u^{m+1}, q) = 0, \forall q \in L^2(\Omega),$$

where $\rho > 0$ is a real number such that $0 < \rho < 2\nu$ and $\nu - \frac{c}{\nu}\|f\|_{E^*} > 0$.

Arrow-Hurwicz Algorithm: Let $u^0 \in V$ and $p^0 \in L^2(\Omega)$ arbitrary given. For known u^m, p^m ($m \in \mathbb{N}$), define $u^{m+1} \in V$ and $p^{m+1} \in L^2(\Omega)$ as the solution of the problem:

$$< u^{m+1} - u^m > + \rho\nu < u^m, v > + \rho\hat{b}(u^m, u^{m+1}, v) =$$
$$= \rho(p^m, \text{div } v) + \rho(f,v), \forall v \in V,$$
$$\alpha(p^{m+1} - p^m, q) + \rho(\text{div } u^{m+1}, q) = 0, \forall q \in L^2(\Omega),$$

where $\nu - \frac{2c}{\nu}\|f\|_{E^*} - \frac{4c^2}{\nu^2}\|f\|^2_{E^*} = \nu^* > 0$ and $\alpha, \rho > 0$ are real numbers such that $0 < \rho < \frac{\alpha\nu^*}{2(1+\nu^2\alpha)}$.

The Uzawa algorithm has as disadvantage that in the equation appears the nonlinear term $\hat{b}(u^{m+1}, u^{m+1}, v)$, which makes difficult the solvability of this equation. On the other hand, in the Arrow-Hurwicz algorithm, with respect to the Uzawa algorithm, appears moreover the real parameter α, which is related by some conditions with the parameter ρ. However, this method has the advantage that the term $\hat{b}(u^m, u^{m+1}, v)$, in which u^{m+1} is unknown, is linear with respect to this unknown function.

Next, we will develop an algorithm of the above type, and we will prove the convergence for it. The advantage of this algorithm against those presented are the usage of a single real parameter, ρ, like in the Uzawa's method, and of the linear term $\hat{b}(u^m, u^{m+1}, v)$, like in the Arrow-Hurwicz method. We succeeded to combine the two above methods to give a new numerical method that presents the mentioned advantages.

The numerical algorithm is the following one:

Initially, we arbitrary give $p^0 \in L^2(\Omega)$ and $u^0 \in V$. For known p^m and u^m, we compute $p^{m+1} \in L^2(\Omega)$ and $u^{m+1} \in V$ by:

$$< u^{m+1} - u^m, v > + \rho\nu < u^{m+1}, v > + \rho\hat{b}(u^m, u^{m+1}, v) - \\ - \rho(p^m, \text{div } v) = \rho(f,v), \forall v \in V, \tag{2}$$

$$(p^{m+1} - p^m, q) + \rho(\text{div } u^{m+1}, q) = 0, \forall q \in L^2(\Omega), \tag{3}$$

where ρ is an arbitrary strictly positive real number.

For an arbitrary fixed $m \in \mathbb{N}^*$, we will prove the existence and the uniqueness of the solution for the previous algorithm.

Theorem 1.1 *If $m \in \mathbb{N}^*$ and $p^m \in L^2(\Omega)$, $u^m \in V$ are known, then the solution of the problem (2)–(3), $(u^{m+1}, p^{m+1}) \in V \times L^2(\Omega)$, exists and it is unique.*

Proof. If we succeed to prove the existence and the uniqueness of u^{m+1}, then the existence and the uniqueness of p^{m+1} will follow immediately, p^{m+1} being determined from relation (3): $p^{m+1} = p^m - \rho \text{div } u^{m+1}$.

Let us prove now the existence and the uniqueness of u^{m+1}. Equation (2) can be written as:

$$< u^{m+1}, v > + \rho\nu < u^{m+1}, v > + \rho\hat{b}(u^m, u^{m+1}, v) =$$
$$= \rho(p^m, \text{div } v) + < u^m, v > + \rho(f, v), \forall v \in V,$$

or, equivalently, $a(u^{m+1}, v) = g(v), \forall v \in V$, where $a : V \times V \to \mathbb{R}$,

$$a(u, v) = (1 + \rho\nu) < u, v > + \frac{\rho}{2} \sum_{i,j=1}^{N} \int_\Omega u_i^m (D_i u_j) v_j dx -$$
$$- \frac{\rho}{2} \sum_{i,j=1}^{N} \int_\Omega u_i^m u_j (D_i v_j) dx, u, v \in V,$$

and we have considered the linear continuous functional $g : V \to \mathbb{R}$ given by

$$g(v) =< u^m, v > + \rho(p^m, \text{div } v) + \rho(f, v), \forall v \in V$$

(we have applied: $\hat{b}(u, v, w) = \frac{1}{2} \sum_{i,j=1}^{N} \int_\Omega u_i (D_i v_j) w_j dx - \frac{1}{2} \sum_{i,j=1}^{N} \int_\Omega u_i v_j (D_i w_j) dx$).

We have that $a(u, v)$ is continuous, bilinear and coercive, because

$$a(u, u) =< u, u >= \|u\|^2, \forall u \in V,$$

so, applying the Lax-Milgram Theorem, we have that there exists a unique u^{m+1} solution of the above equation. □

Theorem 1.2 *If*

$$\nu^2 - c\|f\|_{E^*} > 0 \tag{4}$$

and $\rho \in \mathbb{R}$ satisfies the condition

$$0 < \rho < \frac{\nu\left(\nu^2 - c\|f\|_{E^*}\right)}{c^2\|f\|_{E^*}^2 + \nu^2}, \tag{5}$$

then, for $m \to \infty$, the solution u^m of problem (2) (strongly) converges to u in V, and p^m weakly converges to p in $L^2(\Omega)/\mathbb{R}$, where (u, p) is the solution of problem (1).

Proof. The proof can be done following a similar way to the one used to prove the convergence of the Arrow-Hurwitz algorithm. First, denote by $v^m = u^m - u$ and by $q^m = p^m - p$. We take $v = 2v^{m+1}$ in equation (2) and in equation (1) multiplied by ρ, and we have:

$$< u^{m+1} - u^m, 2v^{m+1} > +\rho\nu < u^{m+1}, 2v^{m+1} > +\rho\hat{b}(u^m, u^{m+1}, 2v^{m+1}) - \\ -\rho(p^m, \text{div } (2v^{m+1})) = (f, 2v^{m+1})$$

and $\rho\nu < u, 2v^{m+1} > +\rho\hat{b}(u, u, 2v^{m+1}) - (p, \text{div } (2v^{m+1})) = (f, 2v^{m+1})$.

Making the difference of these equations term by term, we obtain:

$$< u^{m+1} - u^m, 2v^{m+1} > +2\rho\nu < u^{m+1} - u, v^{m+1} > + \\ +2\rho\left[\hat{b}(u^m, u^{m+1}, v^{m+1}) - \hat{b}(u, u, v^{m+1})\right] - 2\rho(p^m - p, \text{div } v^{m+1}) = 0.$$

On the other hand, we can write:

$$< u^{m+1} - u^m, 2v^{m+1} > = < v^{m+1} - v^m, 2v^m > = \\ = < v^{m+1} - v^m, v^{m+1} > + < v^{m+1} - v^m, v^{m+1} > = \\ = \left\|v^{m+1}\right\|^2 - < v^m, v^{m+1} > + < v^{m+1} - v^m, v^{m+1} - v^m > + \\ + < v^{m+1} - v^m, v^m > = \left\|v^{m+1}\right\|^2 + \left\|v^{m+1} - v^m\right\|^2 - < v^m, v^{m+1} > + \\ + < v^{m+1}, v^m > - < v^m, v^m > = \left\|v^{m+1}\right\|^2 - \left\|v^m\right\|^2 + \left\|v^{m+1} - v^m\right\|^2 \text{ and}$$

$$\hat{b}(u^m, u^{m+1}, v^{m+1}) - \hat{b}(u, u, v^{m+1}) = \\ = \hat{b}(u^{m+1} - u, u, v^{m+1}) + \hat{b}(u^m, u^{m+1}, v^{m+1}) - \hat{b}(u^{m+1}, u, v^{m+1}) = \\ = \hat{b}(v^{m+1}, u, v^{m+1}) + \hat{b}(u^m - u^{m+1}, u, v^{m+1}) + \hat{b}(u^m, u^{m+1}, v^{m+1}) - \\ -\hat{b}(u^m, u, v^{m+1}) = \hat{b}(v^{m+1}, u, v^{m+1}) + \hat{b}(v^m - v^{m+1}, u, v^{m+1}) + \\ +\hat{b}(u^m, u^{m+1} - u, v^{m+1}) = \hat{b}(v^{m+1}, u, v^{m+1}) + \hat{b}(v^m - v^{m+1}, u, v^{m+1}) + \\ +\hat{b}(u^m, v^{m+1}, v^{m+1}) = \hat{b}(v^{m+1}, u, v^{m+1}) + \hat{b}(v^m - v^{m+1}, u, v^{m+1}),$$

because $\hat{b}(w, v, v) = 0, \forall w, v \in V$ (see [6], p. 218).

Using these relations, we have that:

$$\left\|v^{m+1}\right\|^2 - \left\|v^m\right\|^2 + \left\|v^{m+1} - v^m\right\|^2 + 2\rho\nu\left\|v^{m+1}\right\|^2 = \\ -2\rho\hat{b}(v^{m+1}, u, v^{m+1}) - 2\rho\hat{b}(v^m - v^{m+1}, u, v^{m+1}) + 2\rho(q^m, \text{div } v^{m+1}) \leq \\ 2\rho c\left\|v^{m+1}\right\|^2 \cdot \|u\| + 2\rho c\left\|v^m - v^{m+1}\right\| \cdot \left\|v^{m+1}\right\| \cdot \|u\| + 2\rho(q^m, \text{div } v^{m+1}) \leq \\ 2\frac{\rho c}{\nu}\|f\|_{E^*}\left\|v^{m+1}\right\|^2 + 2\frac{\rho c}{\nu}\|f\|_{E^*}\left\|v^m - v^{m+1}\right\| \cdot \left\|v^{m+1}\right\| + 2\rho(q^m, \text{div } v^{m+1}).$$

But, for every $\delta > 0$, we can write:

$2\frac{\rho c}{\nu}\|f\|_{E^*}\left\|v^m - v^{m+1}\right\| \cdot \left\|v^{m+1}\right\| \leq \frac{\rho^2 c^2}{\nu^2\delta}\|f\|_{E^*}^2\left\|v^{m+1}\right\|^2 + \delta\left\|v^{m+1} - v^m\right\|^2$, therefore

$$\left\|v^{m+1}\right\|^2 - \left\|v^m\right\|^2 + \left\|v^{m+1} - v^m\right\|^2 + 2\rho\nu\left\|v^{m+1}\right\|^2 \leq \\ \leq 2\frac{\rho c}{\nu}\|f\|_{E^*}\left\|v^{m+1}\right\|^2 + \frac{\rho^2 c^2}{\nu^2\delta}\|f\|_{E^*}^2\left\|v^{m+1}\right\|^2 + \\ +\delta\left\|v^{m+1} - v^m\right\|^2 + 2\rho(\text{div } v^{m+1}, q^m). \tag{6}$$

Next, taking $q = 2q^{m+1}$ in relation (3), we have that $(p^{m+1}-p^m, 2q^{m+1}) = -\rho(\text{div } u^{m+1}, 2q^{m+1})$, or $(q^{m+1} - q^m, 2q^{m+1}) = -2\rho(\text{div } u^{m+1}, q^{m+1})$. Using the same technique, we obtain $(q^{m+1} - q^m, 2q^{m+1}) = \left|q^{m+1}\right|^2 - |q^m|^2 + \left|q^{m+1} - q^m\right|^2$, and, using that div $u = 0$, it follows:

$$\begin{gathered}(q^{m+1} - q^m, 2q^m) = \left|q^{m+1}\right|^2 - |q^m|^2 + \left|q^{m+1} - q^m\right|^2 = \\ = -2\rho(\text{div } u^{m+1}, q^{m+1}) = -2\rho(\text{div } u^{m+1}, q^{m+1}) - 2\rho(\text{div } u, q^{m+1}) = \\ = -2\rho(\text{div } (u^{m+1} - u), q^{m+1}) = -2\rho(\text{div } v^{m+1}, q^{m+1})\end{gathered}$$

and, therefore,

$$\left|q^{m+1}\right|^2 - |q^m|^2 + \left|q^{m+1} - q^m\right|^2 = -2\rho(\text{div } v^{m+1}, q^{m+1}). \tag{7}$$

Adding relations (6) and (7), we have that:

$$\begin{gathered}\left\|v^{m+1}\right\|^2 - \|v^m\|^2 + \left\|v^{m+1} - v^m\right\|^2 + 2\rho\nu \left\|v^{m+1}\right\|^2 + \\ + \left|q^{m+1}\right|^2 - |q^m|^2 + \left|q^{m+1} - q^m\right|^2 \leq \\ \leq 2\tfrac{\rho c}{\nu} \|f\|_{E^*} \left\|v^{m+1}\right\|^2 + \tfrac{\rho^2 c^2}{\nu^2 \delta} \|f\|_{E^*}^2 \left\|v^{m+1}\right\|^2 + \delta \left\|v^{m+1} - v^m\right\|^2 + \\ +2\rho(\text{div } v^{m+1}, q^m) - 2\rho(\text{div } v^{m+1}, q^{m+1}) = \\ \leq 2\tfrac{\rho c}{\nu} \|f\|_{E^*} \left\|v^{m+1}\right\|^2 + \tfrac{\rho^2 c^2}{\nu^2 \delta} \|f\|_{E^*}^2 \left\|v^{m+1}\right\|^2 + \\ +\delta \left\|v^{m+1} - v^m\right\|^2 + 2\rho(\text{div } v^{m+1}, q^m - q^{m+1}),\end{gathered}$$

and we can evaluate (with the same δ previously considered):

$$\begin{gathered}2\rho(\text{div } v^{m+1}, q^m - q^{m+1}) \leq 2\rho \left\|v^{m+1}\right\| \cdot \left|q^m - q^{m+1}\right| \leq \\ \leq \tfrac{\rho^2}{\delta} \left\|v^{m+1}\right\|^2 + \delta \left|q^m - q^{m+1}\right|^2.\end{gathered}$$

From here it results that

$$\begin{gathered}\left\|v^{m+1}\right\|^2 - \|v^m\|^2 + \left\|v^{m+1} - v^m\right\|^2 + 2\rho\nu \left\|v^{m+1}\right\|^2 + \left|q^{m+1}\right|^2 - |q^m|^2 + \\ + \left|q^{m+1} - q^m\right|^2 \leq 2\tfrac{\rho c}{\nu} \|f\|_{E^*} \left\|v^{m+1}\right\|^2 + \tfrac{\rho^2 c^2}{\nu^2 \delta} \|f\|_{E^*}^2 \left\|v^{m+1}\right\|^2 + \\ +\delta \left\|v^{m+1} - v^m\right\|^2 + \tfrac{\rho^2}{\delta} \left\|v^{m+1}\right\|^2 + \delta \left|q^m - q^{m+1}\right|^2,\end{gathered}$$

or, equivalently,

$$\begin{gathered}\left\|v^{m+1}\right\|^2 - \|v^m\|^2 + (1-\delta) \left\|v^{m+1} - v^m\right\|^2 + \\ + \left(2\rho\nu - 2\tfrac{\rho c}{\nu} \|f\|_{E^*} - \tfrac{\rho^2 c^2}{\nu^2 \delta} \|f\|_{E^*}^2 - \tfrac{\rho^2}{\delta}\right) \left\|v^{m+1}\right\|^2 + \\ + \left|q^{m+1}\right|^2 - |q^m|^2 + (1-\delta) \left|q^m - q^{m+1}\right|^2 \leq 0.\end{gathered} \tag{8}$$

Summing these inequalities for $m = 0, 1, \ldots, n$ (with arbitrary $n \in \mathbb{N}$) we obtain:

$$\begin{gathered}\left\|v^{n+1}\right\|^2 + (1-\delta) \sum_{m=1}^{n} \left\|v^{m+1} - v^m\right\|^2 + \\ + \left(2\rho\nu - 2\tfrac{\rho c}{\nu} \|f\|_{E^*} - \tfrac{\rho^2 c^2}{\nu^2 \delta} \|f\|_{E^*}^2 - \tfrac{\rho^2}{\delta}\right) \sum_{m=1}^{n} \left\|v^{m+1}\right\|^2 + \left|q^{n+1}\right|^2 + \\ +(1-\delta) \sum_{m=1}^{n} \left|q^m - q^{m+1}\right|^2 \leq \left\|v^0\right\|^2 + \left|q^0\right|^2.\end{gathered} \tag{9}$$

From hypotheses (4) and (5), we have that $0 < \frac{\rho(c^2\|f\|_{E^*})+\nu^2}{2\nu(\nu^2-c\|f\|^2_{E^*}} < \frac{1}{2}$, so we can choose a $\delta > 0$ in relation (9) such that $0 < \frac{\rho(c^2\|f\|_{E^*})+\nu^2}{2\nu(\nu^2-c\|f\|^2_{E^*}} < \frac{1}{2} < \delta < 1$, which is equivalent to $2\rho\nu - 2\frac{\rho c}{\nu}\|f\|_{E^*} - \frac{\rho^2 c^2}{\nu^2\delta}\|f\|^2_{E^*} - \frac{\rho^2}{\delta} > 0$, and on the other hand we have that $1-\delta > 0$.

Therefore, in relation (9), all the coefficients from the left-hand side of the inequality are strictly positive, and from here it results that $\lim\limits_{m\to\infty} \left\|v^{m+1}\right\|^2 = 0$ and $\lim\limits_{m\to\infty} \left|q^{m+1} - q^m\right|^2 = 0$.

Since $v^{m+1} = u^{m+1} - u$, we have $\lim\limits_{m\to\infty} \left\|u^{m+1} - u\right\| = 0$, therefore u^{m+1} converges to u in V.

On the other hand, we have from relation (9) that the sequence $(q^{m+1})_{m\in\mathbb{N}}$ is bounded in $L^2(\Omega)$, so $(p^{m+1})_{m\in\mathbb{N}}$ will be also bounded in $L^2(\Omega)$. Therefore it follows that we can extract from $(p^{m+1})_{m\in\mathbb{N}}$ a subsequence $(p^{k_{m+1}})_{m\in\mathbb{N}}$ weakly convergent to $\tilde{p} \in L^2(\Omega)$.

Passing to the limit with $m \to \infty$ in relation (2), we have

$$\nu < u, v > +\rho b(u, u, v) - (\tilde{p}, \text{div } v) = (f, v), \forall v \in E,$$

relation which we subtract from (1) and we obtain $(p - \tilde{p}, \text{div } v) = 0, \forall v \in E$, therefore $\nabla(p - \tilde{p}) = 0$, so $p = \tilde{p} + K$, where $K \in \mathbb{R}$ is arbitrary. □

Remark 1.1 $L^2(\Omega)/\mathbb{R} = \left\{ p \in L^2(\Omega) \mid \int\limits_\Omega p(x)dx = 0 \right\}$ (see [6], p. 15).

Remark 1.2 We know that the pressure p must satisfy the condition:

$$\int\limits_\Omega p(x)dx = 0$$

(see [5], p. 163). Asking this condition (like in [6]), we can obtain the convergence of p^m to p in $L^2(\Omega)$ (instead $L^2(\Omega)/\mathbb{R}$).

Choosing in the previous algorithm $p^0 \in L^2(\Omega)$ such that $\int\limits_\Omega p^0(x)dx = 0$, then using relation (3) we have that $p^m = p^0 - \sum\limits_{k=1}^{m} \text{div } u^k$, and because $\int\limits_\Omega \text{div } v dx = 0, \forall v \in X$ (see [5], p. 69) and we obtain $\int\limits_\Omega p^m(x)dx = 0, \forall m \in \mathbb{N}$. From the previous theorem it results now that p^m weakly converges to p in the space $L^2(\Omega)$.

2 Discretization of the numerical algorithm

In this section we describe the discretization of the numerical algorithm using Galerkin method, for which we prove the existence, the uniqueness and the convergence of the solution. This discretization can be used for the effective calculation of the approximate solution, because it is reduced to the solving of a linear algebraic system.

The discrete forms of the numerical algorithms from Section 1 can be obtained using many methods (finite differences, finite element, Galerkin). For instance, in [6] there are presented discretizations of the Uzawa and Arrow-Hurwicz algorithms using finite differences method.

Next, we will present the discretization of algorithm (2) from Section 1, using the Galerkin method. For the application of this method we will use as discretization basis the orthonormal system of eigenfunctions of the Laplace's operator $-\Delta$, which is the duality mapping between the space $(H_0^1(\Omega))^N$ and its topological dual.

In the same conditions as in Section 1 and in conditions of Theorem 1.2, consider $(\varphi_n)_{n\in\mathbb{N}} \subset V$ the orthonormal system formed by eigenfunctions of the Laplace's operator $-\Delta$, which is an orthonormal basis in the space $V = (H_0^1(\Omega))^N$ (see [4], p. 67). Let $k \in \mathbb{N}^*$ and $S_k(\Omega)$ be the space generated by the eigenfunctions $\varphi_1, \varphi_2, \dots, \varphi_k$. In this case, we have that the matrix $G = (G_{ij})$, $G_{ij} = \langle \varphi_i, \varphi_j \rangle$ is the unity matrix, because $\{\varphi_i\}_{i=1,2,\dots,k}$ forms an orthonormal system.

We want to formulate the discrete problem corresponding to the problem (2), which asks to give an approximative problem whose solution will good enough approximate the solution u^{m+1} of the problem (2) (with known u^m, p^m, for an arbitrary fixed $m \in \mathbb{N}$) (we are at the $(m+1)^{\text{th}}$ step of the considered algorithm).

Denote, like in Section 1, by $a : V \times V \to \mathbb{R}$,

$$\begin{aligned} a(u,v) = (1+\rho\nu) < u, v > &+ \tfrac{\rho}{2} \sum_{i,j=1}^{N} \int_{\Omega} u_i^m (D_i u_j) v_j dx - \\ &- \tfrac{\rho}{2} \sum_{i,j=1}^{N} \int_{\Omega} u_i^m u_j (D_i v_j) dx, u, v \in V. \end{aligned} \tag{10}$$

We have that $a\,(u,v)$ is continuous, bilinear and coercive.

Using the fact that:

$$\hat{b}(u,v,w) = \frac{1}{2} \sum_{i,j=1}^{N} \int_{\Omega} u_i (D_i v_j) w_j dx - \frac{1}{2} \sum_{i,j=1}^{N} \int_{\Omega} u_i v_j (D_i w_j) dx$$

we can now formulate the approximative problem corresponding to problem (2):

Find $u_k^{m+1} \in S_k\,(\Omega)$ *such that:*

$$a\left(u_k^{m+1}, \varphi\right) = g(\varphi), \forall \varphi \in S_k\,(\Omega)\,, \tag{11}$$

where $g : V \to \mathbb{R}$ *is given by* $g(v) =< u_k^m, v > +\rho(p_k^m, div\ v) + \rho(f, v), \forall v \in V$ *and* $p_k^m = p_k^{m-1} - \rho \cdot \text{div}\ u_k^m$.

Since $u_k^{m+1} \in S_k(\Omega)$, we have that

$$u_k^{m+1} = \sum_{i=1}^{k} \alpha_i \varphi_i, \tag{12}$$

with $\alpha_i \in \mathbb{R}$, and relations (11) and (12) lead us to the algebraic linear system were α_i are not known:

$$\sum_{i=1}^{k} \alpha_i \cdot a(\varphi_i, \varphi_j) = g(\varphi_j),\ j = 1, 2, \ldots, k. \tag{13}$$

In the following, we prove the existence and uniqueness of the solution u_k^{m+1} for problem (11), and also its strongly convergence to the solution u^{m+1} of problem (2), using the specific arguments of the Galerkin method.

Theorem 2.1 *In the above conditions, there exists a unique* $u_k^{m+1} \in S_k(\Omega)$ *satisfying problem (11).*

Proof. The proof is immediate using the Lax-Milgram Theorem, because a is continuous, bilinear and coercive, and g is a linear functional on V. □

Theorem 2.2 *The solution* u_k^{m+1} *of the problem (11) strongly converges in* $V = (H_0^1(\Omega))^N$ *when* $k \to \infty$ *to the solution* u^{m+1} *of the problem (2).*

Proof. We have that $a(u_k^{m+1}, v) = g(v), \forall v \in S_k(\Omega)$. Taking $v = u_k^{m+1}$, we obtain

$$a(u_k^{m+1}, u_k^{m+1}) = \left\|u_k^{m+1}\right\|^2 = g(u_k^{m+1}) \leq \|g\|_{V^*} \cdot \left\|u_k^{m+1}\right\|,$$

so $\left\|u_k^{m+1}\right\| \leq \|g\|_{V^*}$, and then the sequence $(u_k^{m+1})_{k\in\mathbb{N}}$ is bounded in V, therefore it exists a subsequence $(u_{l_k}^{m+1})_{k\in\mathbb{N}}$ of this sequence, weakly convergent in V to an element $\tilde{u} \in V$.

First, we prove that $u^{m+1} = \tilde{u}$. For this, we pass to the limit with $k \to \infty$ in the relation $a(u_{l_k}^{m+1}, v) = g(v), \forall v \in S_k(\Omega)$, and obtain $a(\tilde{u}, v) = g(v), \forall v \in V$. But this is in fact the relation (2) (written for $\tilde{u}$ instead u^{m+1}), so $\tilde{u}$ verifies the problem (2). By the uniqueness of the solution of problem (2), it results that $u^{m+1} = \tilde{u}$. Hence, the subsequence $(u_{l_k}^{m+1})_{k\in\mathbb{N}}$ converges to the solution u^{m+1} of the problem (2).

Now, we prove the strong convergence of this solution to u^{m+1}. For this, denote by $r_k u^{m+1}$ the restriction of u^{m+1} to the subspace $S_k(\Omega)$. To simplify the notations, denote also in the following the subsequence $u_{l_k}^{m+1}$ by u_k^{m+1}.

We have that $a(u_k^{m+1} - r_k u^{m+1}, u_k^{m+1} - r_k u^{m+1}) = \left\| u_k^{m+1} - r_k u^{m+1} \right\|^2$. On the other hand,

$$a(u_k^{m+1} - r_k u^{m+1}, u_k^{m+1} - r_k u^{m+1}) = a(u_k^{m+1}, u_k^{m+1}) - \\ - a(u_k^{m+1}, r_k u^{m+1}) - a(r_k u^{m+1}, u_k^{m+1}) + a(r_k u^{m+1}, r_k u^{m+1}),$$

therefore

$$\left\| u_k^{m+1} - r_k u^{m+1} \right\|^2 = a(u_k^{m+1}, u_k^{m+1}) - \\ - a(u_k^{m+1}, r_k u^{m+1}) - a(r_k u^{m+1}, u_k^{m+1}) + a(r_k u^{m+1}, r_k u^{m+1}).$$

Passing to the limit in this relation for $k \to \infty$, we obtain:

$$\lim_{k\to\infty} \left\| u_k^{m+1} - r_k u^{m+1} \right\|^2 = a(u^{m+1}, u^{m+1}) - \\ - a(u^{m+1}, u^{m+1}) - a(u^{m+1}, u^{m+1}) + g(u^{m+1}) = \\ = -a(u^{m+1}, u^{m+1}) + g(u^{m+1}) = 0,$$

and from here it results that:

$$\lim_{k\to\infty} \left\| u_k^{m+1} - u^{m+1} \right\| = 0,$$

therefore u_k^{m+1} strongly converges in V to u^{m+1}. □

Remark 2.1 For $k \to \infty$ and $m \to \infty$, we have that u_k^{m+1} strongly converges in V to the solution u of the problem (1).

3 Numerical Results and Conclusions

We can apply this discretization for the computation of the approximate solution, considering that this is reduced to the solving of the linear algebraic system (13), and that in the coefficients of this system there are involved the scalar products of the orthonormal system $\{\varphi_n\}$ in V. On the other hand, the eigenfunctions of the Laplace's operator, φ_n, are easy to compute for some particular domains Ω, which makes more facile the effective computation of the solution.

Some numerical results were tested for the rectangular domain $\Omega = (0,1) \times (0,1)$ or the ball $\Omega = B(0,1) \subset \mathbb{R}^2$, where $f = 0$ on Ω, and we have experimentally chosen m, k, ν and ρ as in the table from bellow.

Ω	m	k	ν	ρ
$(0,1) \times (0,1)$	10	500	0.4	0.1
$B(0,1)$	20	450	0.3	0.1

The next figure shows the streamlines of the flow for the case $\Omega = B(0,1) \subset \mathbb{R}^2$.

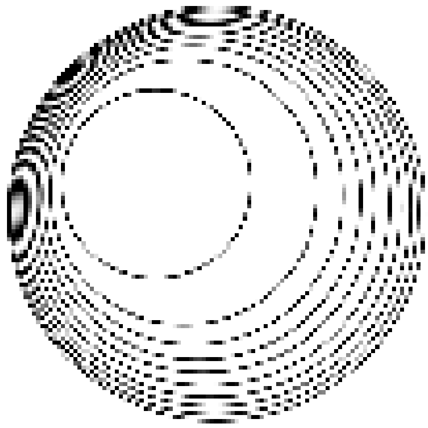

Figure 1: Streamlines of the flow for Problem (2)–(3)

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Cristina Sburlan
Faculty of Mathematics and Informatics
Ovidius University, Constantza, Romania
E-mail: c_sburlan@univ-ovidius.ro