An. Şt. Univ. Ovidius Constanţa Vol. 17(3), 2009, 111–118

MORE ON MATRIX NEAR - RINGS

Mariana Dumitru

Abstract

In this paper, we study matrix near-rings introduced by A.P.J. van der Walt and J.D.P. Meldrum in 1986 and generalized by K.S. Smith ten years later. We find some properties of their ideals and some applications.

1 Introduction

We recall first some definitions and results on matrix near-rings over a right near-ring with identity R. Proofs and other informations could be found in [3], [4], [5].

In our considerations, R will denote a right near-ring with identity $1 \neq 0$, which is 0-symmetric, while M will denote a faithfull left $R-$module, i.e. M is endowed with a group binary operation, $+$, and with a left external multiplications $\cdot : R \times M \longrightarrow M$, such that:

(i) $(a+b) \cdot x = a \cdot x + b \cdot x$, for all $a, b \in R, x \in M$;

(ii) $(a \cdot b) \cdot x = a \cdot (b \cdot x)$, for all $a, b \in R, x \in M$;

(iii) $1 \cdot x = x$, for all $x \in M$;

(iv) $Ann_R(M) = \{0\}$.

Key Words: matrix near-rings.
Mathematics Subject Classification: 16Y30, 16P10.
Received: April 2009
Accepted: October 2009

For $n \in \mathbb{N}^*$, $(M^n, +)$ can be also structured as a faithful left R-module.

K.C. Smith [5] has generalized the construction of matrix near-ring $M_n(R)$ given earlier by Meldrum and van der Walt [3].

The problem in this construction is that the old construction in the case of rings does not fit, since the addition is not commutative and the distributivity law is only on the right side.

Take $n \in \mathbb{N}^*$; we may define functions $f_{ij}^r : M^n \longrightarrow M^n$, for $r \in R$, $i, j \in \{1, 2, ..., n\}$ by $f_{ij}^r(x_1, x_2, ..., x_n) = (0, ..., 0, rx_j, 0, ..., 0)$, where rx_j is on the i-th position in the n-tuple obtained for $(x_1, ..., x_n) \in M^n$. Hence $f_{ij}^r = \sigma_i \lambda_r \pi_j$, where $\sigma_i : M \longrightarrow M^n$ is $\sigma_i(x) = (0, ..., 0, x, 0, ..., 0)$, $\pi_j : M^n \longrightarrow M$, $\pi_j(x_1, x_2, ..., x_n) = x_j$, $\lambda_r : M \longrightarrow M$, $\lambda_r(x) = rx$. $M_n(R, M)$ is the subring of $M_0(M^n)$ (the near-ring of functions preserving zero from M^n to M^n) generated by the set

$$F = \left\{ f_{ij}^r | i, j = 1, 2, ..., n,\ r \in R \right\}.$$

From [3], [5] we know the following rules of calculations on F, for all $i, j, k, l, h = 1, 2, ..., n$, and $r, s \in R$:

(*i*) $f_{ij}^r + f_{ij}^s = f_{ij}^{r+s}$;

(*ii*) $f_{ij}^r + f_{kl}^s = f_{kl}^s + f_{ij}^r$, if $i \neq k$;

(*iii*) $f_{ij}^r \cdot f_{kl}^s = \begin{cases} f_{il}^{rs}, & \text{if } j = k \\ 0, & \text{if } j \neq k \end{cases}$;

(*iv*) $f_{ij}^r \left(\sum\limits_{k=1}^{n} f_{kh}^s \right) = f_{ih}^{rs}$.

There are some properties which are transferred from R to $M_n(R, M)$:

1. If R is distributive (distributively generated), so is $M_n(R, M)$.
2. If R is Abelian, then $M_n(R)$ is Abelian.
3. $\left\{ \sum\limits_{i \in \Delta} f_{ii}^r | r \in R \right\}$, for $\Delta \neq \oslash$, $\Delta \in \{1, 2, ..., n\}$ generates a subnear-ring of $M_n(R, M)$, isomorphic to R.
4. If R is an abstract affine near-ring, then so in $M_n(R, M)$.
5. If R has identity $1 \neq 0$, then, denoting $E_{ij} = f_{ij}^1$, $1 \leq i, j \leq n$, we get the identity of $M_n(R, M)$, $I = \sum\limits_{i=1}^{n} E_{ii}$. Here, $E_{ij} \cdot E_{kl} = \begin{cases} E_{il}, & \text{if } j = k \\ 0, & \text{if } j \neq k \end{cases}$.

We notice that $M_n(R,M)$ is a 0-symmetric near-ring, while, when R is not so, then we have:

$$f_{ij}^{r} \cdot f_{kl}^{s} = f_{il}^{r \cdot 0}, \text{ when } j \neq k.$$

6. The matrices $\sum_{i=1}^{n} E_{ii} r_i$ are called **diagonal matrices**.

The papers in the Reference list pointed out some properties of $M_n(R,M)$ or $M_n(R,R)$ (denoted $M_n(R)$).

So, $r \in R$ is distributive with respect to addition in M (in R), if and only if f_{ij}^{r} is distributive in $M_n(R,M)$ (in $M_n(R)$), or, equivalently, with respect to addition in M^n (i.e. $f_{ij}^{r} \in End(M^n,+)$).

K.C. Smith [5], as well as J.D.P. Meldrum and A.P.J. van der Walt [3] have proved the following interesting proposition.

Proposition 1.1. *For any matrix $f \in M_n(R,M)$ and any i, $1 \leq i \leq n$, and for any $a,b,...,c \in R$, there exist $r,s,...,t \in R$, such that*

$$f\left(f_{1i}^{a} + f_{2i}^{b} + ... + f_{ni}^{c}\right) = f_{1i}^{x} + f_{2i}^{y} + ... + f_{ni}^{z}.$$

This shows that the multiplication in $M_n(R,M)$ has some interesting properties, and $M_n(R,M)$ is additively generated by the set F.

The proof of the Proposition 1.1. is done by induction with respect to $w(f)$, the number of f_{ij}^{r} involved in the expression of f, using the decomposition of f as $f = g + h$ or $f = gh$, where $g,h \in M_n(R,M)$ are matrices with $w(g), w(h) < w(f)$.

2 Ideals in $M_n(R)$

Let L be a left ideal of R.

This means that L is a normal subgroup of $(R,+)$ with the property:

$$r(a+s) - rs \in L, \forall r,s \in R, a \in L.$$

If, in addition, $ar \in L$, for all $r \in R$, $a \in L$, then L is an ideal of R.

Although the construction of matrices is not like in the ring case, we can establish a correspondence of ideals for R and those for $M_n(R)$.

So, we consider the subset of $M_n(R)$:

$$L^* = \{f \in M_n(R) \,|\, f(x) \in L^n, \text{ for all } x \in R^n\}.$$

Then L^* is a (two-sided) ideal of $M_n(R)$. Indeed, if $f,g \in L^*$, we have $(f-g)(x) = f(x) - g(x) \in L^n$, for all $x \in R^n$.

In a similar way, we see that $h+f-h\in L^{*}$, for all $h\in M_{n}\left(R\right)$ and $f\in L^{*}$,then if $f\in L^{*},g,h\in M_{n}\left(R\right)$, then:

$$fg\left(x\right)=f\left(g\left(x\right)\right)\in L^{n},\text{ for all }x\in R^{n}.$$

This means that L^{*} is a right ideal, no matter whether the normal subgroup L is a right ideal or not. Now, since L is a left ideal, we may prove that, for all $g,h\in M_{n}\left(R\right)$ and $f\in L^{*}$, we have $g\left(f+h\right)-gh\in L^{*}$.

This can be proved by induction on $w\left(g\right)$. Indeed, for $g=f_{ij}^{s}=\sigma_{i}\lambda_{s}\pi_{j}$, we get:

$$f_{ij}^{s}\left(f+h\right)\left(x\right)=\sigma_{i}\left(s\pi_{j}\left(f\left(x\right)+h\left(x\right)\right)\right)=\sigma_{i}\left(s\left(\left(f\left(x\right)\right)_{j}+\left(h\left(x\right)\right)_{j}\right)\right),$$

where $\left(f\left(x\right)\right)_{j}\in L$, $s,\left(h\left(x\right)\right)_{j}\in R$, and L is a left ideal in R. Then there is $t_{j}\in L$, such that

$$\begin{aligned}f_{ij}^{s}\left(f+h\right)\left(x\right) &= \sigma_{i}\left(s\left(\left(h\left(x\right)\right)_{j}+t_{j}\right)\right)=\\ &= \sigma_{i}\left(s\left(\left(h\left(x\right)\right)_{j}\right)\right)+\sigma_{i}\left(t_{j}\right)=\left(f_{ij}^{s}h\right)\left(x\right)+\sigma_{i}\left(t_{j}\right),\end{aligned}$$

and $\sigma_{i}\left(t_{j}\right)\in L^{n}$. Hence $f_{ij}^{s}\left(f+h\right)-f_{ij}^{s}h\in L^{*}$.

The next step of induction is clear, therefore L^{*} is a left ideal in $M_{n}\left(R\right)$.

We consider now the ideal generated by

$$F_{1}^{L}=\left\{f_{ij}^{a}|a\in L\right\},$$

denoting it by L^{+}.

Which is the relationship between L^{+} and L^{*}? Let $a\in L$, then, for $x\in R^{n}$,

$$f_{ij}^{a}\left(x\right)=\sigma_{i}\left(a\pi_{j}\left(x\right)\right)\in L^{n},$$

since $a\pi_{j}\left(x\right)\in L$ (L is a right ideal). We have $L^{+}\subseteq L^{*}$.

For two distinct ideals L,L_{1} of R, $L\neq L_{1}$, we have:

$$L^{*}\neq L_{1}^{*}.$$

Indeed, if $a\in L\backslash L_{1}$, $f_{11}^{a}\in L^{*}$, but $f_{11}^{a}\notin L_{1}^{*}$.

We establish now a converse correspondence.

Let T be an ideal of $M_{n}\left(R\right)$; define

$$T_{*}=\left\{x\in Im(\pi_{j}f),\text{for some}f\in T,1\leq j\leq n\right\}.$$

Clearly, $a\in T_{*}$ if and only if $f_{11}^{a}\in T$.

Indeed, for $a \in T_*$, consider $f \in T$ and j, $1 \leq j \leq n$, such that $(\pi_j f)(x) = a$, for $x \in R^n$. Hence $f(x) = \sigma_j(a) = \sigma_j(a \cdot 1) = \sigma_j(\lambda_a(1)) = \sigma_j \lambda_a \pi_j(1)$.

As a consequence of Proposition 1.1., we have that, for $f \in M_n(R)$ and $x, y, ..., z \in R$ there exists $a \in R$ such that

$$E_{11} f\left(f_{11}^x + f_{21}^y + ... + f_{n1}^z\right) = f_{11}^a.$$

Hence, for $x \in R^n$, there is $g \in M_n(R)$ such that $x = g(e_1)$, where $e_1 = (1, 0, ..., 0)$, since we may take $g = f_{11}^{x_1} + f_{21}^{x_2} + ... + f_{n1}^{x_n}$, for $x = (x_1, ..., x_n)$.

From $f(x) = g(e_1)$, we have:

$$g(e_1) = g\left(\left(f_{11}^1 + f_{21}^0 + ... + f_{n1}^0\right)(e_1)\right) = \left(f_{11}^{a_1} + f_{21}^{a_2} + ... + f_{n1}^{a_n}\right)(e_1),$$

where $a_j = a$. But then $E_{1j} g\left(f_{11}^1 + f_{21}^0 + ... + f_{n1}^0\right) = f_{11}^a \in T$.

We have proved:

Proposition 2.2. *There is a correspondence between the set of ideals in $M_n(R)$ and the set of ideals in R, namely:*

(i) $J \longrightarrow J^*$, *for* $J \in L(R)$,

(ii) $T \longrightarrow T_*$, *for* $T \in L(M_n(R))$,

such that:
1) *The first correspondence is injective.*
2) $(T_*)^* \supseteq T$.
3) $(J^*)_* = J$.
4) $\left((T_*)^*\right)_* = T_*$.

Corollary 2.3. *There is a bijection between* $L(R)$ *and* $L_1 \subseteq L(M_n(R))$, *where* $L_1 = \{T \in L(M_n(R)) \mid \exists J \in L(R), T = J^*\}$.

Proof Is obvious from the above result.

3 Application to primitivity in $M_n(R)$

As it has been seen in the previous sections, although there are similarities to the ring case, for matrix near-rings there are some striking differences.

We point out other similarities in the case of connected modules over R and $M_n(R)$ and for the 2-primitivity.

Let R be a right 0-symmetric near-ring with identity and G be a left R-module. Denote the semigroup of all R-endomorphisms of G by $End_R G$. For a group G, if S is a semigroup of all endomorphisms of G acting on the right, the right near-ring of all function $f : G \longrightarrow G$ such that $f(gs) = (f(g))_s$, for all $g \in G, s \in S$, will be called a **bicentralizer** near-ring, denoted by $BiCen_C G$. The $n \times n$ matrix near-ring $M_n(R)$ over R is the subnear-ring of $BiCen_R R^n$ generated by the functions f_{ij}^r, $r \in R, 1 \leq i, j \leq n$.

Definition 3.1. An R-module G is called **connected** if, for any $g_1, g_2 \in G$, there exists $g \in G$ and $r, s \in R$ such that $g_1 = rg$ and $g_2 = sg$.

Obviously, every monogenic module is connected, the converse being false. If G is a connected R-module, then:

(1) If $g_1, ..., g_n \in G, n \geq 2$, then there are $r_1, ..., r_k \in R$ and $g \in G$, such that $g_i = r_i g, i = 1, 2, ..., n$ (by induction on n).

(2) If d is distributive in R, then $d(g+h) = dg + dh$, for any $g, h \in G$.

Indeed, putting $g = rg', h = sg'$, we have
$d(g+h) = d(rg' + sg') = (d(r+s))g' = (d(r+s))g' = (dr + ds)g' =$
$= (dr)g' + (ds)g' = d(rg') + d(sg') = dg + dh.$

Proposition 3.2. (1) *If G is a connected R-module, then G^n is a connected $M_n(R)$-module.*
(2) If G is monogenic as an R-module, then G^n is monogenic viewed as an $M_n(R)$-module.

Proof (1) The action of $M_n(R)$ on G^n is defined as follows. If $A \in M_n(R)$ and $(g_1, ..., g_n) \in G^n$, let $r_1, ..., r_n \in R$ and $g \in G$ be such that $g_i = r_i g$, $i = 1, 2, ..., n$. Then $A(g_1, ..., g_n) = (A(r_1, ..., r_n))g$. Here $(g_1, ..., g_n) = (r_1, ..., r_n)g$ and $A(r_1, ..., r_n)$ means that the action of A as a function on R^n to R^n (we denote the vectors in R^n as rows). If $g_i = s_i h$, with $s_i \in R$ and $h \in G$, $i = 1, 2, ..., n$ then there are $k \in G$ and $r, s \in R$, such that $g = rk$ and $h = sk$, therefore $g_i = r_i rk = s_i sk$, i.e. $r_i r = s_i s + K$, where $K = Ann_R k$. We may show that $(A(r_1, ..., r_n))g = (A(s_1, ..., s_n))h$ and the action of $M_n(R)$ over G^n is well-defined. The axioms in the definition of a of module over $M_n(R)$ are easily proved, so it remains to show only the connectedness of G^n. Let $(g_1, ..., g_n), (h_1, ..., h_n) \in G^n$, and $r_i, s_i \in R, i = 1, 2, ..., n, g \in G$ such that $g_{i_r} = r_i g$ and $h_i = s_i g, i = 1, 2, ..., n$. Then we obtain

$$\begin{aligned}
(f_{11}^{r_1} + ... + f_{n1}^{r_n})(g, 0, ..., 0) &= (g_1, ..., g_n) \text{ and} \\
(f_{11}^{s_1} + ... + f_{n1}^{s_n})(g, 0, ..., 0) &= (h_1, ..., h_n).
\end{aligned}$$

(2) If G is monogenic by g, then the $M_n(R)$-module G^n is generated by $(g, 0, ..., 0)$ as it is obvious from the first part.

Proposition 3.3. *If Γ is a connected $M_n(R)$-module, then there exists an appropriate R-module G, such that $(\Gamma, +) \cong (G^n, +)$ as groups.*

Proof We take $G := f_{11}^1\Gamma = \left\{f_{11}^1\gamma|\gamma \in \Gamma\right\}$; f_{11}^1 being distributive in $M_n(R)$ and connected, G is a subgroup of $(\Gamma, +)$ and we can define an operation of R-module over G by putting $r\left(f_{11}^1\gamma\right) := f_{11}^1\left(f_{11}^r f_{11}^1\gamma\right) = f_{11}^r\gamma$. Of course, $f_{11}^1\gamma = f_{11}^1 f_{1i}^1\gamma \in G$. Defining $\varphi : \Gamma \longrightarrow G^n$ by $\varphi(\gamma) := \left(f_{11}^1\gamma, ..., f_{1n}^1\gamma\right)$ and taking into account that f_{ij}^1 are distributive elements in $M_n(R)$, we obtain a group homomorphism φ which is bijective, because for all $\alpha = \left(f_{11}^1\gamma_1, ..., f_{11}^1\gamma_n\right) \in G^n$, $\varphi\left(\sum_{i=1}^n f_{11}^1\gamma_i\right) = \alpha$ and $\ker\varphi = \{0\}$.

Proposition 3.4. *If G is a connected R-module, then*

$$End_R G \cong End_{M_n(R)} G^n$$

. In particular, $End_{M_n(R)} R^n \cong R$ as semigroups.

Proof We define the isomorphism $\varphi : End_{M_n(R)} G^n \longrightarrow End_R G$ by taking $\varphi(\sigma)$ such that $g\varphi(\sigma) = \pi_1((g, 0, ..., 0)\sigma)$. Note that, since $(g, 0, ..., 0)\sigma = f_{11}^1((g, 0, ..., 0))$, it follows that $(g, 0, ..., 0)\sigma = (g', 0, ..., 0)$, hence $g\varphi(\sigma\tau) = \pi_1((g, 0, ..., 0)\sigma\tau) = \pi_1(((g, 0, ..., 0)\sigma)\tau) = \pi_1((\pi_1(((g, 0, ..., 0)\sigma), 0, ..., 0))\tau) = (g\varphi(\sigma))\varphi(\tau)$, therefore $\varphi(\sigma) = \varphi(\sigma)\varphi(\tau)$. Assume $\sigma \neq \tau$, then there is a $\gamma \in G^n$ such that $\gamma\sigma \neq \gamma\tau$. If $\pi_i(\gamma\sigma) \neq \pi_i(\gamma\tau)$, for some i, we obtain $\varphi(\sigma) \neq \varphi(\tau)$, hence φ is injective. For $s \in End_R G$, we define $\sigma \in End_{M_n(R)} G^n$ by $(g_1, ..., g_n)\sigma = (g_1 s, ..., g_n s)$ (by induction on the weight function, we obtain that $\sigma \in End_{M_n(R)} G^n$), and $\varphi(\sigma) = s$. The last part follows from the isomorphism $End_R R \cong R$ (as additive groups).

Lemma 3.5. *Let G be a connected R-module. Then any $M_n(R)$-submodule of G^n is of the form L^n, where L is an R-submodule of G.*

Proof Putting $L_i := \{\pi_i(\gamma)|\gamma \in T\}$ where T is an $M_n(R)$-submodule of G^n, we see that $L_i = L_j$, $1 \leq i, j \leq n$. But L_1 is an R-submodule of G and $T = L_1^n$.

Some immediate corollaries are the following ones:

Corollary 3.6. *If G is a connected R-module, then G is simple (i.e. it has no trivial submodules) if and only if G^n is simple.*

Corollary 3.7. *If the monogenic R-module G is faithful then G^n is faithful. If R is ν-primitive on G, then $M_n(R)$ is ν-primitive on G^n, $\nu = 0, 2$.*

Moreover, for $\nu = 2$, we have:

Theorem 3.8. *R is 2-primitive if and only if $M_n(R)$ is 2-primitive.*

Proof Assume Γ is a 2-primitive $M_n(R)$-module and let G be the derived R-module in Proposition 3.3. Since $r \in Ann_R G$ if and only if $f_{11}^r \in Ann_{M_n(R)}\Gamma$, G is faithful. To prove that G is 2-*primitive, let* $f_{11}^1\gamma \in G$ (any nonzero element). As Γ is 2-primitive, we have, for any $f_{11}^{1}\gamma \in G$, a matrix $A \in M_n(R)$ such that $A\left(f_{11}^{1}\gamma\right) = f_{11}^{1}\gamma$. But $Af_{11}^{1} = \sum_{i=1}^{n} f_{i1}^{r_i}$, therefore $\left(f_{11}^{1}Af_{11}^{1}\right)\gamma = f_{11}^{r_1}\gamma = r_1\left(f_{11}^{1}\gamma\right) = f_{11}^{1}\delta$. Thus every nonzero element of G generates G and G is 2-primitive.

A nice application of this theorem is the following:

Theorem 3.9. *If R is 2-primitive on G and $S = End_R G$ has only finitely many orbits on G^n, R not being a ring, then $M_n(R)$ $BiCen_S G^n$.*

Proof is based on the following result (Meldrum [2], 3.15, 4.16 and 4.28): "If R is a near-ring with identity which is not a ring and R is 2-primitive on an R-module Γ such that $T := End_R\Gamma$ has only a finite number of orbits on Γ, then $R \cong BiCen_T\Gamma$". We take as R the matrix near-ring $M_n(R)$, $\Gamma = G^n$ and $T = S$.

References

[1] H.E. Heatherly, *Matrix near-rings*, J. London Math.Soc.(2), **7**(1973), 355-356.

[2] J.D.P. Meldrum, J.H. Meyer, *Intermediate ideals in matrix near-rings*, Comm. Algebra **24** (1996), 1601-1619.

[3] J.D.P. Meldrum, A.P.J. van der Walt, *Matrix near-rings*, Arch. Math., **47** (1986), 312-319.

[4] G. Pilz, *Near-rings*, North Holland, Amsterdam, 1983.

[5] K. C. Smith, *Generalized matrix near-rings*, Comm.Algebra, **24** (1996), 2065-2077.

Mariana Dumitru
Maritime University Constanta, Romania
E-mail: darianax@gmail.com