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SOME RESULTS ON AN ITERATIVE METHOD FOR NONEXPANSIVE MAPPINGS IN BANACH SPACES

Sun Young Cho, Shin Min Kang and Xiaolong Qin

Abstract

In this paper, we consider the problem of finding a common fixed point of a nonexpansive mapping and a λ-strict pseudocontraction based on a general iterative process. Strong convergence of iterative sequences generated in the purposed iterative process is obtained in a real Banach space.

1 Introduction and preliminaries

Let C be a nonempty closed and convex subset of a Banach space E and E^* the dual space of E. Let $\langle \cdot, \cdot \rangle$ denote the pairing between E and E^*. For $q > 1$, the generalized duality mapping $J_q : E \to 2^{E^*}$ is defined by

$$J_q(x) = \{f \in E^* : \langle x, f \rangle = \|x\|^q, \|f\| = \|x\|^{q-1}\}$$

for all $x \in E$. In particular, $J = J_2$ is called the normalized duality mapping. It is known that $J_q(x) = \|x\|^{q-2}J(x)$ for all $x \in E$. If E is a Hilbert space, then $J = I$, the identity mapping. Let $U = \{x \in E : \|x\| = 1\}$. E is said to be Gâteaux differentiable if the limit $\lim_{t\to 0} \frac{\|x+ty\|-\|x\|}{t}$ exists for each $x, y \in U$. In this case, E is said to be smooth. The norm of E is said to be uniformly Gâteaux differentiable if for each $y \in U$, the limit is attained uniformly for $x \in U$. The norm of E is said to be Fréchet differentiable, if

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for each $x \in U$, the limit is attained uniformly for $y \in U$. The norm of E is said to be uniformly Fréchet differentiable, if the limit is attained uniformly for $x, y \in U$. It is well-known that (uniform) Fréchet differentiability of the norm of E implies (uniform) Gâteaux differentiability of the norm of E. The modulus of smoothness of E is defined by

$$\rho(t) = \sup\{\frac{1}{2}(\|x+y\| + \|x-y\|) - 1 : x, y \in E, \|x\| = 1, \|y\| \leq t\}.$$

A Banach space E is said to be uniformly smooth if $\lim_{t\to 0} \frac{\rho(t)}{t} = 0$. Let $q > 1$. A Banach space E is said to be q-uniformly smooth, if there exists a fixed constant $c > 0$ such that $\rho(t) \leq ct^q$. It is well-known that E is uniformly smooth if and only if the norm of E is uniformly Fréchet differentiable. If E is q-uniformly smooth, then $q \leq 2$ and E is uniformly smooth, and hence the norm of E is uniformly Fréchet differentiable, in particular, the norm of E is Fréchet differentiable.

Let D be a subset of C and Q be a mapping of C into D. Then Q is said to be sunny if

$$Q(Qx + t(x - Qx)) = Qx,$$

whenever $Qx + t(x - Qx) \in C$ for $x \in C$ and $t \geq 0$. A mapping Q of C into itself is called a retraction if $Q^2 = Q$. If a mapping Q of C into itself is a retraction, then $Qz = z$ for all $z \in R(Q)$, where $R(Q)$ is the range of Q. A subset D of C is called a sunny nonexpansive retract of C if there exists a sunny nonexpansive retraction from C onto D.

The following result describes a characterization of sunny nonexpansive retractions on a smooth Banach space.

Proposition 1 *(Reich [11]). Let E be a smooth Banach space and let C be a nonempty subset of E. Let $Q : E \to C$ be a retraction and let J be the normalized duality mapping on E. Then the following are equivalent:*

(1) *Q is sunny and nonexpansive;*

(2) $\|Qx - Qy\|^2 \leq \langle x - y, J(Qx - Qy)\rangle, \quad \forall x, y \in E$;

(3) $\langle x - Qx, J(y - Qx)\rangle \leq 0, \quad \forall x \in E, y \in C.$

Let $T : C \to C$ be a nonlinear mapping. In this paper, we use $F(T)$ to denote the set of fixed points of T. Recall that T is said to be nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C. \tag{1.1}$$

T is said to be λ-strictly pseudocontractive if there exists a constant $\lambda \in (0, 1)$ such that

$$\langle Tx - Ty, j(x - y)\rangle \leq \|x - y\|^2 - \lambda\|(I - T)x - (I - T)y\|^2 \tag{1.2}$$

for every $x, y \in C$ and for some $j(x-y) \in J(x-y)$. It is clear that (1.2) is equivalent to the following

$$\langle (I-T)x-(I-T)y, j(x-y)\rangle \geq \lambda \|(I-T)x-(I-T)y\|^2$$

We remark that if T is λ-strictly pseudocontractive, then T is Lipschitz with the Lipschitz constant $L = \frac{1+\lambda}{\lambda}$.

We also remark that the class of strictly pseudocontractive mapping was first introduce by Browder and Petryshyn [2] in a real Hilbert space. Let C be a nonempty subset of a real Hilbert space H. Recall that $S : C \to C$ is said to be k-strictly pseudocontractive if there exists a constant $k \in [0, 1)$ such that

$$\|Sx - Sy\|^2 \leq \|x-y\|^2 + k\|(I-T)x-(I-T)y\|^2, \quad \forall x, y \in C. \tag{1.3}$$

It is clear that (1.3) is equivalent to the following

$$\langle Tx-Ty, x-y\rangle \leq \|x-y\|^2 - \frac{1-k}{2}\|(I-T)x-(I-T)y\|^2, \quad \forall x, y \in C. \tag{1.4}$$

For the class of nonexpansive mappings, one classical way to study nonexpansive mappings is to use contractions to approximate a nonexpansive mapping ([1],[12]). More precisely, take $t \in (0, 1)$ and define a contraction $T_t : C \to C$ by

$$T_t x = tu + (1-t)Tx, \quad \forall x \in C,$$

where $u \in C$ is a fixed point. Banach's contraction mapping principle guarantees that T_t has a unique fixed point x_t in C. That is,

$$x_t = tu + (1-t)Tx_t.$$

It is unclear, in general, what the behavior of x_t is as $t \to 0$, even if T has a fixed point. However, in the case of T having a fixed point, Browder [1] proved that if E is a Hilbert space, then x_t converges strongly to a fixed point of T. Reich [12] extended Broweder's result to the setting of Banach spaces and proved that if E is a uniformly smooth Banach space, then x_t converges strongly to a fixed point of T.

Recall that the normal Mann's iterative process was introduced by Mann [5] in 1953. The normal Mann's iterative process generates a sequence $\{x_n\}$ in the following manner:

$$x_1 \in C, \quad x_{n+1} = \alpha_n x_n + (1-\alpha_n)Tx_n, \quad \forall n \geq 1, \tag{M}$$

where the sequence $\{\alpha_n\}_{n=0}^{\infty}$ is in the interval $(0, 1)$. If T is a nonexpansive mapping with a fixed point and the control sequence $\{\alpha_n\}$ is chosen so that

$\sum_{n=0}^{\infty} \alpha_n(1-\alpha_n) = \infty$, then the sequence $\{x_n\}$ generated by the normal Mann iterative process converges weakly to a fixed point of T. In an infinite-dimensional Hilbert space, the normal Mann's iteration process has only weak convergence, in general, even for nonexpansive mappings. Therefore, many authors try to modify the normal Mann iteration process to have strong convergence for nonexpansive mappings; see, e.g., [4, 6-10, 18] and the references therein.

Recall that the Ishikawa iterative process was introduced by Ishikawa [3] in 1974. Ishikawa iterative process generates a sequence $\{x_n\}$ in the following manner:

$$\begin{cases} x_1 \in C, \\ y_n = \beta_n x_n + (1-\beta_n)Tx_n, \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n)Ty_n, \quad \forall n \geq 1, \end{cases} \tag{I}$$

where x_1 is an initial value and $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $[0,1]$. The weak convergence of the Ishikawa iterative process for nonexpansive mappings has been obtained by Tan and Xu [15], see also Schu [13].

In this paper, we introduce the following modified Ishikawa iterative process

$$\begin{cases} x_1 \in C, \\ z_n = \rho x_n + (1-\rho)Sx_n, \\ y_n = \beta_n x_n + (1-\beta_n)Tz_n, \\ x_{n+1} = \alpha_n u + \alpha_n' x_n + \alpha_n'' y_n, \quad n \geq 1. \end{cases} \tag{CKQ}$$

where the initial guess x_1 is taken in C arbitrarily, $S : C \to C$ is a λ-strict pseudocontraction, $T : C \to C$ is a nonexpansive mapping, $u \in C$ is an arbitrary (but fixed) element in C and $\{\alpha_n\}$, $\{\alpha_n'\}$, $\{\alpha_n''\}$ and $\{\beta_n\}$ are sequences in $[0,1]$ and ρ is some constant. We prove, under certain appropriate assumptions on the control sequences that $\{x_n\}$ defined by (CKQ) converges to a fixed point of T.

In order to prove our main results, we need the following lemmas.

Lemma 1 *(Reich [12]). Let E be a uniformly smooth Banach space and let $T : C \to C$ be a nonexpansive mapping with a fixed point $x_t \in C$ of the contraction $C \ni x \mapsto tu + (1-t)Tx$ converging strongly as $t \to 0$ to a fixed point of T. Define $Q : C \to F(T)$ by $Qu = s - \lim_{t \to 0} x_t$. Then Q is the unique sunny nonexpansive retract from C onto $F(T)$; that is, Q satisfies the property*

$$\langle u - Qu, J(z - Qu) \rangle \leq 0, u \in C, \quad \forall z \in F(T).$$

Lemma 2 *(Xu [17]). Let* $\{\alpha_n\}$ *be a sequence of nonnegative real numbers satisfying the property*

$$\alpha_{n+1} \leq (1-\gamma_n)\alpha_n + \gamma_n\sigma_n, \quad n \geq 0,$$

where $\{\gamma_n\} \subset (0,1)$ *and* $\{\sigma_n\}$ *such that*

(a) $\lim_{n\to\infty} \gamma_n = 0$ *and* $\sum_{n=0}^{\infty} \gamma_n = \infty$,

(b) *either* $\limsup_{n\to\infty} \sigma_n \leq 0$ *or* $\sum_{n=0}^{\infty} |\gamma_n\sigma_n| < \infty$.

Then $\{\alpha_n\}$ *converges to zero.*

Lemma 3 *(Suzuki [14]). Let* $\{x_n\}$ *and* $\{y_n\}$ *be bounded sequences in a Banach space* X *and let* $\{\beta_n\}$ *be a sequence in* $[0,1]$ *with* $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$. *Suppose that* $x_{n+1} = (1-\beta_n)y_n + \beta_n x_n$ *for all integers* $n \geq 0$ *and*

$$\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0.$$

Then $\lim_{n\to\infty} \|y_n - x_n\| = 0$.

Lemma 4 *(Xu [16]). Let* E *be a real* 2*-uniformly smooth Banach space* E *with the smooth constant* K. *Then the following inequality holds:*

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, J(x)\rangle + 2K^2\|y\|^2, \quad \forall x, y \in E.$$

Lemma 5 . *Let* C *be a nonempty subset of a real* 2*-uniformly smooth Banach space* E *with the smooth constant* K *and* $S : C \to C$ *be a* λ*-strict pseudo-contraction. For* $a \in (0,1)$, *define* $S_\alpha x = (1-a)x + aSx$. *Then, as* $a \in (0,\rho)$, *where* $\rho = \min\{1, \frac{\lambda}{K^2}\}$, S_a *is a nonexpansive mapping with* $F(S_a) = F(S)$.

Proof. For any $x, y \in C$, we see from Lemma 4 that

$$\begin{aligned}
&\|S_a x - S_a y\|^2 \\
&= \|(1-a)x + aSx - [(1-a)y + aSy]\|^2 \\
&= \|(1-y) + a[Sx - Sy - (x-y)]\|^2 \\
&\leq \|x-y\|^2 + 2a\langle Sx - Sy - (x-y), J(x-y)\rangle + 2K^2a^2\|Sx - Sy - (x-y)\|^2 \\
&= \|x-y\|^2 + 2a\langle Sx - Sy, J(x-y)\rangle - 2a\|x-y\|^2 + 2K^2a^2\|Sx - Sy - (x-y)\|^2 \\
&\leq \|x-y\|^2 + 2a[\|x-y\|^2 - \lambda\|Sx - Sy - (x-y)\|^2] - 2a\|x-y\|^2 \\
&\quad + 2K^2a^2\|Sx - Sy - (x-y)\|^2 \\
&= \|x-y\|^2 - 2a\lambda\|Sx - Sy - (x-y)\|^2 + 2K^2a^2\|Sx - Sy - (x-y)\|^2 \\
&\leq \|x-y\|^2.
\end{aligned}$$

From the assumption, we obtain that S_a is nonexpansive. It is easy to see that $F(S_a) = F(S)$.

2 Main Results

Theorem 1 *Let E be a real 2-uniformly smooth Banach space with the smooth constant K, C a nonempty closed convex subset of E, $T : C \to C$ a nonexpansive mapping and $S : C \to C$ a λ-strict pseudocontraction. Assume that $F(TS) = F(T) \cap F(S) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated in the (CKQ), where $\rho = \min\{1, \frac{\lambda}{K^2}\}$, $\{\alpha_n\}$, $\{\alpha'_n\}$, $\{\alpha'''_n\}$ and $\{\beta_n\}$ are sequences in $[0, 1]$. Assume that the following conditions are satisfied:*

(a) $\alpha_n + \alpha'_n + \alpha''_n = 1$;

(b) $\sum_{n=1}^{\infty} \alpha_n = \infty, \lim_{n\to\infty} \alpha_n = 0$;

(c) $0 < \liminf_{n\to\infty} \alpha'_n \le \limsup_{n\to\infty} \alpha'_n < 1$;

(d) $\lim_{n\to\infty} |\beta_{n+1} - \beta_n| = 0$ *and* $\beta \le b < 1$ *for some* $b \in (0, 1)$.

Then the sequence $\{x_n\}$ converges strongly to a fixed point of T.

Proof. First, we show that $\{x_n\}$ is bounded. Put $S_\rho = (1 - \rho)I + \rho S$. It follows from Lemma 5 that S_ρ is nonexpansive and $F(S_\rho) = F(S)$. We also see that the iterative process (CKQ) is equivalent to

$$\begin{cases} x_1 \in C, \\ y_n = \beta_n x_n + (1 - \beta_n) T S_\rho x_n, \\ x_{n+1} = \alpha_n u + \alpha'_n x_n + \alpha''_n y_n, \quad n \ge 1. \end{cases} \tag{2.1}$$

Fixing $p \in F(T) \cap F(S)$, we have

$$\begin{aligned} \|y_n - p\| &\le \beta_n \|x_n - p\| + (1 - \beta_n) \|T S_\rho x_n - p\| \\ &\le \beta_n \|x_n - p\| + (1 - \beta_n) \|S_\rho x_n - p\| \\ &\le \|x_n - p\|. \end{aligned}$$

It follows that

$$\begin{aligned} \|x_{n+1} - p\| &\le \alpha_n \|u - p\| + \alpha'_n \|x_n - p\| + \alpha''_n \|y_n - p\| \\ &\le \alpha_n \|u - p\| + \alpha'_n \|x_n - p\| + \alpha''_n \|x_n - p\| \\ &= \alpha_n \|u - p\| + (1 - \alpha_n) \|x_n - p\|. \end{aligned}$$

By mathematical induction, we can obtain that

$$\|x_n - p\| \le \max\{\|u - p\|, \|x_1 - p\|\}, \quad \forall n \ge 1.$$

This implies that $\{x_n\}$ is bounded, so are $\{y_n\}$ and $\{z_n\}$.

Next, we claim that

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{2.2}$$

Note that

$$\begin{aligned}\|y_{n+1} - y_n\| &\le \beta_{n+1}\|x_{n+1} - x_n\| + |\beta_{n+1} - \beta_n|\|TS_\rho x_n - x_n\| \\ &\quad + (1-\beta_{n+1})\|TS_\rho x_{n+1} - TS_\rho x_n\| \\ &\le \beta_{n+1}\|x_{n+1} - x_n\| + |\beta_{n+1} - \beta_n|\|TS_\rho x_n - x_n\| \\ &\quad + (1-\beta_{n+1})\|S_\rho x_{n+1} - S_\rho x_n\| \\ &\le \|x_{n+1} - x_n\| + |\beta_{n+1} - \beta_n|M,\end{aligned} \tag{2.3}$$

where M is an appropriate constant such that $M = \sup_{n\ge1}\{\|TS_\rho x_n - x_n\|\}$. Put $l_n = \frac{x_{n+1}-\alpha'_n x_n}{1-\alpha'_n}$ for each $n \ge 1$, that is,

$$x_{n+1} = (1-\alpha'_n)l_n + \alpha'_n x_n, \quad \forall n \ge 1. \tag{2.4}$$

Now, we estimate $\|l_{n-1} - l_n\|$. Note that

$$\begin{aligned}&l_{n+1} - l_n \\ &= \frac{\alpha_{n+1}u + \alpha''_{n+1}y_{n+1}}{1-\alpha'_{n+1}} - \frac{\alpha_n u + \alpha''_n y_n}{1-\alpha'_n} \\ &= (\frac{\alpha_{n+1}}{1-\alpha'_{n+1}} - \frac{\alpha_n}{1-\alpha'_n})u + \frac{\alpha''_{n+1}}{1-\alpha'_{n+1}}(y_{n+1} - y_n) + (\frac{\alpha''_{n+1}}{1-\alpha'_{n+1}} - \frac{\alpha''_n}{1-\alpha'_n})y_n \\ &= (\frac{\alpha_{n+1}}{1-\alpha'_{n+1}} - \frac{\alpha_n}{1-\alpha'_n})(u - y_n) + \frac{\alpha''_{n+1}}{1-\alpha'_{n+1}}(y_{n+1} - y_n).\end{aligned}$$

It follows that

$$\|l_{n-1} - l_n\| \le |\frac{\alpha_{n-1}}{1-\alpha'_{n-1}} - \frac{\alpha_n}{1-\alpha'_n}|\|u - y_n\| + \|y_{n-1} - y_n\|. \tag{2.5}$$

Substitute (2.3) into (2.5); it yields that

$$\|l_{n-1} - l_n\| \le |\frac{\alpha_{n-1}}{1-\alpha'_{n-1}} - \frac{\alpha_n}{1-\alpha'_n}|\|u - y_n\| + \|x_{n+1} - x_n\| + M|\beta_{n+1} - \beta_n|.$$

This implies that

$$\|l_{n-1} - l_n\| - \|x_n - x_{n-1}\| \le |\frac{\alpha_{n-1}}{1-\alpha'_{n-1}} - \frac{\alpha_n}{1-\alpha'_n}|\|u - y_n\| + M|\beta_{n+1} - \beta_n|.$$

In view of the conditions (b), (c) and (d), we see that

$$\limsup_{n\to\infty} \big(\|l_{n+1} - l_n\| - \|x_{n+1} - x_n\|\big) \le 0.$$

We obtain from Lemma 3 that $\lim_{n\to\infty}\|l_n - x_n\| = 0$. From (2.4), we see that

$$x_{n+1} - x_n = (1-\alpha_n')(l_n - x_n).$$

It follows that (2.2) holds.

On the other hand, we have

$$\|y_n - x_n\| \le \frac{\|x_{n+1}-x_n\|}{\alpha_n''} + \frac{\alpha_n}{\alpha_n''}\|x_n - u\|.$$

In view of the condition (c) and (2.2), we obtain that

$$\lim_{n\to\infty}\|y_n - x_n\| = 0. \tag{2.6}$$

Note that

$$y_n - x_n = (1-\beta_n)(TS_\rho x_n - x_n)$$

Since $\beta_n \le b < 1$ for all $n \ge 1$, we obtain from (2.6) that

$$\lim_{n\to\infty}\|TS_\rho x_n - x_n\| = 0. \tag{2.6}$$

Next, we claim that

$$\limsup_{n\to\infty}\langle u - Q(u), j(x_n - Q(u))\rangle \le 0, \tag{2.7}$$

where $Qu = \lim_{t\to 0} z_t$ and z_t solves the fixed point equation

$$z_t = tu + (1-t)TS_\rho z_t, \quad \forall t \in (0,1),$$

from which it follows that

$$\|z_t - x_n\| = \|(1-t)(TS_\rho z_t - x_n) + t(u - x_n)\|.$$

For any $t \in (0,1)$, we see that

$$\begin{aligned}
\|z_t - x_n\|^2 &= (1-t)\langle TS_\rho z_t - x_n, j(z_t - x_n)\rangle + t\langle u - x_n, j(z_t - x_n)\rangle \\
&= (1-t)\big(\langle TS_\rho z_t - TS_\rho x_n, j(z_t - x_n)\rangle + \langle TS_\rho x_n - x_n, j(z_t - x_n)\rangle\big) \\
&\quad + t\langle u - z_t, j(z_t - x_n)\rangle + t\langle z_t - x_n, j(z_t - x_n)\rangle \\
&\le (1-t)\big(\|z_t - x_n\|^2 + \|TS_\rho x_n - x_n\|\|z_t - x_n\|\big) \\
&\quad + t\langle u - z_t, j(z_t - x_n)\rangle + t\|z_t - x_n\|^2 \\
&\le \|z_t - x_n\|^2 + \|TS_\rho x_n - x_n\|\|z_t - x_n\| + t\langle u - z_t, j(z_t - x_n)\rangle.
\end{aligned}$$

It follows that

$$\langle z_t - u, j(z_t - x_n)\rangle \le \frac{1}{t}\|TS_\rho x_n - x_n\|\|z_t - x_n\| \quad \forall t \in (0,1).$$

In view of (2.6), we see that

$$\limsup_{n\to\infty}\langle z_t - u, j(z_t - x_n)\rangle \le 0. \tag{2.8}$$

Since $z_t \to Q(u)$ as $t \to 0$ and the fact that j is strong to weak* uniformly continuous on bounded subsets of E, we see that

$$\begin{aligned}
&|\langle u - Q(u), j(x_n - Q(u))\rangle - \langle z_t - u, j(z_t - x_n)\rangle| \\
&\le |\langle u - Q(u), j(x_n - Q(u))\rangle - \langle u - Q(u), j(x_n - z_t)\rangle| \\
&\quad + |\langle u - Q(u), j(x_n - z_t)\rangle - \langle z_t - u, j(z_t - x_n)\rangle| \\
&\le |\langle u - Q(u), j(x_n - Q(u)) - j(x_n - z_t)\rangle| + |\langle z_t - Q(u), J(x_n - z_t)\rangle| \\
&\le \|u - Q(u)\|\|j(x_n - Q(u)) - j(x_n - z_t)\| + \|z_t - Q(u)\|\|x_n - z_t\| \to 0, \quad \text{as } t \to 0.
\end{aligned}$$

Hence, for any $\epsilon > 0$, there exists $\delta > 0$ such that for $\forall t \in (0, \delta)$ the following inequality holds

$$\langle u - Q(u), j(x_n - Q(u))\rangle \le \langle z_t - u, j(z_t - x_n)\rangle + \epsilon.$$

This implies that

$$\limsup_{n\to\infty}\langle u - Q(u), j(x_n - Q(u))\rangle \le \limsup_{n\to\infty}\langle z_t - u, j(z_t - x_n)\rangle + \epsilon.$$

Since ϵ is arbitrary and (2.8), we see that

$$\limsup_{n\to\infty}\langle u - Q(u), j(x_n - Q(u))\rangle \le 0.$$

This proves that (2.7) holds.

Finally, we show that $x_n \to Q(u)$ as $n \to \infty$. Note that

$$\begin{aligned}
\|y_n - Q(u)\| &= \|\beta_n(x_n - Q(u)) + (1 - \beta_n)(TS_\rho x_n - Q(u)) \\
&\le \beta_n\|x_n - Q(u)\| + (1 - \beta_n)\|S_\rho x_n - Q(u)\| \\
&\le \|x_n - Q(u)\|.
\end{aligned}$$

It follows that

$$\begin{aligned}
&\|x_{n+1} - Q(u)\|^2 \\
&= \alpha_n\langle u - Q(u), j(x_{n+1} - Q(u))\rangle + \alpha_n'\langle x_n - Q(u), j(x_{n+1} - Q(u))\rangle \\
&\quad + \alpha_n''\langle y_n - Q(u), j(x_{n+1} - Q(u))\rangle \\
&\le \alpha_n\langle u - Q(u), j(x_{n+1} - Q(u))\rangle + \alpha_n'\|x_n - Q(u)\|\|x_{n+1} - Q(u)\| \\
&\quad + \alpha_n''\|y_n - Q(u)\|\|x_{n+1} - Q(u)\| \\
&\le \alpha_n\langle u - Q(u), j(x_{n+1} - Q(u))\rangle + (1 - \alpha_n)\|x_n - Q(u)\|\|x_{n+1} - Q(u)\| \\
&\le \alpha_n\langle u - Q(u), j(x_{n+1} - Q(u))\rangle + \frac{1 - \alpha_n}{2}(\|x_n - Q(u)\|^2 + \|x_{n+1} - Q(u)\|^2).
\end{aligned}$$

This shows that

$$\|x_{n+1} - Q(u)\|^2 \leq (1 - \alpha_n)\|x_n - Q(u)\|^2 + 2\alpha_n \langle u - Q(u), j(x_{n+1} - Q(u))\rangle.$$

In view of Lemma 2, we can obtain the desired conclusion easily.

As some corollaries of Theorem 1, we have the following results immediately.

Putting $T = I$, the identity mapping, we have the following result.

Corollary 1 *Let E be a real 2-uniformly smooth Banach space with the smooth constant K, C a nonempty closed convex subset of E and $S : C \to C$ a λ-strict pseudocontraction with a fixed point. Let $\{x_n\}$ be a sequence generated in the the following manner*

$$\begin{cases} x_1 \in C, \\ y_n = \beta_n x_n + (1 - \beta_n)(\rho x_n + (1 - \rho)Sx_n), \\ x_{n+1} = \alpha_n u + \alpha_n' x_n + \alpha_n'' y_n, \quad n \geq 1. \end{cases}$$

where $\rho = \min\{1, \frac{\lambda}{K^2}\}$, $\{\alpha_n\}$, $\{\alpha_n'\}$, $\{\alpha_n'''\}$ and $\{\beta_n\}$ are sequences in $[0, 1]$. Assume that the following satisfied are satisfied.

(a) $\alpha_n + \alpha_n' + \alpha_n'' = 1$;

(b) $\sum_{n=1}^{\infty} \alpha_n = \infty, \lim_{n\to\infty} \alpha_n = 0$;

(c) $0 < \liminf_{n\to\infty} \alpha_n' \leq \limsup_{n\to\infty} \alpha_n' < 1$;

(d) $\lim_{n\to\infty} |\beta_{n+1} - \beta_n| = 0$ *and* $\beta \leq b < 1$ *for some* $b \in (0, 1)$.

Then the sequence $\{x_n\}$ converges strongly to a fixed point of S.

Putting $\beta_n = 0$ for each $n \geq 0$, we have the following result.

Corollary 2 *Let E be a real 2-uniformly smooth Banach space with the smooth constant K, C a nonempty closed convex subset of E, $T : C \to C$ a nonexpansive mapping and $S : C \to C$ a λ-strict pseudocontraction. Assume that $F(TS) = F(T) \cap F(S) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated in the the following manner:*

$$\begin{cases} x_1 \in C, \\ y_n = \rho x_n + (1 - \rho)Sx_n, \\ x_{n+1} = \alpha_n u + \alpha_n' x_n + \alpha_n'' T y_n, \quad n \geq 1. \end{cases}$$

where $\rho = \min\{1, \frac{\lambda}{K^2}\}$, $\{\alpha_n\}$, $\{\alpha_n'\}$ and $\{\alpha_n'''\}$ are sequences in $[0, 1]$. Assume that the following conditions are satisfied:

(a) $\alpha_n + \alpha'_n + \alpha''_n = 1$;

(b) $\sum_{n=1}^{\infty} \alpha_n = \infty, \lim_{n\to\infty} \alpha_n = 0$;

(c) $0 < \liminf_{n\to\infty} \alpha'_n \le \limsup_{n\to\infty} \alpha'_n < 1$.

Then the sequence $\{x_n\}$ *converges strongly to a common fixed point of* T *and* S.

Putting $S = I$, the identity mapping, we have the following result.

Corollary 3 *Let* E *be a real* 2*-uniformly smooth Banach space with the smooth constant* K, C *a nonempty closed convex subset of* E *and* $T : C \to C$ *a non-expansive mapping with a fixed point. Let* $\{x_n\}$ *be a sequence generated in the following manner*

$$\begin{cases} x_1 \in C, \\ y_n = \beta_n x_n + (1-\beta_n)Tx_n, \\ x_{n+1} = \alpha_n u + \alpha'_n x_n + \alpha''_n y_n, \quad n \ge 1, \end{cases}$$

where $\{\alpha_n\}$, $\{\alpha'_n\}$, $\{\alpha'''_n\}$ *and* $\{\beta_n\}$ *are sequences in* $[0,1]$. *Assume that the following conditions are satisfied:*

(a) $\alpha_n + \alpha'_n + \alpha''_n = 1$;

(b) $\sum_{n=1}^{\infty} \alpha_n = \infty, \lim_{n\to\infty} \alpha_n = 0$;

(c) $0 < \liminf_{n\to\infty} \alpha'_n \le \limsup_{n\to\infty} \alpha'_n < 1$;

(d) $\lim_{n\to\infty} |\beta_{n+1} - \beta_n| = 0$ *and* $\beta \le b < 1$ *for some* $b \in (0,1)$.

Then the sequence $\{x_n\}$ *converges strongly to a fixed point of* T.

References

[1] F.E. Browder, *Fixed point theorems for noncompact mappings in Hilbert spaces*, Proc. Natl. Acad. Sci. USA, **53** (1965), 1272-1276.

[2] F.E. Browder, W.V. Petryshyn, *Construction of fixed points of nonlinear mappings in Hilbert space*, J. Math. Anal. Appl., **20** (1967), 197-228.

[3] S. Ishikawa, *Fixed points by a new iteration method*, Proc. Am. Math. Soc., **44** (1974), 147-150.

[4] T.H. Kim, H.K. Xu, *Strong convergence of modified Mann iterations*, Nonlinear Anal., **61** (2005), 51-60.

[5] W.R. Mann, *Mean value methods in iteration*, Proc. Amer. Math. Soc., **4** (1953), 506-510.

[6] G. Marino, V. Colao, X. Qin, S.M. Kang, *Strong convergence of the modified Mann iterative method for strict pseudo-contractions*, Comput. Math. Appl., **57** (2009), 455-465.

[7] K. Nakajo, W. Takahashi, *Strong convergence theorems for nonexpansive mappings and nonexpansive semigroups*, J. Math. Anal. Appl., **279** (2003), 372-379.

[8] X. Qin, Y. Su, M. Shang, *Strong convergence of the composite Halpern iteration*, J. Math. Anal. Appl., **339** (2008), 996-1002.

[9] X. Qin, Y. Su, *Approximation of a zero point of accretive operator in Banach spaces*, J. Math. Anal Appl., **329** (2007), 415-424.

[10] X. Qin, Y. Su, *Strong convergence theorems for relatively nonexpansive mappings in a Banach space*, Nonlinear Anal., **67** (2007), 1958-1965.

[11] S. Reich, *Asymptotic behavior of contractions in Banach spaces*, J. Math. Anal. Appl., **44** (1973), 57-70.

[12] S. Reich, *Strong convergence theorems for resolvents of accretive operators in Banach spaces*, J. Math. Anal. Appl., **75** (1980), 287-292.

[13] J. Schu, *Iterative construction of fixed points of asymptotically nonexpansive mappings*, J. Math. Anal. Appl., **158** (1991), 407-413.

[14] T. Suzuki, *Strong convergence of Krasnoselskii and Mann's type sequences for one-parameter nonexpansive semigroups without Bochne integrals*, J. Math. Anal. Appl., **305** (2005), 227-239.

[15] K.K. Tan, H.K. Xu, *Approximating fixed points of nonexpansive mappings by the Ishikawa iterative process*, J. Math. Anal. Appl., **178** (1993), 301-308.

[16] H.K. Xu, *Inequalities in Banach spaces with applications*, Nonlinear Anal., **16** (1991), 1127-1138.

[17] H.K. Xu, *Iterative algorithms for nonlinear operators*, J. London Math. Soc., **66** (2002), 240-256.

[18] H. Zhou, *Convergence theorems for λ-strict pseudocontraction in 2-uniformly smooth Banach spaces*, Nonlinear Anal., **69** (2008), 3160-3173.

(Sun Young Cho) Department of Mathematics,
Gyeongsang National University,
Jinju 660-701, Korea
E-mail: ooly61@yahoo.co.kr

(Shin Min Kang) Department of Mathematics and the RINS,
Gyeongsang National University,
Jinju 660-701, Korea

(Xiaolong Qin) Department of Mathematics,
Gyeongsang National University,
Jinju 660-701, Korea
E-mail: qxlxajh@163.com