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DYNAMICAL PROPERTIES FOR A RELAXATION SCHEME APPLIED TO A WEAKLY DAMPED NON LOCAL NONLINEAR SCHRÖDINGER EQUATION

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Abstract

We apply a semi-discrete in time relaxation scheme to a weakly damped forced nonlinear Schrödinger system. This provides us with a discrete infinite-dimensional dynamical system. We prove the existence of a global attractor for this dynamical system.

1 Introduction

The Davey-Stewartson systems (DS) are asymptotical models for water waves (see [8], [9], [11]). Loosely speaking, the Davey-Stewartson systems are Schrödinger equations with a non-local nonlinear term. Here we are concerned with a simplified 1-D model of a weakly damped forced Davey-Stewartson equation, which gives an infinite-dimensional dynamical system, in the framework described in [21], [14], [19], [18]. This equation reads as follows

$$iu_t + i\gamma u + u_{xx} = buE(|u|^2) + f(x), \tag{1}$$

where $\gamma > 0, b \in \mathbb{R}$ are parameters, where f the forcing term does not depend on t and belongs to $L^2(\mathbb{R})$, and where $E = E^*$ is a self-adjoint bounded operator in $L^2(\mathbb{R})$ that satisfies the following properties

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- E is a bounded linear mapping from $L^p(\mathbb{R})$ into $L^p(\mathbb{R})$ for any p such that $1 < p < +\infty$.
- For a function ρ in the Schwartz class which takes values in $\mathbb{R}$, then $E(\rho)$ takes values in $\mathbb{R}$.
- E commutes with differential operators, for ρ in the Schwartz class $\mathcal{S}(\mathbb{R})$, $\partial_x E(\rho) = E(\rho_x)$.

E is very similar to the Hilbert transform except for the second assumption; for instance we may consider $E = Id$ (nonlinear cubic Schrödinger equation) or $E(\rho) = \mathcal{F}^{-1}(\frac{|\xi|}{\sqrt{1+\xi^2}}\hat{\rho}(\xi))$, where $\mathcal{F}$ is the Fourier transform.

In [15], using similar methods as in [3], we have proved that the dynamical system provided by (1) and with initial condition in $H^1(\mathbb{R})$ has a global attractor that is a compact subset of $H^2(\mathbb{R})$. In this article, we will consider a time discretization of this PDE and we will study the dynamical properties of the discrete infinite dimensional dynamical system provided by this scheme.

Since the simplified model (1) can be viewed as a generalization of the cubic nonlinear Schrödinger equation (NLS), let us give an overview of numerical studies for NLS equations. For the conservative case i.e where $\gamma = 0$, and $f = 0$, there are many numerical schemes for this equation as the Crank-Nicolson scheme (see [10], [20]), the Runge-Kutta scheme (see [2], [1], [16]), and the splitting scheme (see [22], [6]). For the dissipative case, we refer to [13] where the authors use the Crank-Nicolson scheme to discretize in time the weakly damped forced NLS equation, and then prove the existence of a global attractor for this equation in 1-D, (for 2-D we refer to [12]). Another numerical method, splitting methods, have been adressed in [5] for solving damped nonlinear Schrödinger equation. The semi discrete in time relaxation scheme (for DS and NLS equations) was introduced by C. Besse (see [7]). The C. Besse relaxation scheme consists in considering the DS equation in 2-D, in the conservative case, as a system of two equations

$$\begin{aligned} \varphi &= E(|u|^2), \\ iu_t + \Delta u &= bu\varphi. \end{aligned} \tag{2}$$

Introduce a time step $\tau = \delta t$. Let now consider an order 2 time scheme for this system considering $u^n \sim u(n\tau)$ and $\varphi^{n+\frac{1}{2}} \sim \varphi((n+\frac{1}{2})\tau)$ (hence the two equations are computed in staggered grids). Then the scheme reads

$$\begin{aligned} i\frac{u^{n+1} - u^n}{\tau} + \Delta\frac{u^{n+1} + u^n}{2} &= b\frac{u^{n+1} + u^n}{2}\varphi^{n+\frac{1}{2}}, \\ E(|u^n|^2) &= \frac{\varphi^{n+\frac{1}{2}} + \varphi^{n-\frac{1}{2}}}{2}. \end{aligned} \tag{3}$$

In this article, we are interested into a relaxation scheme in the dissipative case. Let us specify the relaxation scheme under consideration. We write (1) as

$$\begin{aligned} i(e^{\gamma t}u)_t + e^{\gamma t}u_{xx} &= be^{\gamma t}u\varphi + e^{\gamma t}f(x), \\ e^{2\gamma t}\varphi &= e^{2\gamma t}E(|u|^2). \end{aligned} \tag{4}$$

Set $\tau = \delta t$ for the time step, for $u^0 = u_0 \in H^1(\mathbb{R})$, and $\delta = e^{-\gamma\tau}$. Let us pretend that $e^{\gamma t}u|_{t=(n+\frac{1}{2})\tau} \sim e^{\gamma(n+1)\tau}\frac{u^{n+1}+\delta u^n}{2}$ and that $e^{2\gamma t}\varphi|_{t=n\tau} = e^{2\gamma n\tau}\frac{\delta^{-1}\varphi^{n+\frac{1}{2}}+\delta\varphi^{n-\frac{1}{2}}}{2}$. Then the dissipative relaxation scheme reads as

$$i\frac{u^{n+1}-\delta u^n}{\tau} + \Delta\frac{u^{n+1}+\delta u^n}{2} = b\frac{u^{n+1}+\delta u^n}{2}\varphi^{n+\frac{1}{2}} + f, \tag{5}$$

$$E(|u^n|^2) = \frac{\delta^{-1}\varphi^{n+\frac{1}{2}} + \delta\varphi^{n-\frac{1}{2}}}{2}, \tag{6}$$

with $u^0 = u_0$ and $\delta\varphi^{-\frac{1}{2}} = \delta^{-1}\varphi^{+\frac{1}{2}} = E(|u^0|^2)$. This system is order 2 in time.

We now state our main results for the dissipative properties of the scheme. To begin with, we prove that for any τ, the scheme provides a discrete semi-group

$$\begin{aligned} S_\tau : H^1(\mathbb{R}) &\to H^1(\mathbb{R}) \\ u^n &\mapsto u^{n+1}, \end{aligned}$$

that states as follows

Theorem 1 *The operator S_τ is a bounded one-to-one operator in $H^1(\mathbb{R})$.*

Our second result reads as follows

Theorem 2 *Assume without loss of generality that $\gamma\tau$ is small enough. The discrete semi-group S_τ possesses an absorbing set β that captures any trajectory and that is positively invariant by S_τ, i.e for any bounded set B there exists a time $n(B)$ such that for $n \geq n(B)$, $S_\tau^n B \subset \beta$, and $S_\tau^k \beta \subset \beta$ for $k \geq 0$.*

Our third result is concerned with the existence and the regularity of the global attractor.

Theorem 3 *The discrete semi-group S_τ possesses a compact global attractor in $H^1(\mathbb{R})$, that is actually compact in $H^2(\mathbb{R})$.*

This article is organized as follows: in a second section we prove the well-posedeness of the scheme. The third section is devoted to prove Theorem 2. In a fourth section we focus on the proof of Theorem 3.

Throughout this article, we use c to denote a positive numerical constant, and use $k, k_i;\ i = 1, 2, ...$ to denote some positive constants which depend on γ, $|f|_{L^2}$. The constants $c, k, k_i;\ i = 1, 2, ...$ can take different values at different lines. Let us recall that the scalar product of two functions u and v is defined as $\mathrm{Re}\int_{\mathbb{R}} u(x)\overline{v}(x)dx$.

2 Well-posedness of the scheme

In this section, we prove that the dissipative relaxation scheme is well-posed. For that purpose we first prove that the map $S_\tau : u^n \mapsto u^{n+1}$ is one-to-one in $H^1(\mathbb{R})$, and then prove that S_τ is a continuous mapping in $H^1(\mathbb{R})$.

Proposition 1 *For a given $u_0 \in H^1(\mathbb{R})$, the sequence u^{n+1} defined recursively by (5)-(6) is well defined.*

Proof. Assume that u^0 belongs to $H^1(\mathbb{R})$. Then $\varphi^{-\frac{1}{2}}$ and $\varphi^{\frac{1}{2}}$ belongs to $H^1(\mathbb{R})$. We prove by induction on n that $(\varphi^{n-\frac{1}{2}}, u^n) \mapsto (\varphi^{n+\frac{1}{2}}, u^{n+1})$ is one-to-one. Going back to the (6), we see that $\varphi^{n+\frac{1}{2}}$ belongs to $H^1(\mathbb{R})$, since E maps $H^1(\mathbb{R})$ into $H^1(\mathbb{R})$ and since $H^1(\mathbb{R})$ is an algebra. Solving recursively the scheme amounts to solve the linear equation

$$(Id - i\frac{\tau}{2}\Delta + i\frac{\tau}{2}b\varphi^{n+\frac{1}{2}})u^{n+1} = \delta(Id + i\frac{\tau}{2}\Delta - i\frac{\tau}{2}b\varphi^{n+\frac{1}{2}})u^n - i\tau f. \qquad (7)$$

Since $\varphi^{n+\frac{1}{2}}$ belongs to $L^\infty(\mathbb{R})$, then the unbounded self adjoint operator $\frac{\tau}{2}\Delta - \frac{\tau}{2}b\phi^{n+\frac{1}{2}}$ on $L^2(\mathbb{R})$ has is spectrum in $(-\infty, C]$ and then $(Id - i\frac{\tau}{2}\Delta + i\frac{\tau}{2}b\varphi^{n+\frac{1}{2}})$ is invertible in $L^2(\mathbb{R})$. Then (7) reads also

$$u^{n+1} = \delta U^{n+\frac{1}{2}}u^n - i\tau\frac{\delta+1}{2}(Id - i\frac{\tau}{2}\Delta + i\frac{\tau}{2}b\varphi^{n+\frac{1}{2}})^{-1}f, \qquad (8)$$

where $U^{n+\frac{1}{2}} = (Id - i\frac{\tau}{2}\Delta + i\frac{\tau}{2}b\varphi^{n+\frac{1}{2}})^{-1}(Id + i\frac{\tau}{2}\Delta - i\frac{\tau}{2}b\varphi^{n+\frac{1}{2}})$ is a unitary operator on $L^2(\mathbb{R})$; let us check this point. Set $v = U^{n+\frac{1}{2}}u$. Then

$$i\frac{v-u}{\tau} + \Delta\frac{v+u}{2} = b\phi^{n+\frac{1}{2}}\frac{v+u}{2}. \qquad (9)$$

Consider the scalar product of this equation with $i(u+v)$. Then straightforwardly $|v|_{L^2(\mathbb{R})} = |u|_{L^2(\mathbb{R})}$. On the other hand, consider the scalar product of this equation with $v - u$. Then

$$0 = |v_x|^2_{L^2(\mathbb{R})} - |u_x|^2_{L^2(\mathbb{R})} + b\int_{\mathbb{R}} \phi^{n+\frac{1}{2}}(x)(|v(x)|^2 - |u(x)|^2)dx. \quad (10)$$

Then v belongs to $H^1(\mathbb{R})$ if u does and the proof of the proposition is completed.

We now state

Proposition 2 *For any fixed τ $S_\tau : u^n \mapsto u^{n+1}$ is a continuous mapping in $H^1(\mathbb{R})$.*

Proof. We prove by induction on n that if a sequence u^0_ε converges to u^0 in $H^1(\mathbb{R})$, then u^n_ε converges to u^n in $H^1(\mathbb{R})$. For the sake of conciseness, we just prove the first step, the induction step being very similar. Since $\varphi_\varepsilon^{-\frac{1}{2}}$ converges to $\varphi^{-\frac{1}{2}}$ in $H^1(\mathbb{R})$, then going back to (6) $\varphi_\varepsilon^{\frac{1}{2}}$ converges to $\varphi^{\frac{1}{2}}$ in $H^1(\mathbb{R})$. Setting $w^n_\varepsilon = u^n_\varepsilon - u^n$, we then have

$$\frac{i}{\tau}(w^1_\varepsilon - \delta w^0_\varepsilon) + \Delta\frac{w^1_\varepsilon + \delta w^0_\varepsilon}{2} = b(\varphi_\varepsilon^{\frac{1}{2}} - \varphi^{\frac{1}{2}})\frac{u^1 + \delta u^0}{2} + b\varphi^{\frac{1}{2}}\frac{w^1_\varepsilon + \delta w^0_\varepsilon}{2}. \quad (11)$$

Consider first the scalar product of this equation with $i(w^1_\varepsilon + \delta w^0_\varepsilon)$. Then

$$|w^1_\varepsilon|^2_{L^2(\mathbb{R})} = \delta^2|w^0_\varepsilon|^2_{L^2(\mathbb{R})} + \tau\mathrm{Im}b\int_{\mathbb{R}}(\varphi_\varepsilon^{\frac{1}{2}} - \varphi^{\frac{1}{2}})\frac{u^1 + \delta u^0}{2}\overline{w^1_\varepsilon + \delta w^0_\varepsilon}dx.$$

Then $|w^1_\varepsilon|^2_{L^2(\mathbb{R})}$ converges to 0 in $L^2(\mathbb{R})$ since $|w^0_\varepsilon|^2_{L^2(\mathbb{R})}$ does and since $\varphi_\varepsilon^{\frac{1}{2}} - \varphi^{\frac{1}{2}}$ converges uniformly to 0. For the $H^1(\mathbb{R})$ convergence, consider the scalar product of (11) with $w^1_\varepsilon - \delta w^0_\varepsilon$ and proceed similarly, using the $L^2(\mathbb{R})$ convergence.

3 Proof of Theorem 2

In this section we want to prove the existence of an absorbing set in $H^1(\mathbb{R})$, since the existence of such a set is a consequence of the dissipative nature of the equation. To begin with, we prove the following lemma which gives the existence of an absorbing set in $L^2(\mathbb{R})$.

Lemma 1 *The system (5)-(6) admits an absorbing set in $L^2(\mathbb{R})$, namely the ball of radius $2\frac{|f|_{L^2(\mathbb{R})}}{\gamma}$, which is positively invariant by S_τ.*

Proof. Multiply (5) by $\overline{u^{n+1}+\delta u^n}$, and integrate the imaginary part of the resulting equation to obtain

$$|u^{n+1}|_{L^2}^2 - \delta^2|u^n|_{L^2}^2 = \tau Im \int f\overline{u^{n+1}+\delta u^n} \leq \tau |f|_{L^2(\mathbb{R})}|u^{n+1}+\delta u^n|_{L^2(\mathbb{R})}.$$

Then

$$|u^{n+1}|_{L^2} - \delta|u^n|_{L^2} \leq \tau|f|_{L^2(\mathbb{R})}. \tag{12}$$

Assuming now without loss of generality that $\gamma\tau$ is mall enough to ensure that $\gamma\tau \leq 2(1-e^{-\gamma\tau})$, we then have

$$|u^{n+1}|_{L^2} \leq \delta|u^n|_{L^2} + 2(1-\delta)\frac{|f|_{L^2(\mathbb{R})}}{\gamma}. \tag{13}$$

We conclude by the Gronwall lemma.

Now, since we want to get the $H^1(\mathbb{R})$ estimate, we plan to bound $\phi^{n+\frac{1}{2}}$ in $H^{-1}(\mathbb{R})$. Since we are interested in the long time behavior of solutions, we may assume without loss of generality that u^n belongs to the $L^2(\mathbb{R})$ absorbing ball for any $n \geq 0$. So we state

Lemma 2 *For u^n in the absorbing set of $L^2(\mathbb{R})$, there exists a constant k depending on γ, $|f|_{L^2}$ such that*

$$|\varphi^{n+\frac{1}{2}}|_{H^{-1}(\mathbb{R})} \leq \frac{k}{\tau}. \tag{14}$$

Proof. Going back to (6) we have that

$$\delta^{-1}|\varphi^{n+\frac{1}{2}}|_{H^{-1}(\mathbb{R})} \leq \delta|\varphi^{n-\frac{1}{2}}|_{H^{-1}(\mathbb{R})} + 2|E(|u^n|^2)|_{H^{-1}(\mathbb{R})}. \tag{15}$$

Since E is a bounded map in $H^1(\mathbb{R})$ and is self-adjoint, it also extends to a bounded linear operator in $H^{-1}(\mathbb{R})$. Then, using the embedding $L^1(\mathbb{R}) \hookrightarrow H^{-1}(\mathbb{R})$ we thus obtain

$$\begin{aligned} |\varphi^{n+\frac{1}{2}}|_{H^{-1}(\mathbb{R})} \leq \delta^2|\varphi^{n-\frac{1}{2}}|_{H^{-1}(\mathbb{R})} + C\delta||u^n|^2|_{L^1(\mathbb{R})} = \\ \delta^2|\varphi^{n-\frac{1}{2}}|_{H^{-1}(\mathbb{R})} + C\delta|u^n|_{L^2(\mathbb{R})}^2 \leq \delta(|\varphi^{n-\frac{1}{2}}|_{H^{-1}(\mathbb{R})} + k). \end{aligned} \tag{16}$$

We conclude by the Gronwall lemma that this gives

$$|\varphi^{n+\frac{1}{2}}|_{H^{-1}(\mathbb{R})} \leq \frac{k}{1-\delta} \leq \frac{k}{\tau}.$$

We now complete the proof of Theorem 2 by an $L^2(\mathbb{R})$ upper bound on u_x^n. Let us assume once more without loss of generality that u^n for $n \geq 0$ is trapped into the $L^2(\mathbb{R})$ absorbing ball. Consider the scalar product of (5) with $-(u^{n+1} - \delta u^n)$. Then

$$|u_x^{n+1}|^2_{L^2(\mathbb{R})} = \delta^2 |u_x^n|^2_{L^2(\mathbb{R})} - b\int_{\mathbb{R}} \varphi^{n+\frac{1}{2}} (|u^{n+1}|^2 - \delta^2 |u^n|^2) - \mathrm{Re}\int_{\mathbb{R}} \overline{(u^{n+1} - \delta u^n)} f. \tag{17}$$

Using the $H^{-1}(\mathbb{R})$ bound on $\varphi^{n+\frac{1}{2}}$ we then have for instance

$$|\int_{\mathbb{R}} \varphi^{n+\frac{1}{2}} |u^n|^2| \leq \frac{k}{\tau} ||u^n|^2|_{H^1(\mathbb{R})} \leq \frac{k}{\tau} |u^n|_{L^\infty(\mathbb{R})} |u^n|_{H^1(\mathbb{R})}. \tag{18}$$

Using now Agmon's inequality $|u^n|^2_{L^\infty(\mathbb{R})} \leq c|u^n|_{L^2(\mathbb{R})} |u_x^n|_{L^2(\mathbb{R})}$ and the fact that u^n remains bounded in $L^2(\mathbb{R})$ we then have

$$|\int_{\mathbb{R}} \varphi^{n+\frac{1}{2}} |u^n|^2| \leq \frac{k}{\tau} (1 + |u_x^n|_{L^2(\mathbb{R})})^{\frac{3}{2}}. \tag{19}$$

This yields to (with a constant k_1 that depends on the data as $|b|$ and that is independent of τ)

$$|u_x^{n+1}|^2_{L^2(\mathbb{R})} - \delta^2 |u_x^n|^2_{L^2(\mathbb{R})} \leq \frac{k}{\tau} (1 + |u_x^{n+1}|^{\frac{3}{2}}_{L^2(\mathbb{R})} + \delta^2 |u_x^n|^{\frac{3}{2}}_{L^2(\mathbb{R})}). \tag{20}$$

Using the inequality $a^2 - c^2 \geq (\sqrt{a} - \sqrt{c})(a + c)^{\frac{3}{2}}$, with $a = \sqrt{\frac{1}{2} + |u_x^{n+1}|_{L^2(\mathbb{R})}}$ and $c = \delta^{\frac{1}{2}} \sqrt{\frac{1}{2} + |u_x^n|_{L^2(\mathbb{R})}}$, we then infer from (20) that

$$\sqrt{\frac{1}{2} + |u_x^{n+1}|_{L^2(\mathbb{R})}} - \delta^{\frac{1}{2}} \sqrt{\frac{1}{2} + |u_x^n|_{L^2(\mathbb{R})}} \leq \frac{k}{\tau}. \tag{21}$$

Then the proof of Theorem 2 is completed, due to the discrete Gronwall lemma.

Remark 1 *It is worth to point out that the $H^1(\mathbb{R})$ bounds on the absorbing set depends on τ.*

4 Proof of Theorem 3

The proof is divided into several steps. To begin with, we prove that the trajectories are asymptotically compact in $L^2(\mathbb{R})$; actually, we prove that there exists a bounded set in $H^2(\mathbb{R})$ that attracts the solutions. We then apply Theorem I-1-1 in [21] to obtain the existence of the global attractor, which is compact in $H^1(\mathbb{R})$ and bounded in $H^2(\mathbb{R})$. Due to the famous J. Ball argument, it turns out that this global attractor is also a compact subset in $H^2(\mathbb{R})$. In this section, without loss of generality, we will consider trajectories that remain in the $L^2(\mathbb{R})$ absorbing ball.

To begin with, let us state and prove

Proposition 3 *For any $\eta > 0$, a trajectory u^n splits as $u^n = v^n + w^n$ where, for n large enough depending only on the $L^2(\mathbb{R})$ absorbing ball, $|w^n|_{L^2(\mathbb{R})} \leq \frac{3\eta}{\gamma}$, and, for any n, v^n is trapped into a bounded set of $H^1(\mathbb{R}) \cap L^2(\mathbb{R};(1+x^2)dx)$.*

Remark 2 *The Hilbert space $L^2(\mathbb{R};(1+x^2)dx)$ is the space of functions v such that*

$$\int_{\mathbb{R}} (1+x^2)|v(x)|^2 dx < +\infty;$$

since the embedding $H^1(\mathbb{R}) \cap L^2(\mathbb{R};(1+x^2)dx) \hookrightarrow L^2(\mathbb{R})$ is compact, then it follows from Proposition 3 that the semi-group S_τ is asymptotically compact in $L^2(\mathbb{R})$.

Proof. We use a splitting first introduced in [17] in the continuous case. Consider f_η in the Schwartz class $\mathcal{S}(\mathbb{R})$ such that $|f - f_\eta|_{L^2(\mathbb{R})} \leq \eta$. Split $S_\tau^n u^0 = u^n = v^n + w^n$, where

$$i\frac{v^{n+1} - \delta v^n}{\tau} + \Delta\frac{v^{n+1} + \delta v^n}{2} = b\varphi^{n+\frac{1}{2}}\frac{v^{n+1} + \delta v^n}{2} + f_\eta, \tag{22}$$

$$i\frac{w^{n+1} - \delta w^n}{\tau} + \Delta\frac{w^{n+1} + \delta w^n}{2} = b\varphi^{n+\frac{1}{2}}\frac{w^{n+1} + \delta w^n}{2} + f - f_\eta, \tag{23}$$

supplemented with $v^0 = 0$, $w^0 = u^0$, and such that for any n (6) holds true (actually we can prove by induction on n that $u^n = v^n + w^n$). On the one hand, if we take the scalar product of (23) with $i(w^{n+1} + \delta w^n)$, we thus obtain

$$|w^{n+1}|^2_{L^2(\mathbb{R})} - \delta^2|w^n|^2_{L^2(\mathbb{R})} = \tau \mathrm{Im}\int_{\mathbb{R}}(f - f_\eta)\overline{w^{n+1} + \delta w^n}dx. \tag{24}$$

Then, by Cauchy-Schwarz inequality we have

$$|w^{n+1}|_{L^2(\mathbb{R})} \leq \delta|w^n|_{L^2(\mathbb{R})} + \tau\eta. \tag{25}$$

A straightforward application of the discrete Gronwall lemma gives

$$|w^n|_{L^2(\mathbb{R})} \leq \delta^n|u^0|_{L^2(\mathbb{R})} + \frac{\tau\eta}{1-\delta} \leq \frac{3\eta}{\gamma}, \tag{26}$$

for n large enough such that $\delta^n \frac{2|f|_{L^2(\mathbb{R})}}{\gamma} \leq \frac{\eta}{\gamma}$.
The next step is to check that v_x^n remains bounded in $L^2(\mathbb{R})$; for that purpose we can copy line by line the end of the Proof of Theorem 2; we skip the details for the sake of conciseness.
On the other hand, considering the scalar product of (22) with $ix^2(v^{n+1}+\delta v^n)$ leads to

$$\begin{aligned}|xv^{n+1}|^2_{L^2(\mathbb{R})} - \delta^2|xv^n|^2_{L^2(\mathbb{R})} = \tau \mathrm{Im}\int_{\mathbb{R}} (v_x^{n+1} + \delta v_x^n)\overline{xv^{n+1} + \delta xv^n}dx + \\ \tau \mathrm{Im}\int_{\mathbb{R}} (xf_\eta)\overline{xv^{n+1} + \delta xv^n}dx.\end{aligned} \tag{27}$$

We then have, using the fact that v_x^n remains in a bounded set of $L^2(\mathbb{R})$,

$$|xv^n|_{L^2(\mathbb{R})} \leq \frac{\tau}{1-\delta}|xf_\eta|_{L^2(\mathbb{R})} \leq \frac{2}{\gamma}|xf_\eta|_{L^2(\mathbb{R})}. \tag{28}$$

We now move to the estimate on Δv^n. Consider the scalar product of (22) with $\Delta(v^{n+1} - \delta v^n)$. Then

$$\begin{aligned}|\Delta v^{n+1}|^2_{L^2(\mathbb{R})} = \delta^2|\Delta v^n|^2_{L^2(\mathbb{R})} + \\ \mathrm{Re}\int_{\mathbb{R}} \varphi^{n+\frac{1}{2}}(v^{n+1} + \delta v^n)\overline{\Delta(v^{n+1} - \delta v^n)}dx + 2\mathrm{Re}\int_{\mathbb{R}} f_\eta \Delta(\overline{v^{n+1} - \delta v^n})dx.\end{aligned} \tag{29}$$

As soon as u^n is in the $H^1(\mathbb{R})$ absorbing ball, then due to (6) $\varphi^{n+\frac{1}{2}}$ is also trapped into some bounded set in $H^1(\mathbb{R})$. Therefore (29) yields

$$|\Delta v^{n+1}|^2_{L^2(\mathbb{R})} - \delta^2|\Delta v^n|^2_{L^2(\mathbb{R})} \leq k(\tau)|\Delta(\overline{v^{n+1} - \delta v^n})|_{L^2(\mathbb{R})}. \tag{30}$$

Once again, we conclude by the Gronwall lemma.

At this stage we have proven the existence of a global attractor $\mathcal{A}_\tau$ in $H^1(\mathbb{R})$ that is a bounded set in $H^2(\mathbb{R})$. The compactness of the global attractor in $H^2(\mathbb{R})$ follows the J. Ball argument (see [4]). Consider u_j a sequence in $\mathcal{A}_\tau$ and $u_j^n = S_\tau^n u_j$ the corresponding trajectory. Up to a subsequence, we may consider that u_j converges weakly in $H^2(\mathbb{R})$ and strongly in $H^1(\mathbb{R})$ towards u; set $u^n = S_\tau^n u$ that is also a complete trajectory in the global attractor. To begin with, we establish an energy equality. Consider the scalar product of (5) with $\Delta(u_j^{n+1} - \delta u_j^n)$. We thus obtain

$$|\Delta u_j^{n+1}|^2_{L^2(\mathbb{R})} = \delta^2 |\Delta u_j^n|^2_{L^2(\mathbb{R})} + \\ b\mathrm{Re}\int_{\mathbb{R}} \varphi_j^{n+\frac{1}{2}} (\overline{u_j^{n+1} + \delta u_j^n}) \Delta(u_j^{n+1} - \delta u_j^n) dx + 2\mathrm{Re}\int_{\mathbb{R}} \overline{f} \Delta(u_j^{n+1} - \delta u_j^n) dx. \tag{31}$$

Set

$$X_j^n = b\mathrm{Re}\int_{\mathbb{R}} \varphi_j^{n+\frac{1}{2}} (\overline{u_j^{n+1} + \delta u_j^n}) \Delta(u_j^{n+1} - \delta u_j^n) dx + 2\mathrm{Re}\int_{\mathbb{R}} \overline{f} \Delta(u_j^{n+1} - \delta u_j^n) dx.$$

If the sequence u_j converges weakly in $H^2(\mathbb{R})$ and strongly in $H^1(\mathbb{R})$ (then $\varphi_j^{n+\frac{1}{2}}$ converges also in $H^1(\mathbb{R})$), then $X_j^n \to X^n$. Going backward in time, we then have

$$|\Delta u_j|^2_{L^2(\mathbb{R})} = \delta^{2m} |\Delta u_j^{-m}|^2_{L^2(\mathbb{R})} + \sum_{n=-m}^{0} \delta^{-2(n+1)} X_j^n. \tag{32}$$

We then have, since the sequence $|\Delta u_j^{-m}|^2_{L^2(\mathbb{R})}$ is bounded by $k(\tau)$

$$\limsup_{j\to+\infty} |\Delta u_j|^2_{L^2(\mathbb{R})} \leq \delta^{2m} k(\tau) + \sum_{n=-m}^{0} \delta^{-2(n+1)} X^n. \tag{33}$$

Since by the same energy equality as (32)

$$\sum_{n=-m}^{0} \delta^{-2(n+1)} X^n = |\Delta u|^2_{L^2(\mathbb{R})} - \delta^{2m} |\Delta u^{-m}|^2_{L^2(\mathbb{R})},$$

then

$$\limsup_{j\to+\infty} |\Delta u_j|^2_{L^2(\mathbb{R})} \leq 2\delta^{2m} k(\tau) + |\Delta u|^2_{L^2(\mathbb{R})}. \tag{34}$$

Let m goes to the infinity. Then $\limsup_{j\to+\infty} |\Delta u_j|^2_{L^2(\mathbb{R})} \leq |\Delta u|^2_{L^2(\mathbb{R})}$ and u_j converges strongly towards u in $H^2(\mathbb{R})$. The proof of Theorem 3 is complete.

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