EXAMPLES OF CONICS ARISING IN TWO-DIMENSIONAL FINSLER AND LAGRANGE GEOMETRIES *

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Abstract

The well-known invariants of conics are computed for classes of Finsler and Lagrange spaces. For the Finsler case, some (α, β)-metrics namely Randers, Kropina and "Riemann"-type metrics provides conics as indicatrices and a Randers-Funk metric on the unit disk is treated as example. The relations between algebraic and differential invariants of (α, β)-metrics are pointed out as a method to use the formers in terms of the Finsler metric. In the Lagrange framework, a polynomial of third order Lagrangian inspired by Tzitzeica is studied and examples for all three cases (elliptic, hyperbolic, parabolic) are given.

Introduction

This paper is devoted to a study of conics which are naturally associated in two-dimensional Lagrange, particularly Finsler geometry. We are interested in this dimension since the 2D Lagrange geometry may yields in a somehow intrinsic manner a conic and the particular case of 2D Finsler geometry is a subject of continuous research: see [1], [2], [5], [9]. Moreover, conics in two dimensional Finsler geometry were already studied by Matsumoto in [7] from the point of view of geodesics.

The great importance of indicatrices in the Finslerian setting is pointed out by Okubo's technique ([2, p. 13]) which shows that, in a certain sense, not

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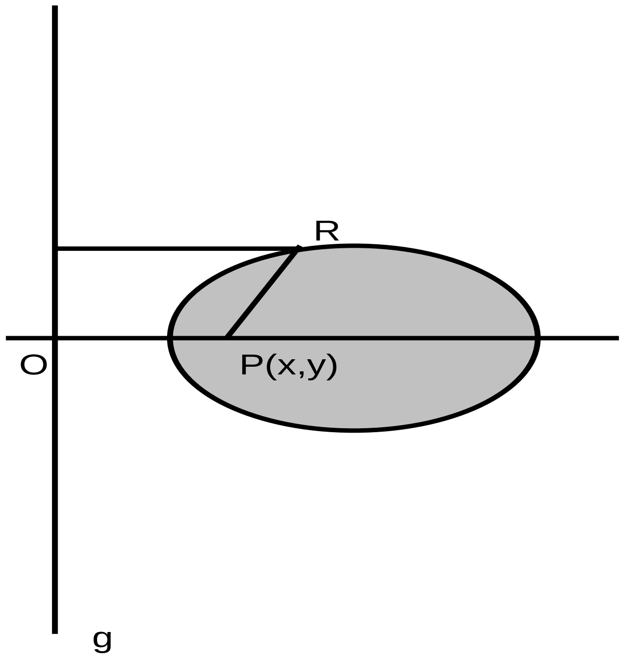
only the fundamental Finsler function F yields the indicatrices but conversely, the indicatrices determine F.

For example, ([2, p. 18]), we consider the Euclidean 2 dimensional space E^2, $\pi = E^2 \backslash \{O\}$, where O is the origin and suppose that the indicatrix of a Finsler space (π, F) in an arbitrary point $P \in \pi$ is an ellipse with the focus P and the directrice g, $O \in g$, orthogonal to OP.



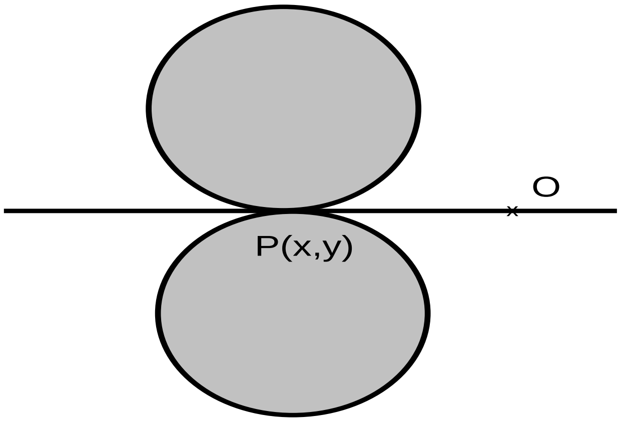
Using Okubo's technique a Finsler function of Randers type is obtained:

$$F(x, \dot{x}, y, \dot{y}) = \frac{1}{e}\sqrt{\frac{\dot{x}^2 + \dot{y}^2}{x^2 + y^2}} - \frac{x\dot{x} + y\dot{y}}{x^2 + y^2}.$$

Here we denoted by (x, y) the local coordinates on π, by $(\dot{x}, \dot{y})$ the local coordinates on $T_P\pi$ and e the eccentricity of the ellipse.

If the indicatrix I_P is the union of two exterior tangent circles, with the common tangent OP, a Finsler metric of Kropina type is obtained [9, p. 585]:

$$F = \frac{\dot{x}^2 + \dot{y}^2}{|y\dot{x} - x\dot{y}|}.$$



So our paper starts with a discussion of indicatrices in the Finsler geometry in the case when these indicatrices are conics.

The general formula of invariants from the Euclidean geometry of conics is applied in the first section to some classes of Finsler metrics belonging to a remarkable set, namely (α, β)-metrics, since this set is a comprehensive one in Finsler geometry including several well-known metrics. A Randers metric on the unit disk, considered by T. Okada in [12] as an example of Finsler metric of Funk type having constant Finslerian-Gaussian (or flag) negative curvature, is included as concrete example.

In the next section we derive relations between algebraic and differential invariants of (α, β)-metrics as a method to use the intrinsic Finsler metric in computations of invariants of conics. As consequence, the invariants of conics from the first section are expressed in terms of F and Finsler metric g.

The last section is devoted to the Lagrange framework where a Lagrangian polynomial of third order in y yields an affine Lagrange metric. The determinant of this Lagrange metric, called by us *Tzitzeica* since it was inspired by the function θ of relation (4) from [17], yields a conic modelled by the notion of g-hypercone [15]. The invariants of this conic are computed in terms of the coefficients of given polynomial Lagrangian. The Lagrange geometry is studied through the Cartan tensor and vertical Christoffel symbols.

1 Finsler spaces with conics as indicatrices

In the two-dimensional Euclidean space $\mathbb{R}^2$ let us consider a conic Γ defined implicitly by $f \in C^\infty\left(\mathbb{R}^2\right)$ as: $\Gamma = \{(x, y) \in \mathbb{R}^2 \mid f(x, y) = 0\}$ where f is a quadratic function namely $f(x, y) = r_{11}x^2 + 2r_{12}xy + r_{22}y^2 + 2r_{10}x + 2r_{20}y + r_{00}$ with $r_{11}^2 + r_{12}^2 + r_{22}^2 \neq 0$. Recall the Euclidean invariants associated to Γ:

$$\begin{cases} D & = \begin{vmatrix} r_{11} & r_{12} & r_{10} \\ r_{12} & r_{22} & r_{20} \\ r_{10} & r_{20} & r_{00} \end{vmatrix} \\ I & = r_{11} + r_{22} \\ \delta & = r_{11}r_{22} - r_{12}^2 \\ \Delta & = \delta + r_{11}r_{00} - r_{10}^2 + r_{22}r_{00} - r_{20}^2. \end{cases} \tag{1}$$

In this paper we consider Γ as the indicatrix of a *Finsler space* $F^2 = (M, F)$ on an open domain M of $\mathbb{R}^2$. More precisely, a function $F : TM \to \mathbb{R}^+$, smooth on TM minus the null section, positively homogeneous of degree 1 with respect to y and such that $L = F^2$ is a regular Lagrangian, defines the pair $F^2 = (M, F)$ called a *Finsler space* and F is called *the fundamental function* of F^2 ([10]). The regularity of $L : TM \to \mathbb{R}$ means that the y-Hessian matrix of L, $g_{ij} = \frac{1}{2}\frac{\partial^2 L}{\partial y^i \partial y^j}$, is of rank 2 i.e. $\det(g_{ij}) \neq 0$ where we use local coordinates $x = \left(x^i\right)_{1 \leq i \leq 2}$ on M and $(x, y) = \left(x^i, y^i\right)$ on the tangent bundle TM. The tensor field $g = \left(g_{ij} = \frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j}\right)$ is called *the Finsler metric* of the Finsler space F^2 or *Lagrange metric* in the general case of a regular Lagrangian. Associated to this Lagrangian we have *the indicatrix of* L: for every $x \in M$, $I_x = \{y \in T_xM; L(x, y) = 1\}$ appears as a curve in $T_xM \simeq \mathbb{R}^2$ defined by: $f(x, y) = L(x, y) - 1$. Since F is strictly positive we derive that

$$I_x = \{y \in T_xM; F(x, y) = 1\}.$$

A Finsler fundamental function is said to be *Minkowskian* if it depends only of y i.e $F = F(y)$.

In order to obtain I_x as a conic in (y^1, y^2) we choose a class of Finsler metrics called (α, β)-*metrics*.

Let us consider: a Riemannian metric $a = (a_{ij}(x))_{1 \leq i,j \leq 2}$ and an 1-form $b = (b_i(x))_{1 \leq i \leq 2}$ both living globally on M and let us associate to these objects the following functions on TM:

- $\alpha(x,y) = \sqrt{a_{ij}y^i y^j}$
- $\beta(x,y) = b_i y^i$
- $F(x,y) = \alpha\phi\left(\frac{\beta}{\alpha}\right)$

with ϕ a C^∞ positive function on some interval $[-r,r]$ big enough such that $r \geq \frac{\beta}{\alpha}$ for all $(x,y) \in TM$. F is a Finsler fundamental function if the following conditions are satisfied ([14, p. 307, eq. (2-11)]):

$$\phi(s) > 0, \phi(s) - s\phi'(s) > 0, (\phi(s) - s\phi'(s)) + \left(b^2 - s^2\right)\phi''(s) > 0$$

for all $|s| \leq b \leq r$, where $s = \frac{\beta}{\alpha}$.

Particular cases:

- **Randers metrics** $\phi(s) = 1 + s$

The expression of the indicatrix is:

$$\begin{aligned} I_x : &(a_{11} - b_1^2)(y^1)^2 + 2(a_{12} - b_1 b_2)y^1 y^2 + (a_{22} - b_2^2)(y^2)^2 \\ &+2b_1 y^1 + 2b_2 y^2 - 1 = 0 \end{aligned} \tag{2}$$

and then:

$$D = -\det a,\ \ I = Tra - \|b\|_e^2,\ \ \delta = \det a(1 - \|b\|_a^2),\ \ \Delta = \delta - Tra, \tag{3}$$

where $\|b\|_e$ is the Euclidean norm of b i.e. $\|b\|_e^2 = b_1^2 + b_2^2$ and $\|b\|_a$ is the norm of the 1-form b with respect to the Riemannian metric a i.e. $\|b\|_a^2 = a^{11}b_1^2 + 2a^{12}b_1 b_2 + a^{22}b_2^2$. Also, $\det a$, respectively Tra is the determinant, respectively the trace of the metric a.

In the particular case of Riemannian metrics from $b = 0$ we get:

$$D = -\det a,\ \ I = Tra,\ \ \delta = \det a,\ \ \Delta = \det a - Tra, \tag{4}$$

for:

$$I_x : a_{11}(y^1)^2 + 2a_{12}y^1 y^2 + a_{22}(y^2)^2 - 1 = 0. \tag{5}$$

- **Kropina metrics** $\phi(s) = \frac{1}{s}$

The expression of the indicatrix is:

$$I_x : a_{11}(y^1)^2 + 2a_{12}y^1 y^2 + a_{22}(y^2)^2 - b_1 y^1 - b_2 y^2 = 0. \tag{6}$$

and then:

$$D = -\frac{1}{4}\det a\|b\|_a^2, \ \ I = Tra, \ \ \delta = \det a, \ \ \Delta = \delta - \frac{1}{4}\|b\|_e^2. \tag{7}$$

Sometimes the notion of *generalized Kropina metric* is useful when ([6]) $\phi(s) = \frac{1}{s^m}, m \neq 0, -1$, but from:

$$I_x : (a_{ij}(x)y^iy^j)^{\frac{m+1}{2}} - (b_i(x)y^i)^m = 0$$

it results that I_x is a conic if and only if $m = 1$.

- **Matsumoto metrics** ([11, p. 553]) $\phi(s) = \frac{1}{1-s}$

The indicatrix is not a conic since:

$$I_x : (a_{ij}(x)y^iy^j + b_i(x)y^i)^2 = a_{ij}(x)y^iy^j.$$

- **"Riemann" type** (α, β)**-metrics** ([11, p. 553]) $\phi(s) = \sqrt{1+s^2}$

The expression of the indicatrix is:

$$I_x : (a_{11} + b_1^2)(y^1)^2 + 2(a_{12} + b_1b_2)y^1y^2 + (a_{22} + b_2^2)(y^2)^2 - 1 = 0. \tag{8}$$

and then:

$$D = -\det a(1 + \|b\|_a^2), \ \ I = Tra + \|b\|_e^2, \ \ \delta = -D, \ \ \Delta = -D - I. \tag{9}$$

Let us remark that g is in fact a Riemannian metric: $g_{ij} = a_{ij} + b_ib_j = g_{ij}(x)$ and this explains the name.

- ([8]) $\phi(s) = 1 + s^2$.

The indicatrix is:

$$I_x : a_{ij}(x)y^iy^j + (b_i(x)y^i)^2 = \sqrt{a_{ij}(x)y^iy^j}$$

and it is not a conic.

Remark 1.1. *Since a is positive definite i.e.* $\det a > 0$ *it results for all above conics to be of elliptic genus. Let us point out that Finsler spaces with elliptic indicatrices has some recent applications in billiard systems, see [13] and [16]. Let us compare the equation of these indicatrices with the canonical form of a nondegenerate elliptic conic:*

$$\varepsilon((y^1)^2 + (y^2)^2) + k = 0, \varepsilon = \pm 1, \ \ k \neq 0.$$

i) For the case I it results the Finsler-Riemannian metric $b_1 = b_2 = 0 = a_{12}, a_{11} = a_{22} = -\frac{\varepsilon}{k}$ *for which we consider the pairs* $(\varepsilon = +1, k < 0), (\varepsilon = -1, k > 0)$. *Therefore the indicatrices in 2D Riemannian geometry are circles and the coresponding Finsler fundamental function is a Minkowskian one:* $F = \sqrt{-\frac{\varepsilon}{k}((y^1)^2 + (y^2)^2)}$.
ii) The case II can not be put in this canonical form.
iii) For the case IV it results the Finsler metric associated to an 1-form b and Riemannian metric $a_{11} = -b_1^2 - \frac{\varepsilon}{k}, a_{12} = -b_1 b_2, a_{22} = -b_2^2 - \frac{\varepsilon}{k}$. *This Riemannian metric is positive definite if and only if* $b_1^2 + \frac{\varepsilon}{k} < 0$ *and* $\det a = b_1^2 + b_2^2 + \frac{\varepsilon}{k} > 0$. *The Finsler fundamental function is exactly the above one.*

It results that $b_2 \neq 0$ *and:*

iii1) for $\varepsilon = 1$ *we get* $\frac{1}{k} \in (-b_1^2 - b_2^2, -b_1^2)$,

iii2) for $\varepsilon = -1$ *we have* $\frac{1}{k} \in (b_1^2, b_1^2 + b_2^2)$.

Example 1.2. *Let us consider after [12, p. 123] the Randers metric on the unit disk* D^1 *determined by:*

$$a_{11} = \frac{1 - \left(x^2\right)^2}{\varphi^2(x)},\ \ a_{12} = \frac{x^1 x^2}{\varphi^2(x)},\ \ a_{22} = \frac{1 - \left(x^1\right)^2}{\varphi^2(x)},\ \ b_1 = \frac{x^1}{\varphi(x)},\ \ b_2 = \frac{x^2}{\varphi(x)},$$

where φ *is the function defining the boundary* $\partial D^1 = S^1$:

$$\varphi(x) = 1 - \left(x^1\right)^2 - \left(x^2\right)^2.$$

In the cited paper it is proved that the associated Randers-Funk metric is of constant negative flag curvature $(-1/4)$; *see also [2, p. 20]. From (3) it results:*

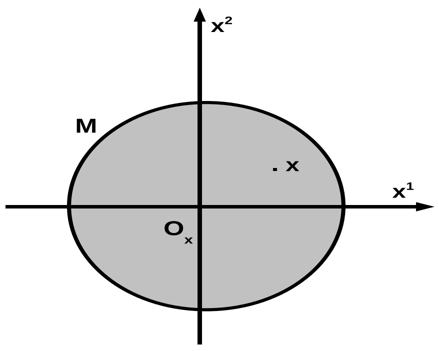
$$D = -\frac{1}{\varphi^3},\ \ I = \frac{2}{\varphi},\ \ \delta = \frac{1}{\varphi^2},\ \ \Delta = -\frac{1}{\varphi}.$$

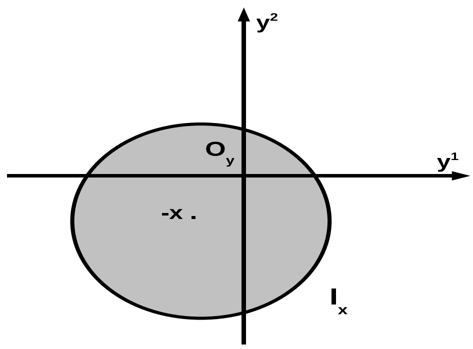
But, a straightforward computation gives the indicatrix

$$I_x : \frac{1}{\varphi}[(x^1 + y^1)^2 + (x^2 + y^2)^2 - 1] = 0$$

and then:

$$D = -1,\ I = 2,\ \delta = 1,\ \Delta = -\varphi.$$





Remark 1.3. *Since we are working in the framework of second order equations providing conics let us pointed out that some remarkable scalar second order equations are giving by the pair* (α, β) *by equaling two types of* (α, β)*-Lagrangians:*
1) Randers=Kropina gives $1 + s = \frac{1}{s}$ *which is the golden mean equation [4], with the solutions* $s_1 = \frac{-1+\sqrt{5}}{2}, s_2 = \frac{-1-\sqrt{5}}{2}$,
2) Randers=Matsumoto: $1 + s = \frac{1}{1-s}$ *with the solutions* $s_{1,2} = \pm 1$*, as well as*

some quartic equations:
3) Kropina="Riemann": $s^2(1+s^2)=1$ *which with* $s^2=u$ *yields again the golden mean equation* $u^2+u=1$,
4) Matsumoto="Riemann": $(1-s)^2(1+s^2)=1$ *with the solutions:*

$$s_1=0, s_2=\frac{1}{3}\left(2-2(17+3\sqrt{33})^{-\frac{1}{3}}+\sqrt[3]{17+3\sqrt{33}}\right)$$

$$s_3=\overline{s}_4=\frac{2}{3}+\frac{1+\sqrt{-3}}{3}\left(17+3\sqrt{33}\right)^{-\frac{1}{3}}-\frac{1-\sqrt{-3}}{6}(17+3\sqrt{33})^{\frac{1}{3}}.$$

5) Randers="Riemann": $1+s=\sqrt{1+s^2}$ *with solution* $s=0$.

It is amazing the occurrence of the prime numbers: 2, 3, 5, 17, 11 (in $33=3\cdot 11$).

2 Relations between the algebraic and the differential invariants

The computations of the first section prove that an (α,β)-metric has two *algebraic* invariants: Tra and $\det a$.

Let us recall that such type of Finsler metrics has four *differential* invariants [9, p. 890]:

$$\begin{cases} p=\frac{1}{2\alpha}\frac{\partial F^2}{\partial \alpha}, & p_0=\frac{1}{2}\frac{\partial^2 F^2}{\partial \beta^2}, \\ p_1=\frac{1}{2\alpha}\frac{\partial^2 F^2}{\partial\alpha\partial\beta}, & p_2=\frac{1}{2\alpha^2}\left(\frac{\partial^2 F^2}{\partial\alpha^2}-\frac{1}{\alpha}\frac{\partial F^2}{\partial\alpha}\right), \end{cases} \tag{10}$$

where the subscripts denote the minus of degree of homogeneity of these invariants and which connect the Riemannian metric a with the Finsler metric $g=(g_{ij})$ through the relation [9, p. 890]:

$$g_{ij}=pa_{ij}+p_0b_ib_j+p_1\left(b_iy^a_j+b_jy^a_i\right)+p_2y^a_iy^a_j, \tag{11}$$

where $y^a_i=a_{ij}(x)y^j$. If $L=y^i\frac{\partial}{\partial y^i}$ is the Liouville vector field on TM then (y^a_i) are the components of L^{*a}, the 1-form dual of L with respect to the Riemannian metric a considered on TM.

We can also express these differential invariants with respect to the function ϕ:

$$\begin{cases} p &= \phi(\phi-s\phi'), \\ p_0 &= (\phi')^2+\phi\phi''=\frac{1}{s}(\phi\phi'-p'), \\ p_1 &= \frac{1}{\alpha}(\phi\phi'-s(\phi')^2-s\phi\phi'')=\frac{p'}{\alpha}, \\ p_2 &= \frac{1}{\alpha^2}(-s\phi\phi'+s^2(\phi')^2+s^2\phi\phi'')=-\frac{s}{\alpha}p_1=-\frac{s}{\alpha^2}p', \end{cases} \tag{12}$$

where prime denotes the derivative with respect to s.

Example. *Let us present some particular cases:*

- Randers:

$$p = 1 + \frac{\beta}{\alpha} = 1 + s, \ \ p_0 = 1, \ \ p_1 = \frac{1}{\alpha}, \ \ p_2 = -\frac{\beta}{\alpha^3} = -\frac{s}{\alpha^2}. \tag{13}$$

- Kropina:

$$p = \frac{2\alpha^2}{\beta^2} = \frac{2}{s^2}, \ \ p_0 = \frac{3\alpha^4}{\beta^4} = \frac{3}{s^4}, \ \ p_1 = -\frac{4\alpha^2}{\beta^3} = -\frac{4}{\alpha s^3}, \ \ p_2 = \frac{4}{\beta^2} = \frac{4}{\alpha^2 s^2}. \tag{14}$$

- "Riemann" type (α, β)-metric:

$$p = p_0 = 1, \quad p_1 = p_2 = 0. \tag{15}$$

- Riemannian metric $F = \alpha$:

$$p = 1, \ \ p_0 = p_1 = p_2 = 0. \tag{16}$$

Performing computations in (11) we derive:

$$Trg = pTra + p_1 \sum_i y_i^a (b_i - \frac{\beta}{\alpha^2} y_i^a) + \sum_i b_i \frac{\partial}{\partial y^i}(\alpha \phi \phi'). \tag{17}$$

Using the results from ([14, p. 307]) we obtain for a general n dimensional (α, β)-metric:

$$detg = \phi^{n+1}(s)\left(\phi(s) - s\phi'(s)\right)^{n-2}\left(\phi(s) - s\phi'(s) + \left(\|\beta\|_x^2 - s^2\right)\phi''(s)\right) deta \tag{18}$$

where $\|\beta\|_x = \sup\{\frac{\beta(x,y)}{\alpha(x,y)}; y \in T_x M\} = \| b \|_a$ which implies:

$$deta = \frac{detg}{\phi^3(s)\left(\phi(s) - s\phi'(s) + (\|b\|_a^2 - s^2)\phi''(s)\right)}. \tag{19}$$

Particular cases:
I Randers metrics
From $\phi' = \phi - s\phi' = 1, \phi'' = 0$ it results:

$$\begin{cases} Tra &= \frac{1}{\alpha F}[\alpha^2 Trg - \sum_i (\alpha b_i + y_i^a)^2] + \frac{1}{\alpha^2}\sum_i (y_i^a)^2, \\ \\ deta &= \frac{detg}{(1+s)^3} = \frac{\alpha^3 detg}{F^3}, \end{cases} \tag{20}$$

and then:

$$\begin{cases} D &= -\frac{\alpha^3 detg}{F^3}, \\ \\ \delta &= \frac{\alpha^3 detg}{F^3}(1- \parallel b \parallel_a^2), \\ \\ I &= \frac{1}{\alpha F}[\alpha^2 Trg - \sum_i(\alpha b_i + y_i^a)^2] + \frac{1}{\alpha^2}\sum_i (y_i^a)^2 - \parallel b \parallel_e^2, \\ \\ \Delta &= \frac{\alpha^3 detg}{F^3} - \frac{1}{\alpha F}[\alpha^2 Trg - \sum_i(\alpha b_i + y_i^a)^2] - \frac{1}{\alpha^2}\sum_i (y_i^a)^2. \end{cases} \tag{21}$$

II Kropina metrics
From $\phi' = -\frac{1}{s^2}, \phi - s\phi' = \frac{2}{s}, \phi'' = \frac{2}{s^3}$ it follows:

$$\begin{cases} Tra &= \frac{s^2}{2}Trg + \frac{2}{\beta}\sum_i y_i^a(b_i - \frac{s}{\alpha}y_i^a) + \frac{s^2}{2}\sum_i b_i \frac{\partial}{\partial y^i}(\frac{\alpha}{s^3}), \\ \\ deta &= \frac{s^6 detg}{2\|b\|_a^2}, \end{cases} \tag{22}$$

and then:

$$\begin{cases} D &= -\frac{s^6 detg}{8}, \\ \\ \delta &= \frac{s^6 detg}{2\|b\|_a^2}, \\ \\ I &= \frac{s^2}{2}Trg + \frac{2}{\beta}\sum_i y_i^a(b_i - \frac{s}{\alpha}y_i^a) + \frac{s^2}{2}\sum_i b_i \frac{\partial}{\partial y^i}(\frac{\alpha^4}{\beta^3}), \\ \\ \Delta &= \frac{s^6 detg}{2\|b\|_a^2} - \frac{1}{4} \parallel b \parallel_e^2 . \end{cases} \tag{23}$$

III "Riemann" type (α, β)-metrics
From $\phi' = \frac{s}{\sqrt{1+s^2}}, \phi - s\phi' = \frac{1}{\sqrt{1+s^2}}, \phi'' = \frac{1}{(1+s^2)^{3/2}}$, we get

$$\begin{cases} Tra &= Trg - \parallel b \parallel_e^2, \\ \\ deta &= \frac{detg}{(1+\|b\|_a^2)}, \end{cases} \tag{24}$$

and then

$$\begin{cases} D &= -\delta = -detg, \\ I &= Trg, \\ \Delta &= detg - Trg. \end{cases} \tag{25}$$

3 Conics for Lagrangians of Tzitzeica type

Let us consider a general Lagrangian on $M = \mathbb{R}^2$ which is polynomial of third order in y:

$$\begin{aligned} L(x,y) &= \frac{1}{3}\left[a(y^1)^3 + 3b(y^1)^2y^2 + 3b_1y^1(y^2)^2 + b_2(y^2)^3\right] \\ &\quad + c(y^1)^2 + 2c_1y^1y^2 + c_2(y^2)^2 + k_1y^1 + k_2y^2 + k_3 \end{aligned} \tag{26}$$

where $a, b, c, b_1, b_2, c_1, c_2, k_1, k_2, k_3 \in C^\infty(\mathbb{R}^2)$. Its associated Lagrange metric:

$$g_{11} = ay^1 + by^2 + c, \quad g_{12} = by^1 + b_1y^2 + c_1, \quad g_{22} = b_1y^1 + b_2y^2 + c_2 \tag{27}$$

is an affine one in y. Since Tzitzeica treated 2D Riemannian metrics of (3.2)-type in [17]-[18] let us call (3.1) a *third order polynomial Lagrangian in Tzitzeica form* and (3.2) a *Lagrange-Tzitzeica metric*. Equaling with zero the determinant of the metric (3.2) we obtain the conic:

$$\begin{aligned} \Gamma_x :\ & (ab_1 - b^2)(y^1)^2 + (ab_2 - bb_1)y_1y_2 + (ac_2 + b_1c - 2bc_1)y^1 \\ & + (bc_2 + b_2c - b_1c_1)y^2 + (cc_2 - c_1)^2 = 0 \end{aligned} \tag{28}$$

called *the absolute of the metric* (3.2). For a real constant $c \neq 0$ the hypersurface $G_x(c)$ of T_xM defined by the equations $g(x,y) = c$ is called a *g-hypercone* in [15, p. 73].

The invariants of Γ_x are:

$$\begin{cases} D &= \frac{1}{4}\begin{vmatrix} a & b & b_1 \\ b & b_1 & b_2 \\ c & c_1 & c_2 \end{vmatrix}^2, \\ I &= ab_1 + bb_2 - b^2 - b_1^2, \\ \delta &= \begin{vmatrix} a & b \\ b & b_1 \end{vmatrix}\begin{vmatrix} b & b_1 \\ b_1 & b_2 \end{vmatrix} - \frac{1}{4}\begin{vmatrix} a & b_1 \\ b & b_2 \end{vmatrix}^2, \\ \Delta &= \delta + (ab_1 - b^2)(cc_2 - c_1^2) - \frac{1}{4}(ac_2 + b_1c - 2bc_1)^2 \\ & \quad + (bb_2 - b_1^2)(cc_2 - c_1^2) - \frac{1}{4}(bc_2 + b_2c - 2b_1c_1)^2. \end{cases} \tag{29}$$

The Cartan tensor field $C_{ijk} = \frac{1}{2}\frac{\partial g_{jk}}{\partial y^i}$ of the Lagrange metric g is constant with respect to y:

$$2C_{111} = a, 2C_{211} = 2C_{112} = b, 2C_{212} = 2C_{122} = b_1, 2C_{222} = b_2 \tag{30}$$

and then, the vertical Christoffel symbols:

$$C^i_{jk} = \frac{1}{2} g^{ia} \left(\frac{\partial g_{ak}}{\partial y^j} + \frac{\partial g_{ja}}{\partial y^k} - \frac{\partial g_{jk}}{\partial y^a} \right)$$

are:

$$\begin{cases} C^1_{11} &= \frac{1}{2\det g} \left[(ab_1 - b^2)y^1 + (ab_2 - bb_1)y^2 + (ac_2 - bc_1) \right], \\ C^2_{11} &= \frac{1}{2\det g} \left[(b^2 - ab_1)y^2 + (bc - ac_1) \right], \\ C^1_{12} &= \frac{1}{2\det g} \left[(bb_2 - b_1^2)y^2 + (bc_2 - b_1c_1) \right], \\ C^2_{12} &= \frac{1}{2\det g} \left[(ab_1 - b^2)y^1 + (b_1c - bc_1) \right], \\ C^1_{22} &= \frac{1}{2\det g} \left[(b_1^2 - bb_2)y^1 + (b_1c_2 - b_2c_1) \right], \\ C^2_{22} &= \frac{1}{2\det g} \left[(ab_2 - bb_1)y^1 + (bb_2 - b_1^2)y^2 + (b_2c - b_1c_1) \right]. \end{cases} \tag{31}$$

The following *canonical forms* of L are to be considered inspired by the canonical forms from [17]:

1. (elliptic) $L = \frac{1}{3}(y^1)^3 - y^1(y^2)^2 + A((y^1)^2 + (y^2)^2), A > 0$, which is a regular Lagrangian:

$$g_{11} = A + y^1, \quad g_{12} = -y^2, \quad g_{22} = A - y^1$$

and then $\Gamma_x : (y^1)^2 + (y^2)^2 = A$.

2. (hyperbolic) $L = \frac{1}{3}[(y^1)^3 + (y^2)^3] + 2Ay^1y^2$ which yields:

$$g_{11} = y^1, \quad g_{12} = A, \quad g_{22} = y^2.$$

The associated conic is $\Gamma_x : y^1y^2 - A = 0$.

3. (parabolic) $L = (y^1)^2y^2 + (y^2)^2$ with:

$$g_{11} = y^2, \quad g_{12} = y^1, \quad g_{22} = 1$$

which implies the conic $\Gamma_x : y^2 - (y^1)^2 = 0$.

In [3, p. 195] is given the general form of Killing tensors of second order for the 2D Euclidean metric $ds^2 = d(y^1)^2 + d(y^2)^2$:

$$aM + bL_1 + cL_2 + eE_1 + dE_2 + gE_3$$

with a, b, c, d, e, g arbitrary real numbers and:

$$M = \begin{pmatrix} (y^2)^2 & -y^1y^2 \\ -y^1y^2 & (y^1)^2 \end{pmatrix}, L_1 = \begin{pmatrix} 0 & -y^2 \\ -y^2 & 2y^1 \end{pmatrix}, L_2 = \begin{pmatrix} 2y^2 & -y^1 \\ -y^1 & 0 \end{pmatrix}$$

$$E_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, E_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, E_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Let us remark that in the cited paper the matrices M, L_1, L_2, E_3 appear times $\frac{1}{2}$ but this factor can be included in the parameters a, b, c, g. Due to the presence of factors 2 and minus in the expressions of L_1 and L_2 a straightforward calculus gives that it does not exist real coefficients A, B, C, D, E such that:

$$g = AL_1 + BL_2 + CE_1 + DE_2 + EE_3$$

i.e. a Lagrange-Tzitzeica metric is not a Killing tensor of second order for the Euclidean 2D metric.

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