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A COMMON FIXED POINT THEOREM FOR MULTIVALUED ĆIRIĆ TYPE MAPPINGS WITH NEW TYPE COMPATIBILITY

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Abstract

Kaneko and Sessa defined the concept of compatibility for multivalued mappings with Hausdorff metric and proved a coincidence point theorem. After then, Pathak defined the concept of weak compatibility and proved a coincidence theorem. In the present work, we define a new type compatibility for multivalued mappings with Hausdorff metric. This new type compatibility is different from compatibility and weak compatibility. We give a common fixed point theorem for multivalued mappings using this new type compatibility.

1 Introduction

Throughout this paper, X stands for a metric space with the metric d whereas $CB(X)$ denotes the family of all nonempty closed bounded subsets of X. Let

$$H(A,B) = \max\{\sup_{x\in A} d(x,B), \sup_{y\in B} d(y,A)\},$$

where $A, B \in CB(X)$ and $d(x,A) = \inf\{d(x,y) : y \in A\}$. The function H is a metric on $CB(X)$ and is called Hausdorff metric. It is well-known that, if X is a complete metric space, then so is the metric space $(CB(X), H)$. Let

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$A, B \in CB(X)$ and $k > 1$. In the sequel the following well-known fact will be used [5]: For each $a \in A$, there is $b \in B$ such that $d(a, b) \leq kH(A, B)$.

Again $B(X)$ stands for the set of all bounded subsets of X. The function δ of $B(X) \times B(X)$ into $[0, \infty)$ is defined as $\delta(A, B) = \sup\{d(a, b) : a \in A, b \in B\}$ for all A, B in $B(X)$. If $A = \{a\}$ is singleton, we write $\delta(A, B) = \delta(a, B)$ and if $B = \{b\}$, then we put $\delta(A, B) = \delta(a, b) = d(a, b)$. It is easily seen that $\delta(A, B) = \delta(B, A) \geq 0$, $\delta(A, B) \leq \delta(A, C) + \delta(C, B)$ and $\delta(A, B) = 0$ implies $A = B = \{a\}$ for all A, B, C in $B(X)$.

The concept of compatibility for single valued mappings, which is defined by Jungck , was extended for multivalued mappings by two ways as follows: Kaneko and Sessa extend it to include multivalued mappings with the metric H on $CB(X)$ (Definition 2 in [4], see also Definition 3.2 in [2]). Jungck and Rhoades extend it to include multivalued mappings with the function δ in $B(X)$ (Definition 3.1 in [2]). After then, this two definitions were weakened as follows: Pathak (Definition 4 in [6]) and Jungck and Rhoades (Definition 2.2 in [3]) introduced the concept of weak compatibility with H and δ, respectively. It is well known that, compatible mappings are weakly compatible, but the converse is not true (see the related papers). Recently, the concept of compatible of type (I) mappings, which was introduced by Pathak et all. [7] for single valued mappings, has been extended to multivalued mappings with δ on $B(X)$ by Altun and Turkoglu (Definition 2 in [1]). Also in [1], there are some mappings such that they are compatible of type (I) but not weakly compatible. The aim of this paper is to prove a common fixed point theorem for multivalued mappings using the metric H on $CB(X)$. For this, we will introduce the definition of compatible of type (I) mappings for multivalued mappings with the metric H on $CB(X)$.

Definition 1 ([4]). *The mappings $f : X \to X$ and $S : X \to CB(X)$ are compatible if $fSx \in CB(X)$ for all $x \in X$ and*

$$\lim_{n\to\infty} H(Sfx_n, fSx_n) = 0,$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = A \in CB(X)$ and $\lim_{n\to\infty} fx_n = t \in A$.

Definition 2 ([6]). *The mappings $f : X \to X$ and $S : X \to CB(X)$ are f-weak compatible if $fSx \in CB(X)$ for all $x \in X$ and the following limits exist and satify*

$$\max\{\lim_{n\to\infty} H(Sfx_n, fSx_n), \lim_{n\to\infty} H(fSx_n, fx_n)\} \leq \lim_{n\to\infty} H(Sfx_n, Sx_n),$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = A \in CB(X)$ and $\lim_{n\to\infty} fx_n = t \in A$.

Kaneko and Sessa [4] and Pathak [6] proved coincidence point theorems using the above definitions, respectively. Also, Turkoglu and Altun [8] proved a common fixed point theorem for multivalued mappings using two conditions instead of compatibility.

Now, we introduce the following definition.

Definition 3. *The mappings $f : X \to X$ and $S : X \to CB(X)$ are compatible of type (I) if*

i) $fSx \in CB(X)$, for all $x \in X$

ii) $d(t, ft) \le \limsup_{n\to\infty} H(A, Sfx_n)$, whenever $\lim_{n\to\infty} Sx_n = A \in CB(X)$ and $\lim_{n\to\infty} fx_n = t \in A$.

The following example shows that f and S are compatible of type (I), but they are not f-weak compatible.

Example 1. *Let $X = [0, \infty)$ be endowed with the Euclidean metric d. Let $fx = 2x$ and $Sx = \{0\} \cup [1, 2x + 2]$ for each $x \in X$. Let $\{x_n\}$ be a sequence in X such that*

$$\lim_{n\to\infty} fx_n = t, \lim_{n\to\infty} Sx_n = A \in CB(X),$$

then $t \in A$ if and only if $t = 0$ or $t \ge 1$. Indeed, if $fx_n \to t$, then $x_n \to \frac{t}{2}$ and so $Sx_n \to A = \{0\} \cup [1, t + 2]$.

Now, if $t \in A$, then $t \in \{0\} \cup [1, t + 2]$, that is, $t = 0$ or $t \ge 1$. On the contrary, if $t = 0$, then $x_n \to 0$ and so $Sx_n \to \{0\} \cup [1, 2] = A$. Thus $t \in A$. If $t \ge 1$, then $x_n \to \frac{t}{2} \ge \frac{1}{2}$ and so $Sx_n \to \{0\} \cup [1, t + 2] = A$, that is, $t \in A$.

Now, for $t = 0$, $d(t, ft) \le \limsup_{n\to\infty} H(A, Sfx_n)$ since $d(t, ft) = 0$. Therefore, f and S are compatible of type (I). On the other hand, for $t = 0$, we consider the sequence $\{x_n\}$ defined by $x_n = \frac{1}{n}$, then $0 \ne \lim_{n\to\infty} H(Sfx_n, fSx_n) \not\le \lim_{n\to\infty} H(Sfx_n, Sx_n) = 0$. Similarly, for $t \ge 1$, we consider the sequence $\{x_n\}$ defined by $x_n = \frac{t}{2} + \frac{1}{n}$, then $0 \ne \lim_{n\to\infty} H(Sfx_n, fSx_n) \not\le \lim_{n\to\infty} H(Sfx_n, Sx_n) = 0$. This shows that f and S are not f-weak compatible as well as compatible.

Proposition 1. *Let $f : X \to X$ and $S : X \to CB(X)$ two mappings. If f and S are compatible of type (I) and $fz \in Sz$ for some $z \in X$, then $d(fz, ffz) \le H(Sz, Sfz)$.*

Proof. Let $\{x_n\}$ be a sequence in X defined by $x_n = z$ for $n = 1, 2, 3, \ldots$ and $fz \in Sz$ for some $z \in X$. Then we have $fx_n \to fz$ and $Sx_n \to Sz$. Since f and S are compatible of type (I), we have

$$d(fz, ffz) \le \limsup_{n\to\infty} H(Sz, Sfx_n) = H(Sz, Sfz).$$

□

2 Main result

Now we give our main theorem.

Theorem 1. *Let (X, d) be a complete metric space. Let $f, g : X \to X$ and $S, T : X \to CB(X)$ be mappings such that f and S as well as g and T are compatible of type (I). Assume $T(X) \subseteq f(X)$, $S(X) \subseteq g(X)$ and, for all $x, y \in X$*

$$\begin{aligned} H(Sx, Ty) \quad &\leq \quad \alpha \max\{d(fx, gy), [d(fx, Sx) + d(gy, Ty)]/2, \\ &\quad\quad [d(fx, Ty) + d(gy, Sx)]/2\}, \end{aligned} \tag{2.1}$$

where $\alpha \in (0, 1)$. If f or g is continuous, then f, g, S and T have a common fixed point.

Proof. Let x_0 be an arbitrary point in X and let $k > 1$ so that $\alpha k < 1$. We shall construct two sequences $\{x_n\}$ and $\{y_n\}$ of elements in X and a sequence $\{A_n\}$ of elements in $CB(X)$. Since $S(X) \subseteq g(X)$, there exists $x_1 \in X$ such that $y_1 = gx_1 \in Sx_0$. Then there exists an element $y_2 = fx_2 \in Tx_1 = A_1$, because $T(X) \subseteq f(X)$, such that

$$d(y_1, y_2) = d(gx_1, fx_2) \leq kH(Sx_0, Tx_1).$$

Since $S(X) \subseteq g(X)$, we may choose $x_3 \in X$ such that $y_3 = gx_3 \in Sx_2 = A_2$ and

$$d(y_2, y_3) \leq kH(Tx_1, Sx_2).$$

By induction we produce the sequences $\{x_n\}, \{y_n\}$ and $\{A_n\}$ such that

$$y_{2n+1} = gx_{2n+1} \in Sx_{2n} = A_{2n}, \tag{2.2}$$

$$y_{2n+2} = fx_{2n+2} \in Tx_{2n+1} = A_{2n+1}, \tag{2.3}$$

$$d(y_{2n+1}, y_{2n}) \leq kH(Sx_{2n}, Tx_{2n-1}), \tag{2.4}$$

$$d(y_{2n+1}, y_{2n+2}) \leq kH(Sx_{2n}, Tx_{2n+1}) \tag{2.5}$$

for every $n \in N$. Letting $x = x_{2n}, y = x_{2n+1}$ in (2.1), we have successively

$$\begin{aligned} H(Sx_{2n}, Tx_{2n+1}) \quad &\leq \quad \alpha \max\{d(fx_{2n}, gx_{2n+1}), \\ &\quad\quad [d(fx_{2n}, Sx_{2n}) + d(gx_{2n+1}, Tx_{2n+1})]/2, \\ &\quad\quad [d(fx_{2n}, Tx_{2n+1}) + d(gx_{2n+1}, Sx_{2n})]/2\} \end{aligned}$$

and so

$$\begin{aligned}
H(Sx_{2n}, Tx_{2n+1}) &= H(A_{2n}, A_{2n+1}) \\
&\leq \alpha \max\{d(y_{2n}, y_{2n+1}), [d(y_{2n}, A_{2n}) + \\
&\quad d(y_{2n+1}, A_{2n+1})]/2, \\
&\quad [d(y_{2n}, A_{2n+1}) + d(y_{2n+1}, A_{2n})]/2\} \\
&\leq \alpha \max\{d(y_{2n}, y_{2n+1}), [d(y_{2n}, y_{2n+1}) + \\
&\quad d(y_{2n+1}, y_{2n+2})]/2, \\
&\quad [d(y_{2n}, y_{2n+2}) + 0]/2\} \\
&\leq \alpha \max\{d(y_{2n}, y_{2n+1}), [d(y_{2n}, y_{2n+1}) + \\
&\quad d(y_{2n+1}, y_{2n+2})]/2\}. \qquad (2.6)
\end{aligned}$$

Thus

$$d(y_{2n+1}, y_{2n+2}) \leq k\alpha d(y_{2n}, y_{2n+1}). \qquad (2.7)$$

Similarly we obtain

$$H(Sx_{2n}, Tx_{2n-1}) = H(A_{2n}, A_{2n-1}) \leq \alpha d(y_{2n-1}, y_{2n}) \qquad (2.8)$$

and so

$$d(y_{2n}, y_{2n+1}) \leq k\alpha d(y_{2n-1}, y_{2n}). \qquad (2.9)$$

Since $k\alpha < 1$ it follows from (2.7), (2.9) that $\{y_n\}$ is a Cauchy sequence. Hence there exists $z \in X$ such that $y_n \to z$. Therefore, $gx_{2n+1} \to z$ and $fx_{2n} \to z$. Also from (2.6) and (2.8) and the fact that $\{y_n\}$ is Cauchy sequence it follows that $\{A_k\}$ is Cauchy sequence in the complete metric space $(CB(X), H)$. Thus $A_k \to A \in CB(X)$. This implies $Tx_{2n+1} \to A$ and $Sx_{2n} \to A$ and therefore $z \in A$, because

$$d(z, A) = \lim_{n\to\infty} d(y_n, A) \leq \lim_{n\to\infty} H(A_{n-1}, A_n) = 0.$$

Now suppose that g is continuous. Since g and T are compatible of type (I), we have

$$d(z, gz) \leq \limsup_{n\to\infty} H(A, Tgx_{2n+1}) \qquad (2.10)$$

and $ggx_{2n+1} \to gz$. Setting $x = x_{2n}$ and $y = gx_{2n+1}$ in (2.1), we have

$$\begin{aligned}
H(Sx_{2n}, Tgx_{2n+1}) &\leq \alpha \max\{d(fx_{2n}, ggx_{2n+1}), \\
&\quad [d(fx_{2n}, Sx_{2n}) + d(ggx_{2n+1}, Tgx_{2n+1})]/2, \\
&\quad [d(fx_{2n}, Tgx_{2n+1}) + d(ggx_{2n+1}, Sx_{2n})]/2\},
\end{aligned}$$

taking limit superior we have

$$\begin{aligned}
\limsup_{n\to\infty} H(A, Tgx_{2n+1}) &\le \alpha \max\{d(z,gz), [d(z,A) + \\
&\quad \limsup_{n\to\infty} d(gz, Tgx_{2n+1})]/2, \\
&\quad [\limsup_{n\to\infty} d(z, Tgx_{2n+1}) + d(gz, A)]/2\}, \\
&\le \alpha \max\{d(z,gz), [0 + d(gz,z) + \\
&\quad \limsup_{n\to\infty} d(z, Tgx_{2n+1})]/2, \\
&\quad [\limsup_{n\to\infty} d(z, Tgx_{2n+1}) + d(gz, z)]/2\}, \\
&\le \alpha \max\{d(z,gz), [d(gz,z) + \\
&\quad \limsup_{n\to\infty} H(A, Tgx_{2n+1})]/2, \\
&\quad [\limsup_{n\to\infty} H(A, Tgx_{2n+1}) + \\
&\quad d(gz,z)]/2\}, \qquad (2.11)
\end{aligned}$$

From (2.10) and (2.11) we have $d(z, gz) = 0$ and so $z = gz$.

Again setting $x = x_{2n}$ and $y = z$ in (2.1) we have

$$\begin{aligned}
H(Sx_{2n}, Tz) &\le \alpha \max\{d(fx_{2n}, gz), [d(fx_{2n}, Sx_{2n}) + d(gz, Tz)]/2, \\
&\quad [d(fx_{2n}, Tz) + d(gz, Sx_{2n})]/2\}, \\
&= \alpha \max\{d(fx_{2n}, z), [d(fx_{2n}, Sx_{2n}) + d(z, Tz)]/2, \\
&\quad [d(fx_{2n}, Tz) + d(z, Sx_{2n})]/2\},
\end{aligned}$$

and allowing $n \to \infty$ we have $H(A, Tz) = 0$ and so $Tz = A$. Since $z \in A$, then $z \in Tz$.

Now since $T(X) \subseteq f(X)$ there exists a point $w \in X$ such that $fw = z \in Tz$. Now setting $x = w$ and $y = z$ in (2.1) we have

$$\begin{aligned}
H(Sw, Tz) &\le \alpha \max\{d(fw, gz), [d(fw, Sw) + d(gz, Tz)]/2, \\
&\quad [d(fw, Tz) + d(gz, Sw)]/2\}
\end{aligned}$$

that is

$$H(Sw, A) = 0.$$

This shows that $fw = z \in Sw = Tz = A$. Since f and S are compatible of type (I) and $fw \in Sw$, then using Preposition 1, we have $d(fw, ffw) \le H(Sw, Sfw)$ and so

$$d(z, fz) \le H(A, Sz). \qquad (2.12)$$

Again setting $x = z = y$ in (2.1), we have

$$\begin{aligned} H(Sz, Tz) &\leq \alpha \max\{d(fz, gz), [d(fz, Sz) + d(gz, Tz)]/2, \\ &\quad [d(fz, Tz) + d(gz, Sz)]/2\} \end{aligned}$$

that is

$$H(Sz, A) \leq \alpha d(fz, z). \tag{2.13}$$

From (2.12) and (2.13), we have $d(z, fz) = 0$ that is $z = fz = gz \in A = Tz = Sz$.

The other case, f is continuous, can be disposed of following a similar argument as above. □

We have the following corollary of the Theorem 1, which is the multivalued version of Corollary 3.1 of [7].

Corollary 1. *Let (X, d) be a complete metric space. Let $f, g : X \to X$ and $S, T : X \to CB(X)$ be mappings such that f and S as well as g and T are compatible of type (I). Assume $T(X) \subseteq f(X)$, $S(X) \subseteq g(X)$ and for all $x, y \in X$*

$$H(Sx, Ty) \leq \alpha d(fx, gy)$$

where $\alpha \in (0, 1)$. If f or g is continuous, then f, g, S and T have a common fixed point.

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