THE FOURIER SPECTRAL METHOD FOR THE SIVASHINSKY EQUATION

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Abstract

In this paper, a Fourier spectral method for solving the Sivashinsky equation with periodic boundary conditions is developed. We establish semi-discrete and fully discrete schemes of the Fourier spectral method. A fully discrete scheme is constructed in such a way that the linear part is treated implicitly and the nonlinear part, explicitly. We use an energy estimation method to obtain the error estimates for the approximate solutions. We perform some numerical experiments as well.

1 Introduction

Spectral methods provide a computational approach which has achieved substantial popularity over the last three decades. They have gained recognition for their highly accurate computations of a wide class of physical problems in the field of computational fluid dynamics. Fourier spectral methods, in particular, have become increasingly popular for solving partial differential

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equations. They are also very useful in obtaining highly accurate solutions to the partial differential equations [2, 9, 1].

The purpose of this paper is to develop a Fourier spectral method for solving numerically the Sivashinsky equation with periodic boundary conditions. We consider one dimension nonlinear evolutionary equation as follows [10, 5, 11]:

$$\frac{\partial u}{\partial t} + \frac{\partial^4 u}{\partial x^4} + \frac{\partial}{\partial x}\left[(2-u)\frac{\partial u}{\partial x}\right] + \alpha u = 0, \tag{1.1}$$

where $\alpha > 0$ is constant.

In particular, we consider the mathematical model of the Sivashinsky equation with initial-boundary value problem, as follows:

$$\frac{\partial u}{\partial t} + \frac{\partial^4 u}{\partial x^4} + \alpha u = \frac{\partial^2 f(u)}{\partial x^2}, \qquad x \in \Omega, \qquad t \in (0, T], \tag{1.2}$$

$$\frac{\partial u}{\partial x}(-L, t) = \frac{\partial u}{\partial x}(L, t), \qquad t > 0, \tag{1.3}$$

$$\frac{\partial^3 u}{\partial x^3}(-L, t) = \frac{\partial^3 u}{\partial x^3}(L, t), \qquad t > 0, \tag{1.4}$$

$$u(x, 0) = u_0(x), \qquad \in \Omega, \tag{1.5}$$

where $f(u) = \frac{1}{2}u^2 - 2u$, u_0 is a given function, $T > 0$, and $\Omega = (-L, L)$.

The Sivashinsky equation arises in the modelling of directional solidification of a dilute binary alloy [10]. The dependant variable $u(x, t)$ is the location of the solid-liquid interface. The equation may be formally derived asymptotically from the one-sided model for directional solidification. The diffusion of the solute in the solid is neglected and thermal conductivities, densities and specific heats of the liquid and solid are assumed to be same.

Numerical methods for Sivashinsiky equation can be found in many references [7, 8]. Omrani [8] studied the error estimates of semidiscrete finite element method for the approximation of the Sivashinsky problem. He also developed a fully discrete scheme based upon the backward Galerkin scheme, a linearized backward Euler method and Crank-Nicolson-Galerkin scheme for the Sivashinsky equation. In [7] Omrani used the finite difference method for the approximate solution of the Sivashinsky equation. Cohen and Peletier [3] calculated the number of steady states for Sivashinsky equation. Daniel [4] reduced the Sivashinsky equation into ordinary differential equation and found the solution in the spherical coordinates.

Some studies have been carried out for the solution of the Sivashinsiky equation using numerical and approximate methods. However to our knowledge, there are no studies of the solution of the Sivashinsky equation using the Fourier spectral method. In this paper, we develop semi discrete and fully discrete Fourier spectral schemes for the Sivashinsky equation (1.1).

The layout of the paper is as follows: We introduce notation and lemmas in section 2, In section 3, we consider a semi discrete Fourier spectral approximation. In section 4, we consider a fully discrete implicit scheme and prove convergence to the solution of the associated continuous problem. In section 5, we perform some numerical experiments as well.

2 Notations and Lemmas

We denote by $(\cdot,\cdot)$ and $\|\cdot\|_0$ the inner product and the norm of $L^2(\Omega)$ defined by

$$(u,v)=\int_\Omega u(x)v(x)dx, \qquad \|u\|_0^2=(u,u),$$

where $\Omega=(0,1)$. Let N be a positive integer and V_N be the set of all trigonometric polynomials of degree at most N, that is

$$V_N=\operatorname{span}\left\{\frac{1}{\sqrt{2L}}e^{i\pi\ell.x/L}:-N/2\le\ell\le N/2\right\}.$$

The L^2 orthogonal projection operator $P_N:L^2(\Omega)\to V_N$ is a mapping such that for any $u\in L^2(\Omega)$,

$$(u-P_Nu,v)=0, \qquad \forall v\in V_N.$$

Let $L^\infty(\Omega)$ denote the Lebesgue space with norm $\|u\|_\infty=\operatorname{ess\,sup}_{x\in\Omega}|u(x)|$ and $H_p^m(\Omega)$ denote the periodic Sobolev space with the norm

$$\|u\|_m=\left(\sum_{|\ell|\le m}\|D^\ell u\|_0^2\right)^{1/2}$$

. Denote

$$L^2\left(0,T;H_p^m(\Omega)\right)=\left\{u(x,t)\in H_p^m(\Omega);\int_0^T\|u\|_m^2dt<+\infty\right\},$$

$$L^{\infty}\left(0,T;H_p^m(\Omega)\right)=\left\{u(x,t)\in H_p^m(\Omega);\ \sup_{0\le t\le T}\|u\|_m^2<+\infty\right\}.$$

Lemma 1. *[2] For any real $0\le\mu\le\sigma$, and $u\in H_p^{\sigma}(\Omega)$, then*

$$\|u-P_Nu\|_{\mu}\le cN^{\mu-\sigma}|u|_{\sigma}.$$

For the discretization in time t, let τ be the mesh spacing of the variable t and we set

$$S_{\tau}=\left\{t=k\tau:1\le k\le\left[\frac{T}{\tau}\right]\right\}.$$

For simplicity $u(x,t)$ is denoted by $u(t)$ or u usually. We define the following difference quotient:

$$u_{\widehat{t}}(t)=\frac{1}{2\tau}[u(t+\tau)-u(t-\tau)],\qquad \widehat{u}(t)=\frac{1}{2}[u(t+\tau)+u(t-\tau)].$$

3 Semi-discrete scheme

The semi-discrete Fourier spectral scheme of the equation (1.2)-(1.5) is to find $u_N(t)\in V_N$, such that, $\forall w_N\in V_N$, we have:

$$\begin{cases}\left(\dfrac{\partial u_N}{\partial t},w_N\right)+\alpha(u_N,w_N)+\left(\dfrac{\partial^2u_N}{\partial x^2},\dfrac{\partial^2w_N}{\partial x^2}\right)=\left(f(u_N),\dfrac{\partial^2w_N}{\partial x^2}\right),\\ u_N(x,0)=P_Nu(0).\end{cases}\tag{3.1}$$

Now we estimate the error $\|u(t)-u_N(t)\|_0$. Denote $e_N=P_Nu(t)-u_N(t)$, from (1.2) and (3.1), $\forall w_N\in V_N$, we have:

$$\begin{cases}\left(\dfrac{\partial e_N}{\partial t},w_N\right)+\alpha(e_N,w_N)+\left(\dfrac{\partial^2e_N}{\partial x^2},\dfrac{\partial^2w_N}{\partial x^2}\right)=\left(f(u)-f(u_N),\dfrac{\partial^2w_N}{\partial x^2}\right),\\ u_N(x,0)=P_Nu(0).\end{cases}\tag{3.2}$$

Choosing $w_N = e_N$ in (3.2) and by applying the Cauchy-Schwartz inequality as well as the algebraic inequality, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|e_N\|_0^2 + \left\|\frac{\partial^2 e_N}{\partial x^2}\right\|_0^2 + \alpha\|e_N\|_0^2 &= \left(f(u) - f(u_N), \frac{\partial^2 e_N}{\partial x^2}\right) \\ &\leq \|(f(u) - f(u_N)\|_0 \left\|\frac{\partial^2 e_N}{\partial x^2}\right\|_0 \\ &\leq \|(f(u) - f(u_N)\|_0^2 + \left\|\frac{\partial^2 e_N}{\partial x^2}\right\|_0^2.\end{aligned} \tag{3.3}$$

But

$$(f(u) - f(u_N) = \phi\prime(u_N - \theta(u - u_N))(u - u_N), \qquad 0 < \theta < 1.$$

Since

$$\|u\|_\infty \leq c, \qquad \|u_N\|_\infty \leq c,$$

we have

$$\phi\prime(u_N - \theta(u - u_N)) \leq c.$$

Thus

$$\begin{aligned}\|f(u) - f(u_N)\|_0 &\leq \|(u - u_N)\|_0 \\ &\leq c\left(\|(u - u_N)\|_0 + \|e_N\|_0\right).\end{aligned} \tag{3.4}$$

Substituting the value of (3.3) and (3.4) in (3.2), we obtain

$$\frac{d}{dt}\|e_N\|_0^2 \leq c_1\|e_N\|_0^2 + c_2\|(u - P_N u)\|_0^2, \tag{3.5}$$

where c_1 and c_2 are constants independent of N. By Gronwall's lemma, we have

$$\|e_N\|_0^2 \leq \|e_N(0)\|_0^2 + c\int_0^t \|(u(s) - P_N u(s))\|_0^2 ds. \tag{3.6}$$

Thus we obtain the following theorem

Theorem 1. *Let $u_0 \in H_0^1(\Omega)$ and $u(t)$ be the solution for periodic initial value problem (1.2-1.5). If $u_N(t)$ is the solution of the semi-discrete approximation (3.1), then there exists a constant c independent of N, such that*

$$\|u(t) - u_N(t)\|_0 \leq cN^{-m}\left(\|u(t)\|_m + \left(\int_0^t \|u(s)\|_m^2 ds\right)^{1/2}\right).$$

4 A fully discrete scheme

In this section, we give a fully discrete scheme of the Fourier spectral method for the Sivashinsky equation. Using the modified leap-frog scheme such that the linear part is treated implicitly and the nonlinear part explicitly, we obtain the fully discrete spectral scheme for solving (1.1). Find $u_N(t) \in S_N$, such that, $\forall w_N \in V_N$, we have:

$$\begin{cases} (u_{N\widehat{t}}, w_N) + \alpha(\widehat{u}_N, w_N) + \left(\dfrac{\partial^2 \widehat{u}_N}{\partial x^2}, \dfrac{\partial^2 w_N}{\partial x^2}\right) = \left(f(u_N), \dfrac{\partial^2 w_N}{\partial x^2}\right), \\ u_N(x,0) = P_N u(0), \qquad u_N(x,\tau) = P_N\left(u(0) + \tau \dfrac{\partial u}{\partial t}(0)\right). \end{cases} \tag{4.1}$$

Let

$$e = u - u_N = (u - P_N u) + (P_N u - u_N) = \xi + \psi,$$

where

$$\xi = u - P_N u, \qquad \psi = P_N u - u_N.$$

Subtracting (1.2) from (4.1), we obtain

$$\begin{cases} (\psi_{\widehat{t}}, w_N) + \alpha\left(\widehat{\psi}, w_N\right) + \left(\dfrac{\partial^2 \widehat{\psi}}{\partial x^2}, \dfrac{\partial^2 w_N}{\partial x^2}\right) \\ \qquad = \left(f(u_N) - f(u), \dfrac{\partial^2 w_N}{\partial x^2}\right) + \left(\dfrac{\partial u}{\partial t} - P_N u_{\widehat{t}}, w\right) \\ \qquad +\alpha(u - P_N \widehat{u}, w_N) + \left(\dfrac{\partial^2}{\partial x^2}(u - P_N \widehat{u}), \dfrac{\partial^2}{\partial x^2} w\right), \quad \forall w_N \in V_N, \\ \psi(0) = P_N u(0) - u_N(0), \qquad \psi(\tau) = P_N\left(u(\tau) - \left(u(o) + \tau \dfrac{\partial u}{\partial t}(0)\right)\right). \end{cases} \tag{4.2}$$

Taking $w = 2\widehat{\psi}(t)$ in (4.2), we obtain

$$\begin{aligned} \left(\|\psi(t)\|_0^2\right)_{\widehat{t}} + 2\alpha \left\|\widehat{\psi}(t)\right\|_0^2 \leq c \Bigg(&\|f(\widehat{u}_N) - f(u)\|_0^2 + \left\|\frac{\partial u}{\partial t} - P_N u_{\widehat{t}}(t)\right\|_0^2 \\ &+ \|u - P_N \widehat{u}\|_0^2 + \left\|\frac{\partial^2}{\partial x^2}(u - P_N \widehat{u}\right\|_0^2 \Bigg). \end{aligned} \tag{4.3}$$

We estimate the right hand side of (4.3)

$$\|f(u_N) - f(u)\|_0 \leq c(\|u - u_N\|_0) \leq c(\|\xi\|_0 + \|\psi\|_0). \tag{4.4}$$

The second term on the right hand side of equation (4.3) can be estimated as below:

$$\left\| \frac{\partial u}{\partial t}(t) - P_N u_{\widehat{t}}(t) \right\|_0^2 \leq \left\| \frac{\partial u}{\partial t}(t) - u_{\widehat{t}}(t) \right\|_0^2 + \|u_{\widehat{t}}(t) - P_N u_{\widehat{t}}(t)\|_0^2 . \tag{4.5}$$

By using Taylor's Theorem with integral remainder

$$\left\| \frac{\partial u}{\partial t}(t) - u_{\widehat{t}}(t) \right\|_0^2 \leq c\tau^{3/2} \left(\int_{t-\tau}^{t+\tau} \left\| \frac{\partial^3 u}{\partial t^3}(s) \right\|_0^2 ds \right)^{1/2} ,$$

$$\begin{aligned} \|u_{\widehat{t}}(t) - P_N u_{\widehat{t}}(t)\|_0^2 &= \frac{1}{2\tau} \|(u(t+\tau) - P_N u(t+\tau)) - (u(t-\tau) - P_N u(t-\tau))\|_0 \\ &= \frac{1}{2\tau} \|\xi(t+\tau) - \xi(t-\tau)\|_0 . \end{aligned}$$

By using Taylor's Theorem with integral remainder

$$\frac{1}{2\tau} \|\xi(t+\tau) - \xi(t-\tau)\|_0 \leq \frac{c}{\tau} \left(\int_{t-\tau}^{t+\tau} \left\| \frac{\partial \xi}{\partial t}(s) \right\|_0^2 ds \right)^{1/2} .$$

Substituting the above estimate in to (4.5), we obtained

$$\left\| \frac{\partial u}{\partial t}(t) - P_N u_{\widehat{t}}(t) \right\|_0 \leq$$

$$\leq \frac{c}{\tau} \left(\int_{t-\tau}^{t+\tau} \left\| \frac{\partial \xi}{\partial t}(s) \right\|_0^2 ds \right)^{1/2} + c\tau^{3/2} \left(\int_{t-\tau}^{t+\tau} \left\| \frac{\partial^3 u}{\partial t^3}(s) \right\|_0^2 ds \right)^{1/2} . \tag{4.6}$$

The third and fourth term on the right hand side of (4.3) can be estimated in a similar manner

$$\|u - P_N \widehat{u}\|_0 \leq \frac{1}{2}(\|\xi(t+\tau) - \xi(t-\tau)\|_0) + \left\| u(t) - \frac{1}{2}u(t+\tau) - \frac{1}{2}u(t-\tau) \right\|_0 ,$$

$$\|u - P_N \widehat{u}\|_0 \leq \frac{1}{2}(\|\xi(t+\tau) - \xi(t-\tau)\|_0) + c\tau^{3/2} \left(\int_{t-\tau}^{t+\tau} \left\| \frac{\partial^2 u}{\partial t^2}(s) \right\|_0^2 ds \right)^{1/2} , \tag{4.7}$$

and

$$\left\|\frac{\partial^2}{\partial x^2}(u-P_N\widehat{u})\right\|_0 \le \frac{1}{2}(\|\xi(t+\tau)-\xi(t-\tau)\|_2)+c\tau^{3/2}\left(\int_{t-\tau}^{t+\tau}\left\|\frac{\partial^2 u}{\partial t^2}(s)\right\|_2^2 ds\right)^{1/2}. \tag{4.8}$$

By applying Lemma 1, we obtain

$$\|\widehat{\xi}(t)\| \le cN^{-m}\left(\|u(t+\tau)+u(t-\tau)\|\right).$$

Putting the above estimates into (4.4), (4.6), (4.7) and (4.8) in to (4.3), we obtain

$$\begin{aligned}\left(\|\psi(t)\|_0^2\right)_{\hat{t}}+2\alpha\left\|\widehat{\psi}(t)\right\|_0^2 &\le cN^{-2m}(\|u(t+\tau)\|_0^2+\|u(t-\tau)\|_0^2)+c\|\widehat{\psi}(t)\|_0^2\\ &+\frac{c}{\tau}\int_{t-\tau}^{t+\tau}\left\|\frac{\partial \xi}{\partial t}(s)\right\|_0^2 ds+cN^{4-2m}\left(\|u(t+\tau)\|_2^2+\|u(t-\tau)\|_2^2\right)\\ &+c\tau^3\left(\int_{t-\tau}^{t+\tau}\left(\left\|\frac{\partial^2 u}{\partial t^2}(s)\right\|_0^2+\left\|\frac{\partial^3 u}{\partial t^3}(s)\right\|_0^2+\left\|\frac{\partial^2 u}{\partial t^2}(s)\right\|_2^2\right)ds\right).\end{aligned} \tag{4.9}$$

By summing up (4.9) for all $t \in S_\tau$ and $t' \le t-\tau$, we obtain

$$\begin{aligned}\|\psi(t)\|_0^2 \le \|\psi(0)\|_0^2+\|\psi(\tau)\|_0^2+c\tau N^{4-2m}\left(\sum_{t'=\tau}^{t-\tau}\|\widehat{u}(t')\|_2^2\right)+c\int_0^T\left\|\frac{\partial \xi}{\partial t}(s)\right\|_0^2 ds\\ +c\tau^4\left(\int_0^T\left(\left\|\frac{\partial^2 u}{\partial t^2}(s)\right\|_0^2+\left\|\frac{\partial^3 u}{\partial t^3}(s)\right\|_0^2+\left\|\frac{\partial^2 u}{\partial t^2}(s)\right\|_2^2\right)ds\right),\end{aligned} \tag{4.10}$$

and

$$\|\psi(\tau)\|_0^2 \le c\tau^4\|u\|_{C^2(0,T;L^2)}^2 \qquad \|\psi(0)\|_0^2 \le cN^{-2m}\|u(0)\|_m^2.$$

By applying Gronwall's Lemma, we have the following theorem

Theorem 2. *Assume τ is sufficiently small, $m>2$ and the solution of (1.2)-(1.5) satisfies $u \in L^2(0,T;H^m(\Omega))\bigcap C^2\left(0,T;L^2(\Omega)\right)$, $\frac{\partial u}{\partial t} \in L^2\left(0,T;L^2(\Omega)\right)$, $\frac{\partial^2 u}{\partial t^2} \in L^2\left(0,T;L^2(\Omega)\right)\bigcap H^2\left(0,T;L^2(\Omega)\right)$, $\frac{\partial^3 u}{\partial t^3} \in L^2\left(0,T;L^2(\Omega)\right)$ where u_N is the solution of (4.1). Then there exists a constant c, independent of τ and N such that*

$$\|u(t)-u_N(t)\|_0 \le c\left(N^{-m}K_1+\tau K_2\right),$$

where

$$K_1 = \left(\|u(0)\|_m^2 + \sum_{t'=\tau}^{t-\tau} \|\widehat{u}(t')\|_2^2 \right)^{1/2},$$

$$K_2 = \left(\int_0^T \left(\left\| \frac{\partial^2 u}{\partial t^2}(s) \right\|_0^2 + \left\| \frac{\partial^3 u}{\partial t^3}(s) \right\|_0^2 + \left\| \frac{\partial^2 u}{\partial t^2}(s) \right\|_2^2 \right) ds \right)^{1/2}.$$

5 Numerical Results

In this section, we present some numerical results of our scheme for the Sivashinsky equation (1.2)-(1.5). The numerical computations were carried out in MATLAB. We employ the standard spectral method (discrete Fourier transform) for the spatial discretization of the equation. We use modified leapfrog scheme for the time integration, in such a way that linear terms are treated implicitly and the nonlinear terms explicitly.

We suppose that the periodic domain for the solution of the Sivashinsky equation is $\Omega = (-L, L)$. First we introduce some notations for the discrete Fourier transformation. We assume that the domain in space is equidiscretized with spacing $h = \frac{2\pi}{N}$, where the integer N is even. The spatial grid points are given by $x_j = L(jh - \pi)/\pi$ of the domain Ω with $j = 0, 1, ..., N$. The formula for discrete Fourier transform is

$$\widehat{u}_k = \frac{1}{N} \sum_{j=0}^{N} u_j e^{-ikx_j}, \qquad k = -N/2 + 1.., -1, 0, 1, ..., N/2, \tag{5.1}$$

and the inverse discrete Fourier transform is given by

$$u_j = \sum_{-N/2+1}^{N/2} \widehat{u}_k e^{ix_j}, \qquad j = 0, 1, ..., N. \tag{5.2}$$

We set the parameters to be $N = 32$, $T = 1.0$ and the step length along the time direction is taken to be $\tau = 0.001$. We consider the Sivashinsky equation (1.2) with initial condition

$$u(x, 0) = cos\left(\frac{x}{2}\right).$$

The exact solution of the equation (1.2)-(1.5) is unknown. Therefore we present the graph of the approximate solution in Figure 1, Figure 2, and in

Figure 3 with different values of α. Momani [11] has used Adomain's decomposition method for the same problem with $\alpha = 0.5$. The results indicate that the wave solution is bifurcated into three waves. However the Fourier spectral method, we have developed, does not indicate the same result (see Figure 2). Otherwise there is good agreement between the results of Fourier spectral method and the Adomain's decomposition method.

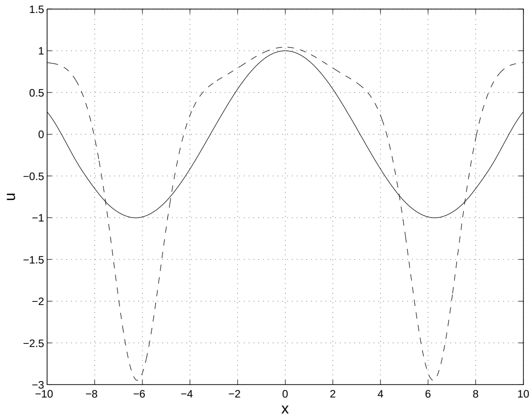

Figure 1: The graph of the approximate solution at t=0 (solid line) and t=1 (dotted line) at α=0.1

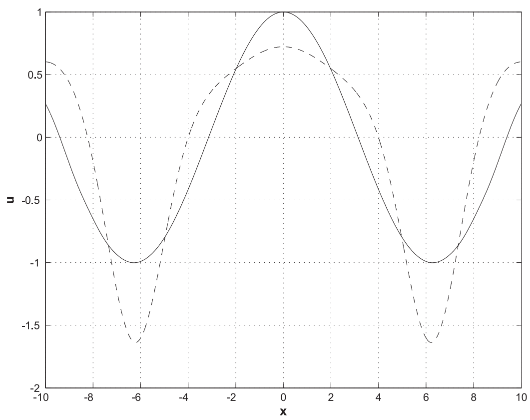

Figure 2: The graph of the approximate solution at t=0 (solid line) and t=1 (dotted line) at α=0.5

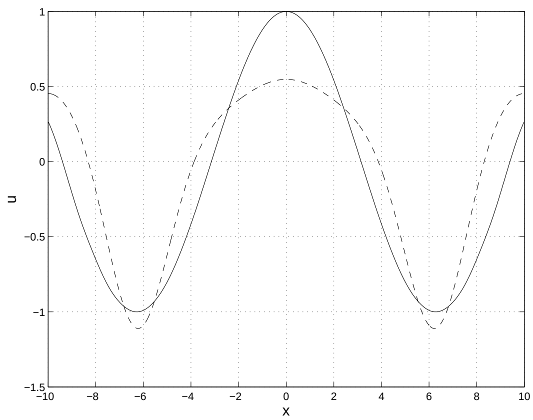

Figure 3: The graph of the approximate solution at t=0 (solid line) and t=1 (dotted line) at α=0.8

6 Conclusion

A Fourier spectral method for the one dimensional Sivashinsky equation has been proposed and the solution is obtained for different values of the parameter α. Error estimates for the semi-discrete and fully discrete schemes are obtained by the energy estimation method. Thus, the proposed method is an efficient technique for the numerical solutions of the partial differential equations.

References

[1] J.P. Boyd, *Chebyshev and Fourier spectral methods*, Springer-Verlag, New York, 2002.

[2] C. Canuto, M.Y. Hussaini, A. Quarteroni, and T.A. Zang, *Spectral methods (fundamental in single domains)* Springer-Verlag, Berlin, 2006.

[3] A.N. Cohen, and L.A. Peletier, *The steady state of one dimensional Sivashinsky equations*, Appl.Math.Lett, **4**(1991), 69-71.

[4] M. Daniel, *Elementary particles as solutions of the Sivashinsky equation*, Physica D: Nonlinear Phenomena, **44**(1990), 502-556.

[5] V.G. Gertsberg and G.I. Sivashinsky, *Large Cells in Nonlinear Rayleigh-Bnard Convection*, Prog. Theo. Phys. **66**(1981), 1219-1229.

[6] S. Momani, *A numerical scheme for the solution of Sivashinsky equation*, Appl. Math. Comp. **168**(2005), 1273-1280.

[7] K. Omrani, *A second order spliting method for a finite difference scheme for the Sivashinsky equation*, Appl. Math. Lett. **16**(2003), 441-445.

[8] K. Omrani, *Numerical methods and error analysis for the nonlinear Sivashinsky equation*, Appl. Math. Comp. **189**(2007), 949-962.

[9] R. Peyret, *Spectral methods in incompressible viscous flow*, Springer-Verlag, New York, 2002.

[10] G.I. Sivashinsky, *On cellular instability in the solidification of a dilute binary alloy*, Physica D: Nonlinear Phenomena, **8**(1983), 243-248.

[11] D.J. Wollkind and L.A. Sigel, *A Nonlinear Stability Analysis of the Freezing of a Dilute Binary Alloy*, Phil. Trans. Roy. Society (London) **268a**(1970), 351-380.

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