An. Şt. Univ. Ovidius Constanţa Vol. 17(2), 2009, 181–190

TWO DIMENSIONAL DIVIDED DIFFERENCES WITH MULTIPLE KNOTS

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Abstract

The notion of two dimensional divided difference was introduced by T. Popoviciu in 1934. Many properties of these differences were obtained by D. V. Ionescu. Other properties of the mentioned differences were obtained by the authors of the present paper.

The focus of the present paper is to establish properties of two dimensional differences in the case of multiple knots. First, we establish some properties of the univariate divided differences with multiple knots: a representation theorem using the determinants and a mean value theorem. Next, one proves the main results of the paper which are a representation theorem for the bivariate divided differences with multiple knots and a mean-value theorem for this kind of divided differences.

1 Introduction

In this section let be $\mathbb{N} = \{1, 2, \dots\}$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $m \in \mathbb{N}_0$, $r_0, r_1, ..., r_m \in \mathbb{N}$, $r_0 + r_1 + \cdots + r_m = M + 1$, $\alpha = \max\{r_0 - 1, r_1 - 1, \dots, r_m - 1\}$, $I \subset \mathbb{R}$ be an interval and $D^\alpha(I)$ the set of all real functions f, α times differentiable on I. If $\alpha = 0$, we consider that $D^0(I) = F(I) = \{f | f : I \to \mathbb{R}\}$. Let $x_0, x_1, \dots, x_m \in I$ be distinct knots.

The m-th order divided differences of $f \in F(I)$ on the knots $x_0, x_1, \dots, x_m$ is defined by

$$[x_0, x_1, \dots, x_m; f] = \sum_{k=0}^{m} \frac{f(x_k)}{u'(x_k)}\,, \tag{1}$$

Key Words: divided differences; two dimensional divided differences with multiple knots.
Mathematics Subject Classification: 41A05, 41A63
Received: January, 2009
Accepted: September, 2009

where $u(x) = (x - x_0)(x - x_1) \cdot \ldots \cdot (x - x_m)$.

In the following, the divided difference with multiple knots

$$\big[\underbrace{x_0, x_0, \ldots, x_0}_{r_0 \text{ times}}, \underbrace{x_1, x_1, \ldots, x_1}_{r_1 \text{ times}}, \ldots, \underbrace{x_m, x_m, \ldots, x_m}_{r_m \text{ times}}; f\big]$$

will be denoted by $[x_0^{(r_0)}, x_1^{(r_1)}, \ldots, x_m^{(r_0)}; f]$, where $f \in D^\alpha(I)$.

It is known that (see [3], [4]) or [6])

$$\big[x_0^{(r_0)}, x_1^{(r_1)}, \ldots, x_m^{(r_m)}; f\big] = \frac{(Wf)\big(x_0^{(r_0)}, x_1^{(r_1)}, \ldots, x_m^{(r_m)}\big)}{V\big(x_0^{(r_0)}, x_1^{(r_1)}, \ldots, x_m^{(r_m)}\big)}, \tag{2}$$

where

$$(Wf)\ x_0^{(r_0)}, x_1^{(r_1)}, \ldots, x_m^{(r_m)} \tag{3}$$

$$= \begin{matrix} 1 & x_0 & \ldots & x_0^{r_0-1} & \ldots & x_0^{M-1} & f(x_0) \\ 0 & 1 & \ldots & (r_0-1)x_0^{r_0-2} & \ldots & (M-1)x_0^{M-2} & f'(x_0) \\ & & & & \ldots & & \\ 0 & 0 & \ldots & (r_0-1)! & \ldots & (M-1)(M-2)\cdot\ldots\cdot(M-r_0+1)x_0^{M-r_0} & f^{(r_0-1)}_{(x_0)} \\ & & & & \ldots & & \\ 1 & x_m & \ldots & x_m^{r_m-1} & \ldots & x_m^{M-1} & f(x_m) \\ 0 & 1 & \ldots & (r_m-1)x_m^{r_m-2} & \ldots & (M-1)x_m^{M-2} & f'(x_m) \\ & & & \ldots & & & \\ 0 & 0 & \ldots & (r_m-1)! & \ldots & (M-1)(M-2)\cdot\ldots\cdot(M-r_m+1)x_m^{M-r_m} & f^{(r_m-1)}_{(x_m)} \end{matrix}$$

and

$$V\big(x_0^{(r_0)}, x_1^{(r_1)}, \ldots, x_m^{(r_m)}\big) \tag{4}$$

$$= \begin{vmatrix} 1 & x_0 & \ldots & x_0^{r_0-1} & \ldots & x_0^{M} \\ 0 & 1 & \ldots & (r_0-1)x_0^{r_0-2} & \ldots & Mx_0^{M-1} \\ & & & \ldots & & \\ 0 & 0 & \ldots & (r_0-1)! & \ldots & M(M-1)\cdot\ldots\cdot(M-r_0+2)x_0^{M-r_0+1} \\ & & & \ldots & & \\ 1 & x_m & \ldots & x_m^{r_m-1} & \ldots & x_m^{M} \\ 0 & 1 & \ldots & (r_{m-1})x_m^{r_m-2} & \ldots & Mx_m^{M-1} \\ & & & \ldots & & \\ 0 & 0 & \ldots & (r_m-1)! & \ldots & M(M-1)\cdot\ldots\cdot(M-r_m+2)x_m^{M-r_m+1} \end{vmatrix}$$

$$= \prod_{k=0}^{m} \prod_{i=0}^{r_k-1} i! \prod_{0\le s<k\le m} (x_k - x_s)^{r_k \cdot r_s}$$

is the generalized determinant of Vandermonde.

If $k \in \{0, 1, \dots, m\}$ and $i \in \{0, 1, \dots, r_k - 1\}$ we denote

$$V_{k,i}\big(x_0^{(r_0)}, x_1^{(r_1)}, \dots, x_m^{(r_m)}\big) \tag{5}$$

$$= (-1)^{M+r_0+r_1+\dots+r_{k-1}+i} \begin{vmatrix} 1 & x_0 & \dots & x_0^{M-1} \\ & & \dots & \\ 0 & 0 & \dots & (M-1)(M-2)\cdot\ldots\cdot(M-i+1)x_k^{M-i} \\ 0 & 0 & \dots & (M-1)(M-2)\cdot\ldots\cdot(M-i-1)x_k^{M-i-2} \\ & & \dots & \\ 0 & 0 & \dots & (M-1)(M-2)\cdot\ldots\cdot(M-r_m+1)x_m^{M-r_m} \end{vmatrix},$$

so the above determinant is obtained from (3) by elimination the line $r_0 + r_1 + \dots + r_{k-1} + i + 1$ and the column $M + 1$.

Theorem 1 *If $f \in D^\alpha(I)$, the identity*

$$\big[x_0^{(r_0)}, x_1^{(r_1)}, \dots, x_m^{(r_m)}; f\big] \tag{6}$$

$$= \frac{1}{V\big(x_0^{(r_0)}, x_1^{(r_1)}, \dots, x_m^{(r_m)}\big)} \sum_{k=0}^{m} \sum_{i=0}^{r_k-1} V_{k,i}\big(x_0^{(r_0)}, x_1^{(r_1)}, \dots, x_m^{(r_m)}\big) f^{(i)}(x_k)$$

holds.

Proof. Taking into account the relation (2), yields

$$\big[x_0^{(r_0)}, x_1^{(r_1)}, \dots, x_m^{(r_m)}; f\big] = \frac{1}{V\big(x_0^{(r_0)}, x_1^{(r_1)}, \dots, x_m^{(r_m)}\big)} \cdot$$

$$\sum_{k=0}^{m} \begin{vmatrix} 1 & x_0 & \dots & x_0^{M-1} & 0 \\ & & \dots & & \\ 0 & 0 & \dots & (M-1)(M-2)\cdot\ldots\cdot(M-r_{k-1}+1)x_{k-1}^{M-r_{k-1}} & 0 \\ 1 & x_k & \dots & x_k^{M-1} & f(x_k) \\ 0 & 1 & \dots & (M-1)x_k^{M-2} & f'(x_k) \\ & & \dots & & \\ 0 & 0 & \dots & (M-1)(M-2)\cdot\ldots\cdot(M-r_k+1)x_k^{M-r_k} & f^{(r_k-1)}_{(x_k)} \\ 1 & x_{k+1} & \dots & x_{k+1}^{M} & 0 \\ & & \dots & & \\ 0 & 0 & \dots & (M-1)(M-2)\cdot\ldots\cdot(M-r_m+1)x_m^{M-r_m} & 0 \end{vmatrix}$$

and developing the above determinant from the column $M + 1$, we obtain the relation (6).

Remark 1 If $r_0 = r_1 = \cdots = r_m = 1$ in Theorem 1 we get the relation (1).

Theorem 2 *The following relation*

$$\left[x_0^{(r_0)}, x_1^{(r_1)}, \ldots, x_m^{(r_m)}; x^k\right] \tag{7}$$

$$= \begin{cases} 0, & \text{if } k \le M-1 \\ 1, & \text{if } k = M \\ r_0x_0 + r_1x_1 + \cdots + r_mx_m, & \text{if } k = M+1 \end{cases}$$

holds, where $x \in I$.

Proof. From (3) the first and the second statement from (7) follow. Taking relation (3) into account, we start from the equality

$$\left[x_0^{(r_0)}, x_1^{(r_1)}, \ldots, x_m^{(r_m)}; (x-x_0)^{r_0}(x-x_1)^{r_1} \cdot \ldots \cdot (x-x_m)^{r_m}\right] = 0.$$

Then, it follows

$$\left[x_0^{(r_0)}, x_1^{(r_1)}, \ldots, x_m^{(r_m)}; x^{M+1} - s_1x^M + s_2x^{M-1} + \cdots + (-1)^{M+1}s_{M+1}\right] = 0,$$

where $s_1 = r_0x_0 + s_1x_1 + \cdots + r_mx_m$, hence

$$\left[x_0^{(r_0)}, x_1^{(r_1)}, \ldots, x_m^{(r_m)}; x^{M+1}\right] = s_1\left[x_0^{(r_0)}, x_1^{(r_1)}, \ldots, x_m^{(r_m)}; x^M\right]$$

and the third statement of (7) follows.

Theorem 3 *Let* $a = \min\{x_0, x_1, \ldots, x_m\}$, $b = \max\{x_0, x_1, \ldots, x_m\}$, $f \in C^{M-1}([a,b])$, *exists* $f^{(M)}$ *on* (a,b) *and* $f^{(l_k)}(x_k) = 0$, *where* $l_k \in \{0, 1, \ldots, r_k - 1\}$, $k \in \{0, 1, \ldots, m\}$. *Then exists* $\xi \in (a,b)$ *such that*

$$\left[x_0^{(r_0)}, x_1^{(r_1)}, \ldots, x_m^{(r_m)}; f\right]\left(1 + (M+1)\xi - \frac{r_0x_0 + r_1x_1 + \cdots + r_mx_m}{V\left(x_0^{(r_0)}, x_1^{(r_1)}, \ldots, x_m^{(r_m)}\right)}\right) \tag{8}$$

$$= \frac{1}{M!}\, f^{(M)}(\xi).$$

Proof. Define the auxiliary function $F : [a,b] \to \mathbb{R}$ by

$$F(x) = (Wf)\left(x, x_0^{(r_0)}, x_1^{(r_1)}, \ldots, x_m^{(r_m)}\right)$$

$$- \left[x_0^{(r_0)}, x_1^{(r_1)}, \ldots, x_m^{(r_m)}; f\right] V\left(x, x_0^{(r_0)}, x_1^{(r_1)}, \ldots, x_m^{(r_m)}\right),$$

for any $x \in [a,b]$, where the first line in $(Wf)\left(x, x_0^{(r_0)}, x_1^{(r_1)}, \ldots, x_m^{(r_m)}\right)$, $V\left(x, x_0^{(r_0)}, x_1^{(r_1)}, \ldots, x_m^{(r_m)}\right)$ is $1 \quad x \quad x^2 \ldots x^M \quad f(x)$, respectively $1 \quad x \quad x^2 \ldots x^M \quad x^{M+1}$. On verifies immediately that $F^{(l_k)}(x_k) = 0$, where $l_k \in \{0,1,\ldots,r_k-1\}$, $k \in \{0,1,\ldots,m\}$, so the function F has $r_0 + r_1 + \cdots + r_m = M+1$ roots. By the generalized Rolle Theorem, there exists a point $\xi \in (a,b)$ such that $F^{(M)}(\xi) = 0$. But

$$F^{(M)}(x) = \begin{vmatrix} 0 & 0 & \ldots & 0 & M! & f^{(M)}(x) \\ 1 & x_0 & \ldots & x_0^{M-1} & x_0^M & f(x_0) \\ & & & \ldots & & \end{vmatrix}$$

$$- \left[x_0^{(r_0)}, x_1^{(r_1)}, \ldots, x_m^{(r_m)}; f\right] \begin{vmatrix} 0 & 0 & \ldots & 0 & M! & (M+1)!x \\ 1 & x_0 & \ldots & x_0^{M-1} & x_0^M & x_0^{M+1} \\ & & & \ldots & & \end{vmatrix}$$

$$= (-1)^{M+2} M! (Wf)\left(x_0^{(r_0)}, x_1^{(r_1)}, \ldots, x_m^{(r_m)}\right)$$

$$+(-1)^{M+3} f^{(M)}(x) V\left(x_0^{(r_0)}, x_1^{(r_1)}, \ldots, x_m^{(r_m)}\right) - \left[x_0^{(r_0)}, x_1^{(r_1)}, \ldots, x_m^{(r_m)}; f\right]$$

$$\cdot \Big((-1)^{M+2} M! (Wx^{M+1})\left(x_0^{(r_0)}, x_1^{(r_1)}, \ldots, x_m^{(r_m)}\right)$$

$$+(-1)^{M+3} (M+1)! x V\left(x_0^{(r_0)}, x_1^{(r_1)}, \ldots, x_m^{(r_m)}\right) \Big)$$

and taking (2) and (7) into account, the relation (8) follows.

In [4] the following mean-value theorem for divided differences with multiple knots is proved.

Theorem 4 *If* $a = \min\{x_0, x_1, \ldots, x_m\}$, $b = \max\{x_0, x_1, \ldots, x_m\}$, $f \in C^{M-1}([a,b])$ *and* $f^{(M)}$ *exists on* (a,b), *then there exists* $\xi \in (a,b)$ *such that*

$$\left[x_0^{(r_0)}, x_1^{(r_1)}, \ldots, x_m^{(r_m)}; f\right] = \frac{1}{M!} f^{(M)}(\xi). \tag{9}$$

In the second section, using this theorem we shall give a mean-value theorem for two dimensional divided differences with multiple knots (Theorem 6).

2 The definition of two dimensional divided differences with multiple knots

In the following let $m, n \in \mathbb{N}_0$, $r_0, r_1, \ldots, r_m, q_0, q_1, \ldots, q_n \in \mathbb{N}$, $r_0 + r_1 + \cdots + r_m = M+1$, $q_0 + q_1 + \cdots + q_n = N+1$, $I, J \subset \mathbb{R}$ intervals, $I \times J$

be a bidimensional interval and $D^{\alpha,\beta}(I \times J)$ the set of all real valued bivariate functions f with the property that exits $\dfrac{\partial f^{i+j}}{\partial x^i \partial y^j}$ on $I \times J$, where $i \in \{0, 1, \dots, \alpha\}$, $j \in \{0, 1, \dots, \beta\}$, $\alpha = \max\{r_0 - 1, r_1 - 1, \dots, r_m - 1\}$ and $p = \max\{q_0 - 1, q_1 - 1, \dots, q_n - 1\}$.

Let $x_0, x_1, \dots, x_m \in I$, $y_0, y_1, \dots, y_n \in J$ be distinct knots. For $y \in J$, we denote by $\left[x_0^{(r_0)}, x_1^{(r_1)}, \dots, x_m^{(r_m)}; f(x,y)\right]_x$ the parametric extension of M-th order divided differences with multiple knots, equivalent the M-th order divided differences of the function $f(\cdot, y) : I \to \mathbb{R}$, $y \in J$, with respect the knots $\underbrace{x_0, x_0, \dots, x_0}_{r_0 \text{ times}}, \underbrace{x_1, x_1, \dots, x_1}_{r_1 \text{ times}}, \dots, \underbrace{x_m, x_m, \dots, x_m}_{r_m \text{ times}}$ is defined by (see (6))

$$\left[x_0^{(r_0)}, x_1^{(r_1)}, \dots, x_m^{(r_m)}; f(x,y)\right]_x = \frac{1}{V\left(x_0^{(r_0)}, x_1^{(r_1)}, \dots, x_m^{(r_m)}\right)} \tag{10}$$
$$\cdot \sum_{k=0}^{M} \sum_{i=0}^{r_k - 1} V_{k,i}\left(x_0^{(r_0)}, x_1^{(r_1)}, \dots, x_m^{(r_m)}\right) \frac{\partial^i f}{\partial x^i}(x_k, y).$$

In a similar way, the parametric extension of N-th order divided differences of the function $f(x, *); J \to \mathbb{R}$, $x \in I$, with respect to the knots

$$\underbrace{y_0, y_0, \dots, y_0}_{q_0 \text{ times}}, \underbrace{y_1, y_1, \dots, y_1}_{q_1 \text{ times}}, \dots, \underbrace{y_n, y_n, \dots, y_n}_{q_n \text{ times}}$$

is defined by

$$\left[y_0^{(q_0)}, y_1^{(q_1)}, \dots, y_n^{(q_n)}; f(x,y)\right]_y \tag{11}$$
$$= \frac{1}{V\left(y_0^{(q_0)}, y_1^{(q_1)}, \dots, y_n^{(q_n)}\right)} \sum_{l=0}^{n} \sum_{j=0}^{q_l - 1} V_{l,j}\left(y_0^{(q_0)}, y_1^{(q_1)}, \dots, y_n^{(q_n)}\right) \frac{\partial^j f}{\partial y^j}(x, y_l).$$

Theorem 5 *The following equalities*

$$\left[y_0^{(q_0)}, y_1^{(q_1)}, \ldots, y_n^{(q_n)}; \left[x_0^{(r_0)}, x_1^{(r_1)}, \ldots, x_m^{(r_m)}; f(x,y)\right]_x\right]_y \tag{12}$$
$$= \left[x_0^{(r_0)}, x_1^{(r_1)}, \ldots, x_m^{(r_m)}; \left[y_0^{(q_0)}, y_1^{(q_1)}, \ldots, y_n^{(q_n)}; f(x,y)\right]_y\right]_x$$
$$= \frac{1}{V\left(x_0^{(r_0)}, x_1^{(r_1)}, \ldots, x_m^{(r_m)}\right) V\left(y_0^{(q_0)}, y_1^{(q_1)}, \ldots, y_n^{(q_n)}\right)} \sum_{k=0}^{m} \sum_{l=0}^{n} \sum_{i=0}^{r_k-1} \sum_{j=0}^{q_l-1}$$
$$\cdot V_{k,i}\left(x_0^{(r_0)}, x_1^{(r_1)}, \ldots, x_m^{(r_m)}\right) V_{l,j}\left(y_0^{(q_0)}, y_1^{(q_1)}, \ldots, y_n^{(q_n)}\right) \frac{\partial^{i+j} f}{\partial x^i \partial y^j}(x_k, y_l)$$

hold, where $(x,y) \in I \times J$.

Proof. Taking into account (10) and (11), we have

$$\left[y_0^{(q_0)}, y_1^{(q_1)}, \ldots, y_n^{(q_n)}; \left[x_0^{(r_0)}, x_1^{(r_1)}, \ldots, x_m^{(r_m)}; f(x,y)\right]_x\right]_y$$
$$= \left[y_0^{(q_0)}, y_1^{(q_1)}, \ldots, y_n^{(q_n)}; \frac{1}{V\left(x_0^{(r_0)}, x_1^{(r_1)}, \ldots, x_m^{(r_m)}\right)} \sum_{k=0}^{m} \sum_{i=0}^{r_k-1}\right.$$
$$\left. \cdot V_{k,i}\left(x_0^{(r_0)}, x_1^{(r_1)}, \ldots, x_m^{(r_m)}\right) \frac{\partial^i f}{\partial x^i}(x_k, y)\right]_y = \frac{1}{V\left(x_0^{(r_0)}, \ldots, x_m^{(r_m)}\right)}$$
$$\cdot \sum_{k=0}^{m} \sum_{i=0}^{r_k-1} V_{k,i}\left(x_0^{(r_0)}, \ldots, x_m^{(r_m)}\right) \left[y_0^{(q_0)}, y_1^{(q_1)}, \ldots, y_n^{(q_n)}; \frac{\partial^i f}{\partial x^i}(x_k, y)\right]_y$$
$$= \frac{1}{V\left(x_0^{(r_0)}, \ldots, x_m^{(r_m)}\right)} \sum_{k=0}^{m} \sum_{i=0}^{r_k-1} V_{k,i}\left(x_0^{(r_0)}, \ldots, x_m^{(r_m)}\right)$$
$$\cdot \frac{1}{V\left(y_0^{(q_0)}, y_1^{(q_1)}, \ldots, y_n^{(q_n)}\right)} \sum_{l=0}^{n} \sum_{j=0}^{q_l-1} V_{l,j}\left(y_0^{(q_0)}, y_1^{(q_1)}, \ldots, y_n^{(q_n)}\right) \frac{\partial^{i+j} f}{\partial x^i \partial y^j}(x_k, y_l),$$

and (12) follows.

Definition 1 The (m,n)-th order divided difference of the function $f \in D^{\alpha,\beta}(I \times J)$ with respect to the distinct knots $(x_i, y_j) \in I \times J$, $i \in \{0, 1, \ldots, m\}$,

$j \in \{0, 1, \ldots, n\}$ is defined by

$$\begin{bmatrix} x_0^{(r_0)}, x_1^{(r_1)}, \ldots, x_m^{(r_m)} & \\ & ; f \\ y_0^{(q_0)}, y_1^{(q_1)}, \ldots, y_n^{(q_n)} & \end{bmatrix} \tag{13}$$

$$= \frac{1}{V\left(x_0^{(r_0)}, x_1^{(r_1)}, \ldots, x_m^{(r_m)}\right) V\left(y_0^{(q_0)}, y_1^{(q_1)}, \ldots, y_n^{(q_n)}\right)} \sum_{k=0}^{m} \sum_{l=0}^{n} \sum_{i=0}^{r_k-1} \sum_{j=0}^{q_l-1}$$

$$V_{k,i}\left(x_0^{(r_0)}, x_1^{(r_1)}, \ldots, x_m^{(r_m)}\right) V_{l,j}\left(y_0^{(q_0)}, y_1^{(q_1)}, \ldots, y_n^{(q_n)}\right) \frac{\partial^{i+j} f}{\partial x^i \partial y^j}(x_k, y_l).$$

In the paper [2], we give a mean-value theorem for two divided differences with simple knots.

Let a, b, c, d be real numbers defined by $a = \min\{x_0, x_1, \ldots, x_m\}$, $b = \max\{x_0, x_1, \ldots, x_m\}$, $c = \min\{y_0, y_1, \ldots, y_n\}$ and $d = \max\{y_0, y_1, \ldots, y_n\}$.

Theorem 6 *If $f(\cdot, y) \in C^{M-1}([a, b])$, $\frac{\partial^M f}{\partial x^M}(\cdot, y)$ exists on (a, b) for any $y \in [c, d]$, $\frac{\partial^M f}{\partial x^M}(x, *) \in C^{N-1}([c, d])$ and $\frac{\partial^{M+N}}{\partial x^M \partial x^N}(x, *)$ exists on (c, d) for any $x \in (a, b)$, then exists $(\xi, \eta) \in (a, b) \times (c, d)$ such that*

$$\begin{bmatrix} x_0^{(r_0)}, x_1^{(r_1)}, \ldots, x_m^{(r_m)} & \\ & ; f \\ y_0^{(q_0)}, y_1^{(q_1)}, \ldots, y_n^{(q_n)} & \end{bmatrix} = \frac{1}{M!\,N!} \frac{\partial^{M+N} f}{\partial x^M \partial y^N}(\xi, \eta), \tag{14}$$

where "$\cdot$" and "$*$" stand for the first and respectively second variable.

Proof. Taking into account (12) and (13) and applying the mean-value theorem for one dimensional divided differences (see Theorem 4), there exist $\xi \in (a, b)$ and respectively $\eta \in (c, d)$ such that

$$\begin{bmatrix} x_0^{(r_0)}, x_1^{(r_1)}, \ldots, x_m^{(r_m)} & \\ & ; f \\ y_0^{(q_0)}, y_1^{(q_1)}, \ldots, y_n^{(q_n)} & \end{bmatrix}$$
$$= \left[y_0^{(q_0)}, y_1^{(q_1)}, \ldots, y_n^{(q_n)}; \left[x_0^{(r_0)}, x_1^{(r_1)}, \ldots, x_m^{(r_m)}; f(x, y)\right]_x\right]_y$$
$$= \left[y_0^{(q_0)}, y_1^{(q_1)}, \ldots, y_n^{(q_n)}; \frac{1}{M!} \frac{\partial^M f}{\partial x^M}(\xi, y)\right]_y$$
$$= \frac{1}{M!}\left[y_0^{(q_0)}, y_1^{(q_1)}, \ldots, y_n^{(q_n)}; \frac{\partial^M f}{\partial x^M}(\xi, y)\right]_y$$
$$= \frac{1}{M!\,N!} \frac{\partial^{M+N} f}{\partial x^M \partial y^N}(\xi, \eta),$$

so the equality (14) holds.

Remark 2 Because $r_0, r_1, \ldots, r_m \in \mathbb{N}$ and $r_0 + r_1 + \cdots + r_m = M + 1$, it results that $M \geq m$, so if m tends to ∞ it results that M also tends to ∞.

We consider a function $f : [a, b] \times [c, d] \to \mathbb{R}$, $f \in C^{\infty,\infty}([a, b] \times [c, d])$, f possessing uniform bounded partial derivatives, so there exists $M > 0$ such that

$$\left| \frac{\partial^{k+l} f}{\partial x^k \partial y^l} (x, y) \right| \leq M \tag{15}$$

for any $(x, y) \in [a, b] \times [c, d]$ and any $(k, l) \in \mathbb{N}_0 \times \mathbb{N}_0$.

Theorem 7 *If $(x_m)_{m\geq 0}$ and $(y_n)_{n\geq 0}$ are sequences of distinct points from $[a, b]$, respectively $[c, d]$, then*

$$\lim_{m,n\to\infty} \left[\begin{matrix} x_0^{(r_0)}, x_1^{(r_1)}, \ldots, x_m^{(r_m)} \\ y_0^{(q_0)}, y_1^{(q_1)}, \ldots, y_n^{(q_n)} \end{matrix} ; f \right] = 0. \tag{16}$$

Proof. Taking into account (14), (15) and Remark 2, relation (16) follows.

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