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DRIFT-FREE LEFT INVARIANT CONTROL SYSTEM ON G_4 WITH FEWER CONTROLS THAN STATE VARIABLES

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Abstract

An optimal control problem on a special nilpotent4-dimensional Lie group is discussed and some of its dynamical and geometrical properties are pointed out.

1 Introduction

Recent work in nonlinear control has drawn attention to drift-free systems with fewer degrees than state variables. These arise naturally in problems of motion planning for wheeled robots subject to nonholonomic controls [9], models of kinematic drift effects in space subjects to appendage vibrations or articulations [9], the molecular dynamics [6], the autonomous underwater vehicle dynamics [1] and spacecraft dynamics [10].

The goal of our paper is to study an optimal control problem on a particular Lie group and to point out some of its dynamical and geometrical properties. Similar problems have been studied on the Lie group $SO(4)$ (see [2].) We consider an optimal control problem on a special nilpotent4-dimensional Lie group, realizing this system as a Hamilton-Poisson system, and then study the system from some standard Hamilton-Poisson geometry points of view. By standard Poisson geometry point of view we mean the classical study of the Lyapunov stability of equilibria by using energy-Casimir type stability tests

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and the study of the existence of periodic solutions by using the Weinstein-Moser theorem. In the third part of the paper we give an explicit integration of the system via elliptic functions. In the sixth section of the paper we give three numerical integrators of the system, and finally the last part of this article discusses some numerics associated with the Poisson geometrical structure of the system.

2 The geometrical picture of the problem

Let G_4 be the Lie group given by:

$$G_4 = \left\{ \begin{bmatrix} 1 & x_2 & x_3 & x_4 \\ 0 & 1 & x_1 & \frac{1}{2}x_1^2 \\ 0 & 0 & 1 & x_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \in \mathcal{M}_4(\mathbb{R}) \middle| \; x_1, x_2, x_3, x_4 \in \mathbb{R} \right\}$$

Proposition 2.1. *The Lie algebra $\mathcal{G}$ of G_4 is generated by:*

$$A_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, A_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$A_3 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, A_4 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and the Lie algebra structure of $\mathcal{G}$ is given by the following table:

$[.,.]$	A_1	A_2	A_3	A_4
A_1	0	$-A_3$	$-A_4$	0
A_2	A_3	0	0	0
A_3	A_4	0	0	0
A_4	0	0	0	0

Proposition 2.2. *The minus-Lie-Poisson structure on* $\mathcal{G}^* \simeq (R^4)^* \simeq R^4$ *is generated by the matrix:*

$$\Pi_- = \begin{bmatrix} 0 & x_3 & x_4 & 0 \\ -x_3 & 0 & 0 & 0 \\ -x_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Proposition 2.3. *The function* C *given by:*

$$C = \frac{1}{2}x_4^2$$

is a Casimir of our configuration.

Proof: Indeed, we have:

$$(\nabla C)^t \Pi = 0$$

as required.

An easy computation leads us via Chow's theorem ([4]) to:

Proposition 2.4. *There exist four drift-free left invariant controllable systems on* G*, namely:*

$$\dot{X} = X(A_1u_1 + A_2u_2), \tag{2.1}$$

$$\dot{X} = X(A_1u_1 + A_2u_2 + A_3u_3), \tag{2.2}$$

$$\dot{X} = X(A_1u_1 + A_2u_2 + A_4u_4), \tag{2.3}$$

$$\dot{X} = X(A_1u_1 + A_2u_2 + A_3u_3 + A_4u_4), \tag{2.4}$$

where $X \in G$*,* A_i *are the matrix defined above and* $u_i \in C^\infty(\mathbb{R}, \mathbb{R}), i = \overline{1,4}$.

3 An optimal control problem for the system (2.2)

Let J be the cost function given by:

$$J(u_1, u_2, u_3) = \frac{1}{2}\int_0^{t_f} \left[c_1u_1^2(t) + c_2u_2^2(t) + c_3u_3^2(t)\right] dt$$

$$c_1 > 0, c_2 > 0, c_3 > 0.$$

Then we have:

Proposition 3.1. *The controls that minimize J and steer the system (2.2) from $X = X_0$ at $t = 0$ to $X = X_f$ at $t = t_f$ are given by:*

$$u_1 = \frac{1}{c_1}x_1, \ u_2 = \frac{1}{c_2}x_2, \ u_3 = \frac{1}{c_3}x_3,$$

where $x_i's$ are solutions of:

$$\begin{cases} \dot{x}_1 = \dfrac{1}{c_2c_3}x_2x_3 + \dfrac{1}{c_3}x_3x_4 \\ \dot{x}_2 = -\dfrac{1}{c_1}x_1x_3 \\ \dot{x}_3 = -\dfrac{1}{c_1}x_1x_4 \\ \dot{x}_4 = 0. \end{cases} \tag{3.1}$$

Remark 3.1. It is easy to see from the equations (3.1) that x_4=constant and so the dynamics (3.1) can be put in the equivalent form:

$$\begin{cases} \dot{x}_1 = \dfrac{1}{c_2c_3}x_2x_3 + \dfrac{k}{c_3}x_3 \\ \dot{x}_2 = -\dfrac{1}{c_1}x_1x_3 \\ \dot{x}_3 = -\dfrac{k}{c_1}x_1 \end{cases} \tag{3.2}$$

The goal of our paper is to study some geometrical and dynamical properties for the system (3.2).

Proposition 3.2. *The dynamics (3.2) has the following Hamilton-Poisson realization:*

$$(\mathbb{R}^3, \Pi, H),$$

where

$$\Pi = \begin{bmatrix} 0 & x_3 & k \\ -x_3 & 0 & 0 \\ -k & 0 & 0 \end{bmatrix}$$

and the Hamiltonian

$$H(x_1, x_2, x_3) = \frac{1}{2}\left(\frac{x_1^2}{2c_1} + \frac{x_2^2}{2c_2} + \frac{x_3^2}{2c_3}\right).$$

Proof. Indeed, it is not hard to see that the dynamics (3.2) can be put in the equivalent form:

$$[\dot{x}_1, \dot{x}_2, \dot{x}_3]^t = \Pi \cdot \nabla H,$$

as required. Moreover, the function C given by:

$$C = -kx_2 + \frac{1}{2}x_3^2$$

is a Casimir of our configuration. Indeed,

$$(\nabla C)^t \Pi = 0$$

as desired.

Remark 3.2. The phase curves of the dynamics (3.2) are intersections of

$$\frac{x_1^2}{2c_1} + \frac{x_2^2}{2c_2} + \frac{x_3^2}{2c_3} = \text{const.}$$

with

$$-kx_2 + \frac{1}{2}x_3^2 = \text{const.},$$

see the Figure 3.1.

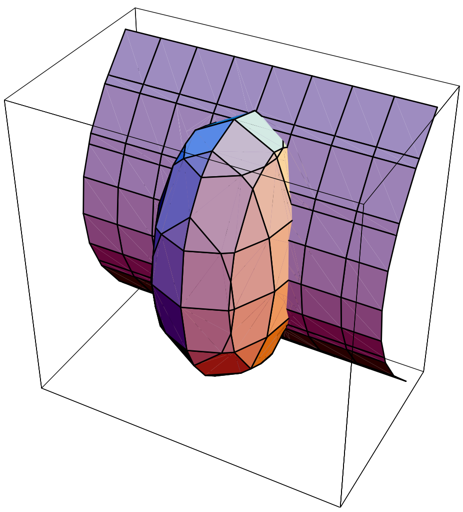

Figure 3.1: The phase curves of the system (3.2)

Proposition 3.3. *The dynamics (3.2) has an infinite number of Hamilton-Poisson realizations.*

Proof. An easy computation shows us that the triples:

$$(\mathbb{R}^3, \{\cdot,\cdot\}_{ab}, H_{cd}),$$

where

$$\begin{aligned}
&\{f,g\}_{ab} = -\nabla C_{ab} \cdot (\nabla f \times \nabla g),\ (\forall) f,g \in C^{\infty}(\mathbb{R}^3,\mathbb{R}),\\
&C_{ab} = aC + bH,\\
&H_{cd} = cC + dH,\\
&a,b,c,d \in \mathbb{R},\ ad - bc = 1,
\end{aligned}$$

define Hamilton-Poisson realizations of the dynamics (3.2), as required.

Remark 3.3. The above proposition tell us in fact that the equation (3.2) is unchanged, so the trajectories of motion in $\mathbb{R}^3$ remain the same when H and C are replaced by G combinations of H and C.

Proposition 3.4. *The dynamics (3.2) can be reduced to the pendulum dynamics.*

Proof. It is known that H and C are constants of motion, i.e.

$$\frac{x_1^2}{c_1} + \frac{x_2^2}{c_2} + \frac{x_3^2}{c_3} = l^2$$

and

$$-kx_2 + \frac{1}{2}x_3^2 = p$$

and then

$$\frac{x_1^2}{c_1} + \left(\frac{x_2}{\sqrt{c_2}} + \frac{k}{c_1}\sqrt{c_2}\right)^2 = l^2 + \frac{c_2k^2}{c_1^2} = r^2.$$

If we take now:

$$\begin{cases} x_1 = r\sqrt{c_1} cos\ \theta \\ x_2 = r\sqrt{c_2} sin\ \theta - \dfrac{kc_2}{c_1} \end{cases}$$

then

$$\dot{x}_2 = \sqrt{\frac{c_2}{c_1}} x_1 \cdot \dot{\theta}$$

and so:

$$\dot{\theta} = -\frac{1}{\sqrt{c_1 c_2}} x_3.$$

Differentiating again, we obtain:

$$\ddot{\theta} = \frac{kr}{c_1 \sqrt{c_2}} cos\ \theta$$

which is nothing else than the pendulum dynamics, as required.

4 Stability

It is not hard to see that the equilibrium states of our dynamics (3.2) are:

$$e_1^M = (0, M, 0),\ M \in \mathbb{R},$$
$$e_2^M = (0, -\frac{kc_2}{c_3}, M),\ M \in \mathbb{R}.$$

First, let us recall very briefly the definitions of spectral stability and nonlinear stability of an equilibria point of an Hamilton-Poisson system. For more information, see [7]. The laws of dynamics are usually presented as equations of motion which we write in the abstract form: $\dot{x} = f(x)$, where $f : D \to \mathbb{R}$ is a C^1 - map on an open set $D \in \mathbb{R}^n$.

Definition 4.1. *An equilibrium state x_e is said to be* **nonlinear stable** *if for each neighborhood U of x_e in D there is a neighbourhood V of x_e in U such that trajectory $x(t)$ initially in V never leaves U.*

Definition 4.2. *An equilibrium state x_e is said to be* **spectral stable** *if all the eigenvalues of the linearized matrix of the system have negative real parts.*

About the spectral stability of these equilibrium states, we have the following result:

Proposition 4.1. *(i) The equilibrium states $e_1^M,\ M \in \mathbb{R}^*$ are spectrally stable if $kM > 0$ and unstable if $kM < 0$.*

(ii) The equilibrium states $e_2^M,\ M \in \mathbb{R}^*$ are spectrally stable for any $M \in R^*$.

We can now pass to discuss the nonlinear stability of the equilibrium states e_1^M and e_2^M, $M \in \mathbb{R}$.

Proposition 4.2. *(i) The equilibrium states e_1^M, $M \in \mathbb{R}^*$ are nonlinearlly stable if $kM > 0$.*

(ii) The equilibrium states e_2^M, $M \in \mathbb{R}^*$ are nonlinearlly stable for any $M \in \mathbb{R}$.

Proof. We shall make the proof using energy-Casimir method (see [3]). Let

$$H_\varphi = H + \varphi(C) = \frac{x_1^2}{2c_1} + \frac{x_2^2}{2c_2} + \frac{x_3^2}{2c_3} + \varphi(-kx_2 + \frac{1}{2}x_3^2)$$

be the energy-Casimir function, where $\varphi : R \to R$ is a smooth real valued function defined on R.

Now, the first variation of H_φ is given by:

$$\delta H_\varphi = \frac{x_1}{c_1}\delta x_1 + \frac{x_2}{c_2}\delta x_2 + \frac{x_3}{c_3}\delta x_3 + \dot{\varphi} \cdot (-k\delta x_2 + x_3 \delta x_3),$$

where

$$\dot{\varphi} = \frac{\partial \varphi}{\partial(-kx_2 + \frac{1}{2}x_3^2)}.$$

This equals zero at the equilibrium of interest if and only if

$$\dot{\varphi}(-kM) = \frac{M}{kc_2}.$$

The second variation of H_φ is given by:

$$\delta^2 H_\varphi = \frac{1}{c_1}(\delta x_1)^2 + \frac{1}{c_2}(\delta x_2)^2 + \frac{1}{c_3}(\delta x_3)^2 + \ddot{\varphi} \cdot (-k\delta x_2 + x_3\delta x_3)^2 + \dot{\varphi} \cdot (\delta x_3)^2,$$

Since $kM > 0$ and having choosing φ such that:

$$\begin{cases} \dot{\varphi}(-kM) = \dfrac{M}{kc_2} \\[2ex] \ddot{\varphi}(-kM) < \dfrac{1}{kc_2} \end{cases}$$

we can conclude that the second variation of H_φ at the equilibrium of interest is positive define and thus e_1 is nonlinearly stable.

Similar arguments lead us to the second result.

5 The existence of periodic solutions

Proposition 5.1. *Near* $e_1^M = (0, M, 0)$*,* $M \in \mathbb{R}^*$*, the reduced dynamics has, for each sufficiently small value of the reduced energy, at least 1-periodic solution whose period is close to:*

$$\frac{2\pi\sqrt{c_1c_2c_3}}{\sqrt{k^2c_2 + kMc_3}}.$$

Proof. Indeed, we have successively:

(i) The restriction of our dynamics (3.2) to the coadjoint orbit:

$$-kx_2 + \frac{1}{2}x_3^2 = -kM \tag{5.1}$$

gives rise to a classical Hamiltonian system.

(ii) The matrix of the linear part of the reduced dynamics has purely imaginary roots. More exactly:

$$\lambda_{2,3} = \pm i\frac{\sqrt{k^2c_2 + kMc_3}}{\sqrt{c_1c_2c_3}}.$$

(iii) $\text{span}(\nabla C(e_1^M)) = V_0$,
where

$$V_0 = \ker(A(e_1^M)).$$

(iv) The smooth function $F \in C^\infty(\mathbb{R}^3, \mathbb{R})$ given by:

$$F(x_1, x_2, x_3) = \frac{x_1^2}{2c_1} + \frac{x_2^2}{2c_2} + \frac{x_3^2}{2c_3} + \frac{M}{kc_2}(-kx_2 + \frac{x_3^2}{2})$$

has the following properties:

- It is a constant of motion for the dynamics (3.2).
- $\nabla F(e_1^M) = 0$.
- $\nabla^2 F(e_1^M)\big|_{W\times W} > 0$,

where

$$W := ker\, dC(e_1^M) = span_{\mathbb{R}}\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right).$$

Then our assertion follows via the Moser-Weinstein theorem with zero eigenvalue, see for details [4].

6 Numerical integration of the dynamics (3.2)

It is easy to see that for the equations (3.2), Kahan's integrator can be written in the following form:

$$\begin{cases} x_1^{n+1} - x_1^n = \dfrac{h}{2c_2c_3}(x_3^{n+1}x_2^n + x_2^{n+1}x_3^n) + \dfrac{hk}{2c_3}(x_3^{n+1} + x_3^n) \\ x_2^{n+1} - x_2^n = -\dfrac{h}{2c_1}(x_1^{n+1}x_3^n - x_3^{n+1}x_1^n) \\ x_3^{n+1} - x_3^n = -\dfrac{hk}{2c_1}(x_1^{n+1} + x_1^n) \end{cases} \tag{6.1}$$

A long but straightforward computation or alternatively, by using MATHEMATICA, lead us to:

Proposition 6.1. *Kahan's integrator (6.1) has the following properties:*
(i) It is not Poisson preserving.
(ii) It does not preserve the Casimir C of our Poisson configuration $(\mathbb{R}^3, \Pi)$.
(iii) It does not preserve the Hamiltonian H of our system (3.2).

We shall discuss now the numerical integration of the dynamics (3.2) via the Lie-Trotter integrator [11].

To begin with, let us observe that the Hamiltonian vector field X_H splits as follows:

$$X_H = X_{H_1} + X_{H_2} + X_{H_3}.$$

where

$$H_1 = \frac{1}{2c_1}x_1^2,\ H_2 = \frac{1}{2c_3}x_2^2,\ H_3 = \frac{1}{2c_3}x_3^2.$$

Following [11], we obtain the Lie-Trotter integrator:

$$\begin{cases} x_1^{n+1} = x_1^n + \dfrac{k}{c_3}tx_3^n \\ x_2^{n+1} = \dfrac{ak}{2}t^2x_1^n + x_2^n + (\dfrac{ak^2}{2c_3}t^3 + \dfrac{abk}{2}t^2 - at)x_3^n \\ x_3^{n+1} = -ktx_1^n - (\dfrac{k^2}{c_3} + bk)t^2x_3^n \end{cases} \tag{6.2}$$

Now, a direct computation or using MATHEMATICA leads us to:

Proposition 6.2. *The Lie-Trotter integrator (6.2) has the following properties:*

(i) It preserves the Poisson structure Π.

(ii) It preserves the Casimir C of our Poisson configuration $(\mathbb{R}^3, \Pi)$.

(iii) It doesn't preserve the Hamiltonian H of our system (3.2).

(iv) Its restriction to the coadjoint orbit $(\mathcal{O}_k, \omega_k)$, where

$$\mathcal{O}_k = \{(x_1, x_2, x_3) \in \mathbb{R}^3 | -kx_2 + \frac{1}{2}x_3^2 = const.\}$$

and ω_k is the Kirilov-Kostant-Souriau symplectic structure on $\mathcal{O}_k$ gives rise to a symplectic integrator.

Remark 6.1. If we compare this method to the 4th-step Runge-Kutta method we can see that Lie-Trotter integrator and Kahan's integrator give us a weak approximation of our dynamics. In fact, Lie-Trotter integrator has failed in this example. This is an open problem which is responsable for this. However, Kahan's integrator and the Lie-Trotter integrator have the advantage of being easier implemented, see Figures 6.1, 6.2 and 6.3.

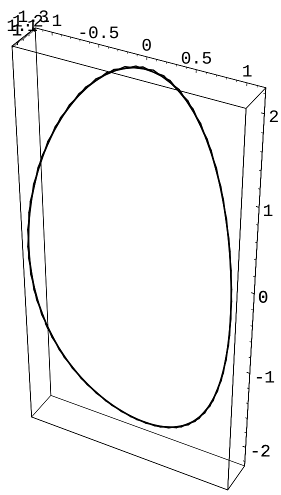

Figure 6.1: The 4th-step Runge-Kutta

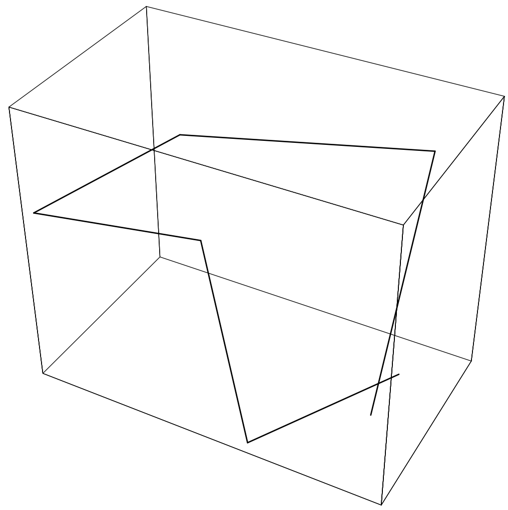

Figure 6.2: The Kahan integrator

Figure 6.3: The Lie-Trotter integrator

7 Conclusion

The paper presents the left invariant controllable systems on a particular Lie group; this arises naturally from the study of the car's dynamics for which the Lie group G_4 represents the phase space ([11]). In addition, we have studied the existence of the periodic orbits around the nonlinear stable states and a comparison between three numerical integration methods. Despite the simplicity of the studied system, we have seen that two of the three methods give us a week approximation of the movement trajectory, unlike some other examples for whitch all the three methods provide the same results ($SL(2, \mathbb{R})$, 3-Dimensional Toda Laticce.)

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