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SOME CHARACTERIZATION OF INFINITE ANTIMATROIDS*

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Abstract

We provide some characterizations of an infinite antimatroid. Some of them are the generalization of the corresponding results of finite antimatroids. Every extension is not straightforward because certain characterizations of a finite antimatroid are lost in the infinite case. Additionally, some characterizations of infinite antimatroids in this paper are in lattices. These lattices axioms are also true for finite antimatorids.

1 Introduction and preliminaries

[1-4] point out that convex geometry is the branch of geometry studying convex sets. The phrase convex geometry is also used in combinatorics as the name for an abstract model of convex sets based on antimatroids. In mathematics, an antimatroid is a formal system that describes processes in which a set is built up by including elements one at a time, and in which an element, once available for inclusion, remains available until it is included.

Antimatroid theory is more and more attracted people's eyes(cf.[1-5]). Unfortunately, for infinite case, the results are much less than that of finite case though infinite case is an important part in antimatroid theory. This article will study on infinite case as Wahl in [5]. We will provide some characterizations for infinite antimatroids. In the future, we hope to see that the work in this article is the foundation to research on infinite antimatroids.

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We begin this article by giving some basic definitions and lemmas. In what follows, E denotes some arbitrary–possibly infinite–set.

Definition 1[5] A *closure system* on a ground set E is a pair $(E, \mathcal{L})$ where $\mathcal{L}$ is a collection of possibly infinite subsets of E such that

(csi) $\emptyset, E \in \mathcal{L}$;

(csii) for all $\mathcal{C} \subseteq \mathcal{L}$, the set $\cap\mathcal{C} \in \mathcal{L}$.

We will call the elements of $\mathcal{L}$ *closed sets.*

A *closure operator* τ on E is a map from E to E such that

(coi) $A \subseteq \tau(A)$;

(coii) $A \subseteq B$ implies $\tau(A) \subseteq \tau(B)$;

(coiii) $\tau(\tau(A)) = \tau(A)$;

(coiv) $\tau(\emptyset) = \emptyset$.

A closure system is called an *antimatroid* if the following *anti-exchange property* is satisfied: for all $A \in \mathfrak{L} = \{Y \subseteq E | \tau(Y) = Y\}$, for all distinct $x, y \in E \setminus A$ and $x \neq y$, $y \in \tau(A \cup x) \Rightarrow x \notin \tau(A \cup y)$.

Antimatroids are also known as *convex geometries.*

Lemma 1[5] A closure system $(E, \mathcal{L})$ provides the following closure operator: for all $A \subseteq E, \tau(A) = \cap\{F \in \mathcal{L} | A \subseteq F\}$. $\tau(A)$ is called the *closure* of A. The notions of closure system and closure operator are in fact equivalent.

Based on Lemma 1, in this paper, no different to say a closure system as $(E, \mathfrak{L})$ or (E, τ), where τ is the corresponding closure operator to $\mathfrak{L}$.

Remark 1 (1) The antimatroid provided above, in finite case, [1,2,4] calls it a convex geometry. Besides, [1,2,4] calls an "antimatroid" in finite case as the complement of a convex geometry. On the other hand, [1,p.23, Theorem 1.3&2,4] means the equivalence of finite antimatroids and convex geometries. The two hands tells us that in finite case, under equivalence, it is no different to say that a construction is an antimatroid or a convex geometry. However, this paper calls the antimatroid in Definition 1 as *infinite antimatroid* though it is true for finite case.

(2) In this article, the knowledge about lattice theory is seen [6,7], and for more details about antimatroids(or say convex geometry), it could be referring to [5,8].

We extend some definitions from finite(cf.[1,p.21&2]) to infinite antimatroid as follows.

Definition 2 Let τ be a closure operator on E. An element x of a subset $A \subseteq E$ an *extreme point* of A if $x \notin \tau(A \setminus x)$. The set of extreme points of A is denoted by $ex(A)$. A *minimal spanning set* of a set X is a minimal subset

$Y \subseteq X$ such that $\tau(Y) = \tau(X)$.

About extreme points, we have the following result.

Lemma 2 Let τ be a closure operator on E and $\mathfrak{L}$ be the set of closed sets of τ. Let $A \in \mathfrak{L}$. Then $x \in ex(A)$ is equivalent to $A \setminus x \in \mathfrak{L}$.

Proof If $x \in ex(A)$ and $A \setminus x \notin \mathfrak{L}$, then there is $y \in \tau(A \setminus x) \setminus (A \setminus x) = \{x\} \subseteq \tau(A) = A$, that is, $x \in \tau(A \setminus x)$. This is a contradiction to $x \notin \tau(A \setminus x)$. Conversely, if for $y \in A$ and $A \setminus y \in \mathfrak{L}$, i.e. $\tau(A \setminus y) = A \setminus y$. Obviously, $y \notin A \setminus y = \tau(A \setminus y)$, that is, $y \in ex(A)$.

Section 2 of the text provides some characterizations for infinite antimatroids. Theorem 2 are the generalizations of finite cases; Theorem 1 are provided by lattice theory. In the last of Section 2, we give a characterization of an infinite antimatroid associated with orderly. All of characterizations are true for finite and infinite.

2 Characterizations

Throughout this section, we want to characterize an infinite antimatroid. Some relative results have been studied by [1,2] for finite cases such as Theorem 2. Using lattice theory, Theorem 1 characterizes an infinite antimatroid, and further, Theorem 3 provides a method to judge an infinite antimatroid to be orderly.

[1,2] said that for finite antimatorids, lattice theory was one of the key sources. So it is not surprising that lattices play an important role in the study of wider classes such as infinite antimatroids. The following theorem 1 will show the point.

Theorem 1 Let τ be a closure operator on E and $\mathfrak{L}$ be the set of closed sets of τ. Then the following conditions are equivalent.

(1) (E, τ) is an infinite antimatroid.

(2) $(\mathfrak{L}, \subseteq)$ satisfies that

(i) it is a complete and bounded lattice with $\emptyset, E \in \mathfrak{L}$;

(ii) for any $A \subseteq E$, $x, y \in E \setminus \tau(A)$ and $x \neq y$, if $\tau(A \cup x \cup y) = \tau(A \cup x)$, then there is a chain $\tau(A) \subset \tau(A \cup y) \subset \tau(A \cup x \cup y)$ in $(\mathfrak{L}, \subseteq)$.

Proof (1)$\Rightarrow$(2) Since for any $A, B \in \mathfrak{L}$, (csii) pledges $A \cap B \in \mathfrak{L}$. Thus, $A \wedge B$, the infimum of A, B in $\mathfrak{L}$, is existed as $A \cap B$. Though $A \cup B$ could not ensure to be a closed set, (csii) assures $\cap\{X \in \mathfrak{L} | A \cup B \subseteq X\} \in \mathfrak{L}$. Evidently $\cap\{X \in \mathfrak{L} | A \cup B \subseteq X\}$ is the supremum $A \vee B$ of A and B in $\mathfrak{L}$. Therefore, $(\mathfrak{L}, \subseteq)$ is a lattice.

Analogously to the above with (csii), the complete of $(\mathfrak{L}, \subseteq)$ is followed.

The hold of (ii) is followed from anti-exchange property.

(csi) shows $\emptyset, E \in \mathfrak{L}$. $\emptyset, E \in \mathfrak{L}$ brings about the bounded property of $(\mathfrak{L}, \subseteq)$.

(2)⇒(1) Let σ be defined on E as $\sigma(A) = \wedge\{X \in \mathfrak{L} | A \subseteq X\}$ for $A \subseteq E$. Because $(\mathfrak{L}, \subseteq)$ is a complete lattice, it leads that the definition of σ is significant, and besides, $\sigma(A) \in \mathfrak{L}$.

Obviously, $A \subseteq \sigma(A)$ is true. Let $A \subseteq B \subseteq E$. It is easy to see that $\sigma(A) \subseteq \sigma(B)$, and $\sigma(\sigma(A)) = \wedge\{X \in \mathfrak{L} | \sigma(A) \subseteq X\} = \sigma(A)$ is correct in view of $\sigma(A) \in \mathfrak{L}$.

$\sigma(\emptyset) = \wedge\{X \in \mathfrak{L} | \emptyset \subseteq X\}$ and the known $\emptyset \in \mathfrak{L}$ in (2) together follows $\sigma(\emptyset) = \emptyset$.

Hence, σ is a closure operator on E.

According to the definition of σ, it hints that $A \in \mathfrak{L}$ if and only if $A = \sigma(A)$. Moreover, the set $\{X \subseteq E | X = \sigma(X)\}$ of closed sets of σ is precisely $\mathfrak{L}$.

Recalling back the relation between closure operator and closure system in Section 2, we get that σ is a closure operator and $\tau = \sigma$. (ii) compels that τ satisfies the anti-exchange property. Therefore, $(E, \mathfrak{L})$ is an infinite antimatroid.

We give two examples to show some difference between finite and infinite antimatroids.

Example 1 Let $E = \{1, 2, \ldots\}$. Define $\tau : 2^E \to 2^E$ as $\tau(A) = A$ if A is a finite subset of E; $\tau(A) = E$ if A is an infinite subset of E. It is easy to see the hold of (csi) and (csii) for τ, and also, the correct of (coi)-(coiv) and the anti-exchange property for τ. Namely, (E, τ) is an infinite antimatroid and the set of all closed sets $\mathfrak{L} = \{E\} \cup \{X | X \text{ is a finite set of } E\}$. Let $A, B \in \mathfrak{L}$. It is easy to see that for all closed sets $A \subset B$, there exists $x \in B \setminus A$ such that $A \cup x$ is closed.

Example 2 Let E be all the real numbers and ρ be the ordinary topology on E, i.e. $\rho(x, y) = |x - y|$ for $x, y \in E$. Let $\mathcal{F}$ be all the closed sets of (E, ρ). Define $\tau : 2^E \to 2^E$ as $\tau(A) = \bigcap_{A \subseteq X \in \mathcal{F}} X$. With the help of topology knowledge (cf.[9&10]), we obtain that (E, τ) is an infinite antimatroid. Let $A = [0, 1]$ and $B = [0, 2]$. Then both A and B are closed sets in (E, τ), and $A \subset B$ holds. But for any $x \in B \setminus A = (1, 2]$, $A \cup x$ is not a closed set in (E, τ).

The two examples imply that for an infinite antimatroid (E, τ) and two closed sets A, B,
if E is infinite, $A \subset B$ can not pledge that there is $x \in B \setminus A$ such that $A \cup x$ is a closed set.

But if E is finite, $A \subset B$ pledges that $A \cup x$ is a closed set for some $x \in B \setminus A$.(cf.[1,2]).

Even though, we still have the following theorem.

Theorem 2 Let τ be a closure operator on E and $\mathfrak{L}$ be the set of closed sets such that for $A, B \in \mathfrak{L}$, if $A \subset B$, then there is a minimal closed set C meeting $A \subset C \subseteq B$. Then the following (1)-(4) are equivalent.

(1) (E, τ) is an infinite antimatroid.

(2) For all closed sets $A \subset B$, there exists $x \in B \setminus A$ such that $A \cup x$ is closed.

(3) $A = \tau(ex(A))$, for every closed set A.

(4) Every $A \subseteq E$ has a unique minimal spanning subset.

Proof (1)$\Rightarrow$(2) Similarly to finite case in [2], suppose C is a minimal closed set such that $A \subset C \subseteq B$, and let $x \in C \setminus A$. Then $x \notin \tau(A) = A$. If $y \in \tau(A \cup x) \setminus (A \cup x)$, then according to anti-exchange property, it gets $x \notin \tau(A \cup y)$. Thus, $A \subset \tau(A \cup y) \subset C$, which contradicts the minimality of C. Namely, there exists $x \in B \setminus A$ such that $A \cup x$ is closed.

(2)$\Rightarrow$(3) Clearly, $ex(A) \subseteq A$ and (coii) leads to $\tau(ex(A)) \subseteq \tau(A) = A$.

By Theorem 1, $(\mathfrak{L}, \subseteq)$ is a lattice. Let $A, B \in \mathfrak{L}$ and A cover B, in notation, $B \prec A$. Recalling back (2), there is $a \in A \setminus B$ satisfying $B \cup a = A$. This means $a \notin A \setminus a = B = \tau(A \setminus a)$. Hence $a \in ex(A)$ according to Lemma 2.

If $\tau(ex(A)) \subset A$. This implies that there is $D \in \mathfrak{L}$, $D \prec A$ and $\tau(ex(A)) \subseteq D \subset A$. Hence, there is $d \in A \setminus D$ so as to $d \in ex(A)$ in view of the above, which is a contradiction to $d \in A \setminus \tau(ex(A))$. Namely, (3) holds.

(3)$\Rightarrow$(4) We will prove that $ex(\tau(A))$ is the unique minimal spanning subset.

For any $A \subseteq E$, by (3), $\tau(A) = \tau(ex(\tau(A)))$ is true. It obtains that $ex(\tau(A))$ is a spanning subset of A. Let Y be a spanning set for A, i.e. $\tau(Y) = \tau(A)$. Suppose $x \notin Y$ for some $x \in ex(\tau(A))$. Then $x \in \tau(Y) = \tau(Y \setminus x) \subseteq \tau(\tau(Y) \setminus x) = \tau(\tau(A) \setminus x)$, a contradiction to $\tau(Y) = \tau(A)$. Hence $ex(\tau(A))$ is contained in every spanning set of A. Furthermore, $ex(\tau(A))$ is the unique minimal spanning subset of A.

(4)$\Rightarrow$(1) Suppose it has $x, y \notin \tau(A), x \neq y, y \in \tau(A \cup x)$, and $x \in \tau(A \cup y)$. Then $\tau(A \cup x) = \tau(A \cup y) = \tau(A \cup x \cup y)$. Let S be the unique minimal spanning set of $\tau(A \cup y)$. Since both $A \cup x$ and $A \cup y$ are spanning set of A, it must get $S \subseteq (A \cup x) \cap (A \cup y) = A$, and thus, $\tau(S) \subseteq \tau(A) \subset \tau(A \cup x)$, a contradiction. This shows the correct of (1).

Theorem 2 is true for finite case because for any closed sets, if $A \subset B$, it must have a minimal closed set C satisfying $A \subset C \subseteq B$. This also could be found in [2].

With the known results, we analyze our characterizations–Theorem 1 and Theorem 2.

(α) Let $E = \{1, 2, \ldots\}$ in which $1 > 2 > \ldots > n > n+1 > \ldots$. Define, for $A \subset E$, $\mathcal{D}_{\leq}(A) = \{x \in E | \exists y \in A : x \leq y\}$. By [5], $(E, \mathcal{D}_{\leq})$ is an infinite antimatroid. Evidently, $\emptyset \prec \ldots \prec D_{\leq}(\{n+1\}) \prec D_{\leq}(\{n\}) \prec \ldots \prec D_{\leq}(\{3\}) \prec D_{\leq}(\{2\}) \prec D_{\leq}(\{1\}) = E$ is not a finite chain in the lattice $(\mathcal{D}_{\leq}, \subseteq)$.

(β) For finite case, the similar result to Theorem 2 is [1,p.21,Theorem 1.1&2,8.7.2 Proposition]. The proofs [1,p.21,Theorem 1.1&2,8.7.2 Proposition] use finite property of the ground set E for finite convex geometry (E, τ). In other words, their talking are only true for finite convex geometries.

(γ) [2] says that when E is finite, (E, τ) is a convex geometry if and only if all maximal chains of closed sets, $\tau(\emptyset) = A_0 \subset A_1 \subset \ldots \subset A_k = E$, have the same length $k = |E - \tau(\emptyset)|$.

Considering (α), we see that this is not true for infinite antimatroid.

(η) [1,p.21,Theorem 1.1 and 2,8.7.2 Proposition] seldom directly discuss the characterizations in lattice theory. In the proof of Theorem 2, we use Theorem 1 which is a lattice characterization of infinite antimatroid.

(ζ) Combining $(\alpha) - (\eta)$, we conclude that our proof of Theorem 2 is different from [1,p.21,Theorem 1.1] and [2,8.7.2 Proposition], though sometimes it seems a little similar. That is to say, our characterizations especially Theorem 2 do not simply generalize the known facts; they have advantages and developments from finite to infinite.

Analogously to [5], we give a definition as follows.

Definition 2 An infinite antimatroid $(E, \mathfrak{L})$ is said to be *orderly* whenever

$$\forall A, B \in \mathfrak{L} \text{ with } A \cup B \in \mathfrak{L}.$$

We state now an easy result using Theorem 1.

Theorem 3 Let τ be a closure operator on E and $\mathfrak{L}$ be the set of closed sets of τ. Then $(E, \mathfrak{L})$ is an orderly infinite antimatroid if and only if $(\mathfrak{L}, \subseteq)$ is a complete and bounded lattice with $A \vee B = A \cup B$ for any $A, B \in \mathfrak{L}$ and has the hold of (ii).

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