An. Şt. Univ. Ovidius Constanţa Vol. 17(1), 2009, 15–22

An isoparametric function on almost k-contact manifolds

Adara M. BLAGA

Abstract

The aim of this paper is to point out an isoparametric function on an almost k-contact manifold.

1 Introduction

Almost 3-contact manifolds were introduced by Kuo [2] and independently, by Udrişte [5]. To their class belong also 3-Sasakian and 3-cosymplectic manifolds studied by Boyer and Galicki [1], whose properties were also analyzed by Montano and De Nicola [4]. In this paper, starting with a proposal for the notion of *almost k-contact structure*, we shall point out an isoparametric function which can be associated in this framework, by generalizing a similar construction initiated by Mihai and Rosca [3].

2 Almost k-contact manifolds

Recall that ***an almost contact manifold*** is an odd-dimensional manifold (M, Φ, ξ, η), where

1. Φ is a field of endomorphisms of the tangent space;
2. ξ is a vector field (called the *Reeb vector field*);
3. η is a 1-form, such that

Key Words: almost k-contact manifold; isoparametric function; Riemannian metric.
Mathematics Subject Classification: 53C15.
Received: December, 2008
Accepted: March, 2009

- $\Phi^2 = -I_{\Gamma(TM)} + \eta \otimes \xi$
- $\eta(\xi) = 1$,

where $I_{\Gamma(TM)}$ denotes the identity on the Lie algebra of vector fields.

Proposition 1 *Any almost contact manifold (M, Φ, ξ, η) admits a Riemannian metric g (called compatible metric) with the properties:*

$$g(\Phi(X), \Phi(Y)) = g(X, Y) - \eta(X)\eta(Y),$$

$$g(\xi, X) = \eta(X),$$

for any $X, Y \in \Gamma(TM)$.

We call (M, Φ, ξ, η, g) ***almost contact metric manifold***. In this case, the Reeb vector field ξ is orthonormal with respect to g $[g(\xi, \xi) = \eta(\xi) = 1]$.

A natural generalization of *almost* 3*-contact manifold* [4] is given by the following definition:

Definition 1 ***An almost k-contact manifold*** *is an $(n + k + nk)$-dimensional manifold M with k almost contact structures (Φ_1, ξ_1, η_1),...,(Φ_k, ξ_k, η_k) such that:*

- $\Phi_i \circ \Phi_j = -\delta_{ij} I_{\Gamma(TM)} + \eta_j \otimes \xi_i + \sum_{l=1}^{k} \varepsilon_{ijl} \Phi_l$
- $\eta_i(\xi_j) = \delta_{ij}$,

for any $i, j, l \in \{1, ..., k\}$, where ε_{ijl} is the totally antisymmetric symbol.

It follows that $\Phi_i(\xi_j) = \sum_{l=1}^{k} \varepsilon_{ijl} \xi_l$ and $\eta_i \circ \Phi_j = \sum_{l=1}^{k} \varepsilon_{ijl} \eta_l$, for any $i, j \in \{1, ..., k\}$. A similar computation like in the almost contact case leads us to $\Phi_i(\xi_i) = 0$ and $\eta_i \circ \Phi_i = 0$, for any $i \in \{1, ..., k\}$. Consider now the case $i \neq j$. Then

$$\Phi_i \circ \Phi_j = \eta_j \otimes \xi_i + \sum_{l=1}^{k} \varepsilon_{ijl} \Phi_l$$

and computing this relation on ξ_i and respectively ξ_j, we obtain

$$\sum_{l=1}^{k} \varepsilon_{ijl} \Phi_l(\xi_i) = \Phi_i(\Phi_j(\xi_i)) = \xi_j,$$

for any $i \neq j$. Multiplying $\xi_l = \Phi_j(\Phi_l(\xi_j))$ with ε_{ijl} and summing over l, we get

$$\sum_{l=1}^{k} \varepsilon_{ijl} \xi_l = \sum_{l=1}^{k} \varepsilon_{ijl} \Phi_j(\Phi_l(\xi_j)) = \Phi_j(\sum_{l=1}^{k} \varepsilon_{ijl} \Phi_l(\xi_j)) = -\Phi_j(\xi_i).$$

Then

$$\Phi_i(\xi_j) = -\sum_{l=1}^{k} \varepsilon_{jil}\xi_l = \sum_{l=1}^{k} \varepsilon_{ijl}\xi_l.$$

Computing $\Phi_i^2 = -I_{\Gamma(TM)} + \eta_i \otimes \xi_i$ for $\Phi_j(X)$, with arbitrary $X \in \Gamma(TM)$, we obtain

$$\begin{aligned} -\Phi_j(X) + \eta_i(\Phi_j(X))\xi_i &= \Phi_i^2(\Phi_j(X)) = \Phi_i[(\Phi_i \circ \Phi_j)(X)] \\ &= \Phi_i[\eta_j(X)\xi_i + \sum_{l=1}^{k} \varepsilon_{ijl}\Phi_l(X)] \\ &= \sum_{l=1}^{k} \varepsilon_{ijl}(\Phi_i \circ \Phi_l)(X). \end{aligned}$$

It follows that

$$\begin{aligned} (\eta_i \circ \Phi_j)(X)\xi_i &= \Phi_j(X) + \sum_{l=1}^{k} \varepsilon_{ijl}(\Phi_i \circ \Phi_l)(X) \\ &= \Phi_j(X) + \sum_{l=1}^{k} \varepsilon_{ijl}[-\delta_{il}X + \eta_l(X)\xi_i + \sum_{p=1}^{k} \varepsilon_{ilp}\Phi_p(X)] \\ &= \Phi_j(X) + \sum_{l=1}^{k} \varepsilon_{ijl}\eta_l(X)\xi_i - \Phi_j(X) = \sum_{l=1}^{k} \varepsilon_{ijl}\eta_l(X)\xi_i, \end{aligned}$$

for any $X \in \Gamma(TM)$. Applying η_i, we find

$$(\eta_i \circ \Phi_j)(X) = \sum_{l=1}^{k} \varepsilon_{ijl}\eta_l(X),$$

for any $X \in \Gamma(TM)$.

Proposition 2 *Any almost k-contact manifold $(M, \Phi_i, \xi_i, \eta_i)_{1 \le i \le k}$ admits a Riemannian metric g compatible with each of the k almost contact structures:*

$$g(\Phi_i(X), \Phi_i(Y)) = g(X, Y) - \eta_i(X)\eta_i(Y),$$

$$g(\xi_i, X) = \eta_i(X), \tag{1}$$

for any $X, Y \in \Gamma(TM), i \in \{1, ..., k\}$.

We call $(M, \Phi_i, \xi_i, \eta_i, g)_{1 \le i \le k}$ ***almost k-contact metric manifold***. In this case, the Reeb vector fields ξ_1,...,ξ_k are orthonormal with respect to g [$g(\xi_i, \xi_j) = \eta_i(\xi_j) = \delta_{ij}$, for any $i, j \in \{1, ..., k\}$].

3 Isoparametric function

Let $(M, \Phi_i, \xi_i, \eta_i, g)_{1\leq i\leq k}$ be an almost k-contact metric manifold and define $\mathcal{H} := \cap_{i=1}^k \ker \eta_i$ the horizontal distribution. Then the tangent bundle splits into the orthogonal sum of the horizontal and vertical distributions,

$$TM = \mathcal{H} \oplus < \xi_1, ..., \xi_k > .$$

Consider the vector field $\xi := \sum_{i=1}^k \lambda_i \xi_i$, $\lambda_i \in C^\infty(M)$ and define the 1-form $\eta := i_\xi g$. Then

$$\eta(X) = i_\xi g(X) = g(\xi, X) = g(\sum_{i=1}^k \lambda_i \xi_i, X) = \sum_{i=1}^k \lambda_i g(\xi_i, X) = \sum_{i=1}^k \lambda_i \eta_i(X),$$

for any $X \in \Gamma(TM)$ and in particular for $X = \xi$,

$$\eta(\xi) = \sum_{i=1}^k \lambda_i \eta_i(\xi) = \sum_{i=1}^k \lambda_i g(\xi_i, \xi) = g(\xi, \xi) = \parallel \xi \parallel^2 . \qquad (2)$$

Let ∇ be the Levi-Civita connection associated to g. From Cartan's structure equations, for $\{e_i\}_{1\leq i\leq k}$ an orthonormal frame and θ the local connection form, we have $\nabla e = \theta \otimes e$, with $\theta_i^j = \lambda_i \eta_j - \lambda_j \eta_i$, $i, j \in \{1, ..., k\}$. If we assume that ξ defines a skew symmetric connection, then $\theta_i^j(\xi) = 0$ and $d\eta_i = \eta \wedge \eta_i$, $i \in \{1, ..., k\}$. It follows

$$0 = d^2\eta_i = d(\eta \wedge \eta_i) = d\eta \wedge \eta_i - \eta \wedge d\eta_i = d\eta \wedge \eta_i - \eta \wedge (\eta \wedge \eta_i) = d\eta \wedge \eta_i,$$

so

$$0 = (d\eta \wedge \eta_i)(X, Y) = d\eta(X)\eta_i(Y) - d\eta(Y)\eta_i(X),$$

for any $X, Y \in \Gamma(TM)$. In particular, for $X = \xi_i$, $Y = \xi_j$, $i \neq j$,

$$0 = d\eta(\xi_i)\eta_i(\xi_j) - d\eta(\xi_j)\eta_i(\xi_i),$$

we find $d\eta(\xi_j) = 0$, for any $j \in \{1, ..., k\}$. Now, for $Y = \xi_i$,

$$0 = d\eta(X)\eta_i(\xi_i) - d\eta(\xi_i)\eta_i(X) = d\eta(X),$$

for any $X \in \Gamma(TM)$ and so $d\eta = 0$.

Following the ideas of Mihai and Rosca [3], we shall prove that on an almost k-contact manifold, $\parallel \xi \parallel^2$ is an isoparametric function. Let $\flat(X) := i_X g$ and $\sharp := \flat^{-1}$ be the musical isomorphisms.

Assume that $\nabla\lambda_i = f\xi_i$, $f \in C^\infty(M)$. Then $\sharp(d\lambda_i) = f\xi_i \Leftrightarrow \flat^{-1}(d\lambda_i) = f\xi_i \Leftrightarrow d\lambda_i = \flat(f\xi_i) = i_{f\xi_i}g = fi_{\xi_i}g = f\eta_i$ and

$$\begin{aligned} 0 &= d^2\lambda_i = d(f\eta_i) = df \wedge \eta_i + fd\eta_i \\ &= df \wedge \eta_i + f\eta \wedge \eta_i = (df + f\eta) \wedge \eta_i \end{aligned}$$

implies $df + f\eta = 0$.

Set $2\lambda = \parallel \xi \parallel^2$ $[= g(\xi,\xi)]$. Then

$$\begin{aligned} d\lambda &= d(\frac{g(\xi,\xi)}{2}) = \frac{1}{2}d[g(\sum_{i=1}^{k}\lambda_i\xi_i, \sum_{j=1}^{k}\lambda_j\xi_j)] = \frac{1}{2}d[\sum_{1\le i,j\le k}\lambda_i\lambda_j g(\xi_i,\xi_j)] \\ &= \frac{1}{2}d[\sum_{1\le i,j\le k}\lambda_i\lambda_j\eta_i(\xi_j)] = \frac{1}{2}d[\sum_{1\le i,j\le k}\lambda_i\lambda_j\delta_{ij}] = \frac{1}{2}d[\sum_{1\le i\le k}\lambda_i^2] \\ &= \frac{1}{2}(\sum_{1\le i\le k}2\lambda_i d\lambda_i) = \sum_{1\le i\le k}\lambda_i d\lambda_i = \sum_{1\le i\le k}\lambda_i f\eta_i \\ &= f\sum_{1\le i\le k}\lambda_i\eta_i = f\eta \end{aligned}$$

and $d(f+\lambda) = df + d\lambda = df + f\eta = 0$ implies $f + \lambda = c(constant)$.

From the structure's equations follows that

$$\nabla_Z\xi_i = \lambda_i\sum_{j=1}^{k}\eta_j(Z)\xi_j - \eta_i(Z)\xi, \tag{3}$$

for any $Z \in \Gamma(TM)$, $i \in \{1,...,k\}$ [3]. Therefore,

Lemma 1 *For any* $Z \in \Gamma(TM)$, $\nabla_Z\xi = (2\lambda + f)\sum_{j=1}^{k}\eta_j(Z)\xi_j - \eta(Z)\xi$.

Proof. Indeed,

$$\begin{aligned}\nabla_Z\xi &= \sum_{i=1}^{k}[\lambda_i\nabla_Z\xi_i + Z(\lambda_i)\xi_i]\\ &= \sum_{i=1}^{k}[\lambda_i(\lambda_i\sum_{j=1}^{k}\eta_j(Z)\xi_j - \eta_i(Z)\xi) + Z(\lambda_i)\xi_i]\\ &= [\sum_{i=1}^{k}\lambda_i^2][\sum_{j=1}^{k}\eta_j(Z)\xi_j] - \eta(Z)\xi + \sum_{i=1}^{k}d\lambda_i(Z)\xi_i\\ &= \parallel \xi \parallel^2 [\sum_{j=1}^{k}\eta_j(Z)\xi_j] - \eta(Z)\xi + \sum_{i=1}^{k}f\eta_i(Z)\xi_i\\ &= (2\lambda + f)[\sum_{i=1}^{k}\eta_i(Z)\xi_i] - \eta(Z)\xi,\end{aligned}$$

for any $Z \in \Gamma(TM)$.

Theorem 1 *Let on an almost k-contact metric manifold M a number of k-smooth functions λ_i such that for all i, the gradient vector field $\nabla\lambda_i$ is parallel with ξ_i with the same factor $f \in C^\infty(M)$. Then, for the vector field $\xi := \sum_{i=1}^{k}\lambda_i\xi_i$, its norm is an isoparametric function on M.*

Proof. Since $\{\xi_i\}$ is an orthonormal set for g we have:

$$2\lambda = \sum_{i=1}^{k}\lambda_i^2$$

and then:

$$\nabla\lambda = f(\sum_{i=1}^{k}\lambda_i\xi_i) = (c-\lambda)(\sum_{i=1}^{k}\lambda_i\xi_i) = (c-\lambda)\xi. \tag{4}$$

Therefore,

$$\parallel \nabla\lambda \parallel^2 = (c-\lambda)^2 2\lambda. \tag{5}$$

Then,

$$div(\nabla\lambda) = (c-\lambda)div\xi - \xi(\lambda), \tag{6}$$

but

$$\xi(\lambda) = \frac{1}{2}\xi(g(\xi,\xi)) = g(\nabla_\xi\xi,\xi). \tag{7}$$

From:

$$\nabla_\xi \xi = (\lambda + c)\xi - \eta(\xi)\xi = (c - \lambda)\xi, \tag{8}$$

it results:

$$div(\nabla\lambda) = (c - \lambda)[kc + \frac{k-2}{2}2\lambda - 2\lambda] = (c - \lambda)[kc + (k-4)\lambda], \tag{9}$$

which, for $k = 3$ gives the relation (2.24) of Rosca-Mihai.

Acknowledgments

I wish to express my thanks to the referees for their useful remarks.

References

[1] C. Boyer, K. Galicki, 3-*Sasakian manifolds. Surveys in differential geometry: essays on Einstein manifolds*, Surv. Differ. Geom., VI, Int. Press, Boston, MA 1999, pp. 123–184.

[2] Y. Y. Kuo, *On almost contact* 3-*structure*, Tohoku Math. J., **22** (1970), 235–332.

[3] A. Mihai, R. Roşca, *On vertical skew symmetric almost contact* 3-*structures*, J. Geom., **82**(2005), 146–155.

[4] B. Cappelletti Montano, A. De Nicola, *3-Sasakian manifolds, 3-cosymplectic manifolds and Darboux theorem*, J. Geom. Phys., **57**(2007), 2509–2520.

[5] C. Udrişte, *Structures presque coquaternioniennes*, Bull. Math. R. S. Roumanie, **13**(61)(1969), 487–507.

Adara M. Blaga
Department of Mathematics and Computer Science
West University of Timişoara
Bld. V. Pârvan nr.4, 300223 Timişoara, România
adara@math.uvt.ro