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Variational statements on KST-metric structures

M. Turinici *

Abstract

Structural refinements of the variational result in Kada, Suzuki and Takahashi [Math. Japonica, 44 (1996), 381-391] are given. Applications to equilibrium points theory (over the considered setting) are also provided.

1 Introduction

Let (M, d) be a complete metric space; and $\varphi : M \to R \cup \{\infty\}$, some function with

(1a) φ is inf-proper ($\mathrm{Dom}(\varphi) \neq \emptyset$ and $\varphi_* := \inf[\varphi(M)] > -\infty$)

(1b) φ is d-lsc ($\liminf_n \varphi(x_n) \geq \varphi(x)$, whenever $x_n \xrightarrow{d} x$).

The following 1979 statement in Ekeland [12] (referred to as Ekeland's variational principle; in short: EVP) is well known.

Theorem 1 *Let the precise conditions hold. Then,*

a) *for each $u \in \mathrm{Dom}(\varphi)$ there exists $v = v(u) \in \mathrm{Dom}(\varphi)$ with*

$$d(u, v) \leq \varphi(u) - \varphi(v) \text{ (hence } \varphi(u) \geq \varphi(v)\text{)} \tag{1}$$

Key Words: Complete metric space; Inf-proper lsc function; Variational principle; w-distance; Pseudometric; Cauchy/asymptotic sequence; Transitive relation; (strong) KST-metric; τ-distance/function; equilibrium point.

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$$\begin{array}{l} v \textit{ is E-variational (modulo } (d,\varphi)\textit{):} \\ d(v,x) > \varphi(v) - \varphi(x), \quad \textit{for all } x \in M \setminus \{v\} \end{array} \tag{2}$$

aa) *if* $u \in \mathrm{Dom}(\varphi)$, $\rho > 0$ *fulfill* $\varphi(u) - \varphi_* \leq \rho$, *then (1) gives*

$$(\varphi(u) \geq \varphi(v) \textit{ and}) \quad d(u,v) \leq \rho. \tag{3}$$

This principle found some basic applications to control and optimization, generalized differential calculus, critical point theory and global analysis; we refer to the quoted paper for details. So, it cannot be surprising that, soon after, many extensions of EVP were proposed. For example, the abstract (order) one starts from the fact that, with respect to the (quasi-)order

(a1) $(x, y \in M)$ $x \leq y$ iff $d(x,y) + \varphi(y) \leq \varphi(x)$

the point $v \in M$ appearing in (2) is *maximal*; so that, EVP is nothing but a variant of the Zorn-Bourbaki maximality principle [7]. The dimensional way of extension refers to the support space (R) of $\mathrm{Codom}(\varphi)$ being substituted by a (topological or not) vector space. An account of the results in this area is to be found in the 2003 monograph by Goepfert, Riahi, Tammer and Zălinescu [13, Ch 3]; see also Rozoveanu [23] and Turinici [28]. Finally, the metrical one consists in the conditions imposed to the ambient metric over M being relaxed. The basic 1996 result in this direction obtained by Kada, Suzuki and Takahashi [16] may be stated as follows. By a *pseudometric* over M we shall mean any map $(x,y) \mapsto e(x,y)$ from $M \times M$ to $R_+ := [0, \infty[$. Suppose that we fixed such an object; which in addition, is *triangular* $[e(x,z) \leq e(x,y) + e(y,z), \forall x,y,z \in M]$. We say that it is a *w-distance* (modulo d) over M provided

(b1) e is strongly d-sufficient: for each $\varepsilon > 0$, there exists $\delta > 0$ such that: $e(z,x), e(z,y) \leq \delta \Longrightarrow d(x,y) \leq \varepsilon$.

(c1) $y \mapsto e(x,y)$ is d-lsc on M (see above), $\forall x \in M$.

Theorem 2 *Let the conditions in Theorem 1 be admitted; and e be some w-distance (modulo d) over M. Then*

b) *For each* $u \in \mathrm{Dom}(\varphi)$, *there exists an E-variational (modulo* (e,φ)*)* $v = v(u) \in \mathrm{Dom}(\varphi)$ *with* $\varphi(u) \geq \varphi(v)$

bb) *For each* $\rho > 0$ *and each* $u \in \mathrm{Dom}(\varphi)$ *with* $e(u,u) = 0$, $\varphi(u) \leq \varphi_* + \rho$ *there exists an E-variational (modulo* (e,φ)*)* $v = v(u,\rho) \in \mathrm{Dom}(\varphi)$ *with* $\varphi(u) \geq \varphi(v)$ *and* $e(u,v) \leq \rho$.

In particular, when $e = d$, these regularity conditions hold; and Theorem 2 includes the local version of Theorem 1 based upon (3). The relative form of the same, based upon (1) also holds, but indirectly; see Bao and Khanh [5] for details. Note that the (rather involved) authors' argument relies on the nonconvex minimization theorem in Takahashi [26]. It is our aim in the present exposition to show (in Section 3) that a simplification of this is possible. The basic tool of our investigations is (cf. Section 2) a pseudometric variational principle (extending the one in Tataru [27]) deductible, in fact, by the original Ekeland's argument. Finally, in Section 4, an application of the obtained facts is given to equilibrium points theory over such structures.

2 Pseudometric variational statements

Let M be some nonempty set. Remember that, by a *pseudometric* over it we shall mean any map $e : M \times M \to R_+$. Suppose that we fixed such an object; which in addition, is *triangular* (cf. Section 1). Let also $\varphi : M \to R \cup \{\infty\}$ be some inf-proper function (cf. (1a)). For an easy reference, we shall formulate the basic regularity conditions involving our data. Call the sequence (x_n) in M, *strongly e-asymptotic* when the series $\sum_n e(x_n, x_{n+1})$ converges (in R). Further, let the *e-Cauchy* property of this object be the usual one: $\forall \delta > 0, \exists n(\delta)$, such that $n(\delta) \leq p < q \Longrightarrow e(x_p, x_q) \leq \delta$. By the triangular property of e we have (for each sequence): strongly e-asymptotic $\Longrightarrow$ e-Cauchy; but the converse is not in general true. Nevertheless, in many conditions involving *all* such objects, this is retainable. A concrete example is to be constructed under the lines below. Let us introduce an *e-convergence* structure over M by: $x_n \xrightarrow{e} x$ iff $e(x_n, x) \to 0$ as $n \to \infty$. We consider the regularity condition

(2a) (e, φ) is weakly descending complete: for each strongly e-asymptotic sequence (x_n) in $\mathrm{Dom}(\varphi)$ with $(\varphi(x_n))$ descending there exists $x \in M$ with $x_n \xrightarrow{e} x$ and $\lim_n \varphi(x_n) \geq \varphi(x)$.

By the remark above, it is implied by its (stronger) counterpart

(2b) (e, φ) is descending complete: for each e-Cauchy sequence (x_n) in $\mathrm{Dom}(\varphi)$ with $(\varphi(x_n))$ descending there exists $x \in M$ with $x_n \xrightarrow{e} x$ and $\lim_n \varphi(x_n) \geq \varphi(x)$.

A remarkable fact to be added is that the reciprocal inclusion also holds:

Lemma 1 *We have (2a) $\Longrightarrow$ (2b); hence (2a) $\Longleftrightarrow$ (2b).*

Proof Assume that (2a) holds; and let (x_n) be an e-Cauchy sequence in $\mathrm{Dom}(\varphi)$ with $(\varphi(x_n))$ descending. By the very definition of this property, there must be a strongly e-asymptotic subsequence $(y_n = x_{i(n)})$ of (x_n) with $(\varphi(y_n))$ descending. This, along with (2a), yields an element $y \in M$ fulfilling

$y_n \xrightarrow{e} y$ and $\lim_n \varphi(y_n) \geq \varphi(y)$. It is now clear (by the choice of (x_n)) that the point y has all desired in (2b) properties. ■

Now, let $\nabla = \nabla_{(e,\varphi)}$ stand for the relation (over M)

(a2) $(x, y \in M)$ $x\nabla y$ iff $e(x, y) + \varphi(y) \leq \varphi(x)$.

This is clearly transitive ($x\nabla y, y\nabla z \Longrightarrow x\nabla z$); but not in general reflexive ($x\nabla x$ may be false for certain $x \in M$). So, for $u \in M$, it is possible that $M(u, \nabla) := \{x \in M; u\nabla x\}$ be empty. If this does not hold ($M(u, \nabla) \neq \emptyset$), we say that u is ∇-*starting*; i.e.,

(b2) $e(u, x) + \varphi(x) \leq \varphi(u)$, for at least one $x \in M$;

also referred to as: u is *starting* (modulo (e, φ)). Finally, call $v \in \mathrm{Dom}(\varphi)$, BB-*variational* (modulo (e, φ)) provided

(c2) $e(v, x) \leq \varphi(v) - \varphi(x) \Longrightarrow \varphi(v) = \varphi(x)$ $[\Longrightarrow e(v, x) = 0]$.

(This terminology is related to the methods introduced by Brezis and Browder [8] and developed by Kang and Park [17]; see also Altman [2], Anisiu [3] and Szaz [25]). For a non-trivial concept, we must take v as starting (modulo (e, φ)) (cf. (b2)); for, otherwise, (c2) is vacuously fulfilled. Some basic properties of such points are given in

Lemma 2 *Assume $v \in \mathrm{Dom}(\varphi)$ is BB-variational (modulo (e, φ)). Then*

$$e(v, x) \geq \varphi(v) - \varphi(x), \text{ for all } x \in M \tag{1}$$

$$e(v, x) > \varphi(v) - \varphi(x), \text{ for each } x \in M \text{ with } e(v, x) > 0. \tag{2}$$

Proof The latter part is clear, by definition; so, it remains to establish the former one. Assume this would be false: $e(v, x) < \varphi(v) - \varphi(x)$, for some $x \in M$. This, along with (c2), yields $\varphi(v) = \varphi(x)$; wherefrom $0 \leq e(v, x) < 0$, contradiction; hence the claim. ■

We may now state a useful pseudometric variational principle.

Theorem 3 *Let the general conditions upon (e, φ) be accepted; as well as (2a)/(2b). Then, for each starting (modulo (e, φ)) $u \in \mathrm{Dom}(\varphi)$ there exists a BB-variational (modulo (e, φ)) $v = v(u) \in \mathrm{Dom}(\varphi)$ with the property (1) (modulo e).*

Proof Let (∇) stand for the transitive relation (a2); and $u \in \mathrm{Dom}(\varphi)$ be as in the statement. By definition, $M(u, \nabla) \neq \emptyset$; hence $u\nabla u_0$, for some $u_0 \in \mathrm{Dom}(\varphi)$. Put $n = 0$. If the alternative below holds

(2c) u_n is BB-variational (modulo (e, φ)) [see above]

we are done, with $v = u_n$. Otherwise, one has (for $n = 0$)

(2d) u_n is not BB-variational (modulo (e, φ)).

In this case, the relations below hold (for $n = 0$)

$$S_n := M(u_n, \nabla) \neq \emptyset, \quad \inf[\varphi(S_n)] < \varphi(u_n); \tag{3}$$

wherefrom, by definition (again for $n = 0$)

$$\exists u_{n+1} \in S_n: \ \varphi(u_{n+1}) \leq (1/2)(\varphi(u_n) + \inf[\varphi(S_n)]) < \varphi(u_n). \tag{4}$$

Now, like before, the couple (2c)+(2d) (with $n+1$ in place of n) comes into discussion, etc. The only point to be clarified is that of (2d) taking place for all n. Then, a sequence (u_n) in $M(u, \nabla)$ may be found so as (cf. (4))

$$e(u_n, u_m) \leq \varphi(u_n) - \varphi(u_m), \text{ whenever } n < m. \tag{5}$$

The sequence $(\varphi(u_n))$ is (strictly) descending and bounded below in R; hence a Cauchy one. This, along with (5), tells us that (u_n) is an e-Cauchy sequence in $\mathrm{Dom}(\varphi)$. Putting these together, it follows via (2b) that there must be some $v \in M$ with

$$u_n \xrightarrow{e} v \text{ and } \lambda := \lim_n \varphi(u_n) \geq \varphi(v). \tag{6}$$

The second half of this gives $v \in \mathrm{Dom}(\varphi)$; since $(u_n) \subseteq \mathrm{Dom}(\varphi)$. On the other hand, fix some rank n. By (5) (and the triangular property of e)

$$e(u_n, v) \leq e(u_n, u_m) + e(u_m, v) \leq \varphi(u_n) - \varphi(u_m) + e(u_m, v), \forall m > n.$$

This, along with (6), yields by a limit process (relative to m)

$$e(u_n, v) \leq \varphi(u_n) - \lim_m \varphi(u_m) \leq \varphi(u_n) - \varphi(v) \quad \text{(i.e.: } u_n \nabla v). \tag{7}$$

Let $x \in M(v, \nabla)$ be arbitrary fixed. By (7) (and the definition of (∇))

$$\varphi(u_n) \geq \varphi(v) \geq \varphi(x), \forall n; \quad \text{hence } \lambda \geq \varphi(v) \geq \varphi(x).$$

On the other hand, (4) yields

$$\varphi(u_{n+1}) \leq (1/2)[\varphi(u_n) + \varphi(x)], \forall n; \ \text{ hence } \lambda \leq \varphi(x) \leq \varphi(v).$$

This gives $\varphi(v) = \varphi(x)$; which (by the arbitrariness of x) tells us that v is BB-variational (modulo (e, φ)). The proof is complete. ■

Now, the regularity condition (2a) holds under

(2e) (e, φ) is weakly complete:
for each strongly e-asymptotic sequence (x_n) in $\mathrm{Dom}(\varphi)$
there exists $x \in M$ with $x_n \xrightarrow{e} x$ and $\liminf_n \varphi(x_n) \geq \varphi(x)$.

For example, this is retainable whenever

(2f) e is weakly complete:
each strongly e-asymptotic sequence is e-convergent

(2g) φ is weakly e-lsc: $\liminf_n \varphi(x_n) \geq \varphi(x)$ whenever the
strongly e-asymptotic sequence (x_n) fulfills $x_n \xrightarrow{e} x$.

In particular, when e is (in addition) *reflexive* $[e(x,x) = 0, \forall x \in M]$, Theorem 3 includes the variational principle in Tataru [27]. The question of the converse inclusion being also true remains open; we conjecture that the answer is positive. Note finally that, by the proposed proof, Theorem 3 is logically reducible to the Principle of Dependent Choices (cf. Wolk [31]). In fact, a similar conclusion is true for the Brezis-Browder ordering principle [8]; see, for instance Cârjă, Necula and Vrabie [9, Ch 2, Sect 2.1]. So, it is natural asking of Theorem 3 being deductible from the quoted principle. A positive answer for this is available; we shall develop such facts elsewhere. Further aspects were delineated in Hyers, Isac and Rassias [15, Ch 5]; see also Bae, Cho and Yeom [4].

Let us now return to our initial setting. An interesting completion of Theorem 3 is the following. Let the concept of E-*variational* (modulo (e, φ)) point be that of (2) (with e in place of d). This is stronger than the concept of BB-variational (modulo (e, φ)) element introduced via (c2). To get a corresponding form of Theorem 3 involving such points we have to impose (in addition to (2a)/(2b))

(2h) e is *transitively sufficient* $(e(z,x) = e(z,y) = 0 \Longrightarrow x = y)$.

Theorem 4 *Let the precise conditions be in force. Then, for each starting (modulo* (e, φ)*)* $u \in \mathrm{Dom}(\varphi)$ *there exists an E-variational (modulo* (e, φ)*)* $w = w(u) \in \mathrm{Dom}(\varphi)$ *with the property (1) (modulo* (e, w)*).*

Proof Let $u \in \mathrm{Dom}(\varphi)$ be taken as in the statement. By Theorem 3, we have promised a BB-variational (modulo (e, φ)) $v = v(u) \in \mathrm{Dom}(\varphi)$ with the property (1) (modulo e). If v is E-variational (modulo (e, φ)) we are done (with $w = v$); so, it remains the alternative of v fulfilling the opposite property: $v \nabla w$ (hence $\varphi(v) = \varphi(w)$), for some $w \in M \setminus \{v\}$. In this case, w is our desired element. Assume not: $w \nabla y$, for some $y \in M$, $y \neq w$. By the preceding relation, $v \nabla y$ (hence $\varphi(v) = \varphi(y)$). Summing up, $v \nabla w$, $v \nabla y$ and

$\varphi(v) = \varphi(w) = \varphi(y)$; wherefrom (by (2h) and the definition of (∇)) $w = y$, contradiction. This ends the argument. ■

As before, the regularity condition (2b) holds under

(2i) (e, φ) is complete: for each e-Cauchy sequence (x_n) in $\mathrm{Dom}(\varphi)$ there exists $x \in M$ with $x_n \xrightarrow{e} x$ and $\liminf_n \varphi(x_n) \geq \varphi(x)$.

For example, this is retainable whenever

(2j) e is complete: each e-Cauchy sequence is e-convergent

(2k) φ is Cauchy e-lsc: $\liminf_n \varphi(x_n) \geq \varphi(x)$, whenever the e-Cauchy sequence (x_n) fulfills $x_n \xrightarrow{e} x$.

On the other hand, (2h) holds whenever e is *sufficient* $[e(x,y) = 0 \Longrightarrow x = y]$. Note that, in such a case, Theorem 4 becomes the variational principle in Turinici [29]; see also Conserva and Rizzo [10]. In particular, when e is (in addition) reflexive and *symmetric* $[e(x,y) = e(y,x), \forall x, y \in M]$ (hence, a (standard) *metric* on M) Theorem 4 is nothing but Ekeland's variational principle (EVP). Further aspects were delineated in Dancs, Hegedus and Medvegyev [11]; see also Hamel [14, Ch 4].

3 Extended KST principles

We are now in position to get an appropriate answer to the questions in Section 1. Let M be a nonempty set; and $d : M \times M \to R_+$ be a pseudometric over it; supposed to be triangular (cf. Section 1) and reflexive sufficient ($d(x,y) = 0 \Longleftrightarrow x = y$). This map has all properties of a metric, except symmetry; we shall term it, an *almost metric* on M. Given such an object, the d-Cauchy and d-convergence properties for a sequence in M are those in Section 2. Let us consider the condition

(a3) d is complete: each d-Cauchy sequence is d-convergent.

Note that, by the lack of symmetry, a d-convergent sequence need not be d-Cauchy; hence, this concept has a technical motivation only.

Let $e : M \times M \to R_+$ be a triangular pseudometric over M. We shall say that this object is a KST-*metric* (modulo d) provided

(b3) e is Cauchy subordinated to d: each e-Cauchy sequence is d-Cauchy (hence d-convergent)

(c3) e is Cauchy d-lsc in the second variable: (y_n) is e-Cauchy and $y_n \xrightarrow{d} y$ imply $\liminf_n e(x, y_n) \geq e(x,y), \forall x \in M$.

Further, let $\varphi : M \to R \cup \{\infty\}$ be some inf-proper, d-lsc function (cf. (1a)+(1b)). The following auxiliary fact will be useful for us.

Lemma 3 *Assume that e is some KST-metric (modulo d) over M. Then, (e, φ) is complete (in the sense of (2i)); hence (a fortiori), descending complete (in the sense of (2b)).*

Proof Let (x_n) be some e-Cauchy sequence in $\mathrm{Dom}(\varphi)$. From (b3), (x_n) is d-Cauchy; so, by completeness, $x_n \xrightarrow{d} y$ as $n \to \infty$, for some $y \in M$. We claim that this is our desired point for (2i). In fact, let $\gamma > 0$ be arbitrary fixed. By the choice of (x_n), there exists $k = k(\gamma)$ so that $e(x_p, x_m) \leq \gamma$, for each $p \geq k$ and each $m > p$. Passing to limit upon m one gets (via (c3)) $e(x_p, y) \leq \gamma$, for each $p \geq k$; and since $\gamma > 0$ was arbitrarily chosen, $x_n \xrightarrow{e} y$. This, along with (1b), yields the needed conclusion. ■

Now, combining these with Theorem 3 (and Lemma 2), we derive

Theorem 5 *Let e be some KST-metric (modulo d) and φ be inf-proper, d-lsc. Then, for each starting (modulo (e, φ)) $u \in \mathrm{Dom}(\varphi)$ there exists $v = v(u) \in \mathrm{Dom}(\varphi)$ with the properties (1) (modulo e) and (1)-(2).*

An interesting problem to be posed is that of getting a corresponding form of this result involving E-variational (modulo (e, φ)) points. The appropriate answer to this is obtainable via Theorem 4. Call the triangular pseudometric $e : M \times M \to R_+$, a *strong* KST-metric (modulo d) when it is a KST-metric (modulo d) and fulfills (2h).

Theorem 6 *Assume that e is some strong KST-metric (modulo d) and φ is inf-proper, d-lsc. Then, for each starting (modulo (e, φ)) $u \in \mathrm{Dom}(\varphi)$ there exists $w = w(u) \in \mathrm{Dom}(\varphi)$ fulfilling (1)+(2) (with (e, w) in place of (d, v)).*

In the following, we shall give some particular cases of our statements.

I) Let d be a metric (i.e.: symmetric almost metric) on M; supposed to be complete (cf. (a3)). Further, let $e : M \times M \to R_+$ be some w-distance (modulo d). By (c1), one gets at once (c3) (see above). On the other hand, (b1) yields (b3) as well as the extra condition (2h); because (as d=metric) $e(z, x) = e(z, y) = 0 \Longrightarrow [d(x, y) \leq \varepsilon, \forall \varepsilon > 0] \Longrightarrow x = y$. Summing up, any w-distance is a strong KST-metric (modulo d). Hence the variational statement in Kada, Suzuki and Takahashi [16] (subsumed to Theorem 2) is a particular case of Theorem 6. But, as explicitly stated by these authors, their contribution extends the one due to T. H. Kim, E. S. Kim and S. S. Shin [18]; hence so does our statement. This also shows that any recursion to the nonconvex minimization theorem in Takahashi [26] is avoidable in such approaches. Some related facts may be found in Ume [30]; see also Zhu, Zhong and Wang [33].

II) Let (M, d) be a complete metric space; and $e : M \times M \to R_+$ be a triangular pseudometric over M. According to Suzuki [24], we say that this object is a *τ-distance* (modulo d) over M when there exists a function $\eta = \eta(e)$ from $M \times R_+$ to R_+ with the properties

(3a) $t \mapsto \eta(x, t)$ is increasing and $\lim_{t\to 0} \eta(x, t) = 0 = \eta(x, 0), \forall x \in M$

(3b) $\lim_n[\sup\{\eta(z_n, e(z_n, y_m)); m \geq n\}] = 0$ and $y_n \xrightarrow{d} y$ imply $\liminf_n e(x, y_n) \geq e(x, y)$, for each $x \in M$

(3c) $\lim_n[\sup\{e(x_n, y_m); m \geq n\}] = 0$ and $\lim_n \eta(x_n, t_n) = 0$ imply $\lim_n \eta(y_n, t_n) = 0$

(3d) $\lim_n \eta(z_n, e(z_n, x_n)) = 0$ and $\lim_n \eta(z_n, e(z_n, y_n)) = 0$ imply $\lim_n d(x_n, y_n) = 0$.

Clearly, any w-distance is a τ-distance too; just take $\eta(x, t) = t, \forall x \in M, \forall t \in R_+$. On the other hand (as we already remarked), any w-distance is a strong KST-metric. So, it is natural asking of what can be said about the relationships between these enlargements of our initial concept. The answer to this is given in

Lemma 4 *Let the precise conventions be in use. Then, each τ-distance (modulo d) is necessarily a strong KST-metric (modulo d); so that,*

$$w\text{-distance} \Longrightarrow \tau\text{-distance} \Longrightarrow \text{strong KST-metric (modulo } d). \qquad (1)$$

Proof Let $e : M \times M \to R_+$ be a τ-distance; and $\eta = \eta(e)$ stand for some associated map fulfilling (3a)-(3d). **i)** Let $x, y, z \in M$ be such that $e(z, x) = e(z, y) = 0$. By (3a), we have $\eta(z, e(z, x)) = \eta(z, e(z, y)) = 0$; and this, added to (3d), gives $d(x, y) = 0$ (hence $x = y$); wherefrom (2h) holds. **ii)** Call the sequence (x_n) in M, (η, e)-*Cauchy* provided

(d3) $\lim_n[\sup\{\eta(z_n, e(z_n, x_m)); m \geq n\}] = 0$, for some $(z_n) \subseteq M$.

By [24, Lemma 3], the generic inclusion holds

$$\text{[for each sequence] } e\text{-Cauchy} \Longrightarrow (\eta, e)\text{-Cauchy}. \qquad (2)$$

On the other hand, note that (3b) may be written as

(3e) (y_n) is (η, e)-Cauchy and $y_n \xrightarrow{d} y$ imply
$\liminf_n e(x, y_n) \geq e(x, y)$, for each $x \in M$.

Combining these gives (c3). **iii)** Finally, note that by [24, Lemma 1]

$$\text{[for each sequence]} \quad (\eta, e)\text{-Cauchy} \Longrightarrow d\text{-Cauchy}; \tag{3}$$

This, via (2), yields (b3). The proof is thereby complete. ∎

Now, by simply adding this to Theorem 6, one gets

Theorem 7 *Assume that e is some τ-distance (modulo d), and φ is inf-proper, d-lsc. Then, for each starting (modulo (e, φ)) $u \in \mathrm{Dom}(\varphi)$ there exists an E-variational (modulo (e, φ)) $w = w(u) \in \mathrm{Dom}(\varphi)$ with the property (1) (modulo e).*

The original proof of this result was provided in Suzuki [24]; but, it is rather involved. The proposed reasoning (based on the developments in Section 2) may be viewed as a refinement of this one; however, it still depends on Suzuki's arguments concerning the inclusions (2)+(3). It would be interesting to have alternate proofs of these, so as to avoid Lemma 4 above. Further structural aspects may be found in G. M. Lee, B. S. Lee, J. S. Jung and S. S. Chang [19]; see also Zhang, Chen and Guo [32].

III) Let again d be a metric over M; supposed to be complete (cf. (a3)); and $e : M \times M \to R_+$ be a triangular pseudometric. According to Lin and Du [20] we say that it is a *τ-function* (modulo d) provided (2h) holds and

(3f) $x \in M$, $y_n \to y$ and $e(x, y_n) \leq M$, $\forall n$ (for some $M = M(x) > 0$) imply $e(x, y) \leq M$

(3g) $\lim_n [\sup\{e(x_n, x_m); m > n\}] = 0$ and $\lim_n e(x_n, y_n) = 0$ imply
$\lim_n d(x_n, y_n) = 0$.

By [20, Remark 1] each w-distance (modulo d) is a τ-function (modulo d). So, it is natural asking of the relationships between this last concept and that of strong KST-metric (modulo d). The answer is contained in

Lemma 5 *Let the precise conventions hold. Then, each τ-function (modulo d) is a strong KST-metric (modulo d); so (combining with the above)*

$$w\text{-distance} \Longrightarrow \tau\text{-function} \Longrightarrow \text{strong KST-metric (modulo } d). \tag{4}$$

Proof Let $e : M \times M \to R_+$ be some τ-function (modulo d). By definition, it fulfills (2h); so, it remains to prove that (b3)+(c3) hold. The latter of these is just (3f). To verify the former, call the sequence (x_n), *almost e-Cauchy* when $\lim_n [\sup\{e(x_n, x_m); m > n\}] = 0$. By definition,

$$\text{[for each sequence]} \ \ e\text{-Cauchy} \Longrightarrow \text{almost } e\text{-Cauchy}.$$

On the other hand, [20, Lemma 2.1] tells us that

$$\text{[for each sequence]} \ \ \text{almost } e\text{-Cauchy} \Longrightarrow d\text{-Cauchy}. \tag{5}$$

Combining with the above gives (b3); and the claim follows. ■

As a consequence of this, the variational statement (involving τ-functions (modulo d)) obtained by the quoted authors is deductible from Theorem 6 above. Note that (as before) the proposed proofs are still depending on Lin-Du's reasoning involving the relation (5); so, it would be interesting to have alternate proofs of it. Further aspects may be found in Lin and Du [21].

IV) Let (M, d) be a complete almost metric space (see above); and $e : M \times M \to R_+$ be a triangular pseudometric over M. According to Al-Homidan Ansari and Yao [1], we say that it is a *Q-function* (modulo d) provided (b1) and (3f) hold. Clearly, the latter condition is just (c1); and from this, (c3) is fulfilled. On the other hand, (b1) yields (b3) and (2h). [The argument is similar to the "metrical" one in **I)**]. We therefore proved

$$Q\text{-function} \Longrightarrow \text{strong KST-metric} \ \ (\text{modulo } d).$$

As a direct consequence, the variational statement (involving Q-functions (modulo d)) obtained by the quoted authors [1, Theorem 3.1] is deductible from Theorem 6. The converse question is still open; we conjecture that a positive answer is ultimately available.

An interesting particular case refers to d being (in addition) symmetric; hence, a metric on M. By the previous remark involving (c1), it follows that

j) each τ-function is a Q-function (modulo d) (cf. [1, Remark 2.1])

jj) the concepts of Q-function and w-distance are *identical*.

This, along with Lemma 5, gives us the inclusions

$$w\text{-distance} \Longrightarrow \tau\text{-function} \Longrightarrow w\text{-distance (modulo } d);$$

i.e., the concepts of w-distance and τ-function are identical (within the metric framework). This, however, seems to be in contradiction with [1, Example 2.1]; further aspects will be delineated elsewhere.

4 Application (equilibrium points)

Roughly speaking, the extension of Theorem 2 given in Section 3 consists in taking $e : M \times M \to R_+$ as a strong KST-metric (modulo d) in place of a w-distance (modulo d) at the level of the comparative relation (1) and the variational property (2). It is therefore natural to ask whether the right member of these could be also extended in some way. An appropriate answer to this may be given along the following lines. Let M be some nonempty set. Take a pseudometric $e : M \times M \to R_+$; which in addition, is triangular (cf. Section 1) and transitively sufficient (cf. (2h)). By a *relative pseudometric* over M we shall mean any map $(x, y) \mapsto F(x, y)$ from $M \times M$ to $\bar{R} = R \cup \{-\infty, \infty\}$. Fix such an object; supposed to be *triangular* $[F(x, z) \leq F(x, y) + F(y, z)$, whenever the right member exists] and *reflexive* $[F(x, x) = 0, \forall x \in M]$. Let $\mu = \mu_F$ stand for the associated function

$$\mu(x) = \sup\{-F(x, y); y \in M\}, \quad x \in M.$$

By reflexivity, $\mu(x) \geq 0$, for each $x \in M$; hence its values are in $R_+ \cup \{\infty\} = [0, \infty]$. Note that the alternative $\mu(M) = \{\infty\}$ cannot be excluded; so, to avoid this, we must assume

(4a) μ is proper $(\text{Dom}(\mu) := \{x \in M; \mu(x) < \infty\} \neq \emptyset)$;

referred to as: F is *semi-proper*. For the arbitrary fixed $u \in \text{Dom}(\mu)$ put $F_u := F(u, .)$. We have by definition

$$F_u(u) = 0; \quad F_u(x) \geq \inf\{F(u, x); x \in M\} = -\mu(u) > -\infty; \tag{1}$$

wherefrom, F_u is inf-proper (cf. (1a)) for all such u; we shall refer to such a property as: F is *semi inf-proper*. In this context, u is starting (modulo (e, F_u)) whenever

(a4) $e(u, x) \leq -F_u(x) = -F(u, x)$, for some $x \in M$;

this will be termed as: u is starting (modulo (e, F)). [In particular, (a4) holds whenever $e(u, u) = 0$, if one takes the reflexive property of F into account; otherwise, it may be false]. Finally, let us say that F is *e-complete* if

(b4) (e, F_u) is complete (cf. (2i)), for all $u \in \text{Dom}(\mu)$.

The following variational statement is available.

Theorem 8 *Let the triangular transitively sufficient pseudometric e and the triangular reflexive relative pseudometric F be such that (4a)+(b4) hold. Then,* *for each starting (modulo (e, F)) $u \in \mathrm{Dom}(\mu)$ there exists $v = v(u) \in \mathrm{Dom}(F_u)$ with*

$$e(u, v) \leq -F(u, v) \leq \mu(u)(< \infty) \tag{2}$$

$$e(v, x) > -F(v, x), \quad \text{for all } x \in M \setminus \{v\}. \tag{3}$$

Proof Let u be taken as before. By the imposed conditions, Theorem 4 is applicable to (e, F_u); moreover (see above) $u \in \mathrm{Dom}(F_u)$ and (by its very choice) u is starting (modulo (e, F_u)). It then follows that, there must be some $v = v(u) \in \mathrm{Dom}(F_u)$ with the properties

$$e(u, v) \leq F_u(u) - F_u(v)(= F(u, u) - F(u, v)) \tag{4}$$

$$e(v, x) > F_u(v) - F_u(x)(= F(u, v) - F(u, x)), \quad \forall x \in M \setminus \{v\}. \tag{5}$$

The former of these is just (2), by the reflexivity of F. And the latter one gives (3); because (from the triangular property of F and (2)) $F(u, v) - F(u, x) \geq F(u, v) - (F(u, v) + F(v, x)) = -F(v, x)$. ■

A basic particular case of this is the one precise by the constructions in Section 3. Precisely, let (M, d) be a complete almost metric space; and $e : M \times M \to R_+$ be a strong KST-metric (modulo d) over it. Further, let $F : M \times M \to \bar{R}$ be a triangular and reflexive relative pseudometric over M, fulfilling (4a). Remember that F is then semi inf-proper, in the sense: F_u is inf-proper, for each $u \in \mathrm{Dom}(\mu)$. Suppose in addition that

(4b) F is semi d-lsc: F_u is d-lsc, for each $u \in \mathrm{Dom}(\mu)$.

Note that, as a consequence of this, (b4) holds, if one takes Lemma 3 into account. By Theorem 8, we then have

Theorem 9 *Let the couple (d, e) be taken as before; and the (triangular reflexive) relative pseudometric F be such that (4a)+(4b) hold. Then, for each starting (modulo (e, F)) $u \in \mathrm{Dom}(\mu)$ there exists $v = v(u) \in \mathrm{Dom}(F_u)$ with the properties (2)+(3).*

Some remarks are in order. Let $\varphi : M \to R \cup \{\infty\}$ be some inf-proper function. The relative pseudometric

(c4) $F(x, y) = \varphi(y) - \varphi(x), \quad x, y \in M \quad$ (where $\infty - \infty = 0$)

is triangular and reflexive, as it can be directly seen. In addition, (4a) holds, because $\mu(x) = \varphi(x) - \varphi_*, x \in M$ (hence $\mathrm{Dom}(\mu) = \mathrm{Dom}(\varphi)$). This, along with $F_u(.) = \varphi(.) - \varphi(u)$, $u \in \mathrm{Dom}(\mu)$, tells us that

i) F is e-complete iff (e, φ) is complete (cf. (2i))

ii) F is semi d-lsc iff φ is d-lsc (according to (1b)).

As a consequence, the results above extend the corresponding ones in Sections 2 and 3. The reciprocal inclusions are also true, by the very argument above; so that Theorem 4 $\Longleftrightarrow$ Theorem 8 and Theorem 6 $\Longleftrightarrow$ Theorem 9. In particular, when $e = d$, this last statement includes directly the one in Oettli and Thera [22]. Some useful applications of these to equilibrium problems may be found in Bianchi, Kassay and Pini [6].

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"A. Myller" Mathematical Seminar,
"A. I. Cuza" University 11,
Copou Boulevard, 700506
Iaşi, Romania
e-mail: mturi@uaic.ro