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The edge fixed geodomination number of a graph

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Abstract

For a vertex x in a connected graph $G = (V(G), E(G))$ of order $p \geq 3$, a set $S \subseteq V(G)$ is an x-geodominating set of G if each vertex $v \in V(G)$ lies on an x-y geodesic for some element y in S. The minimum cardinality of an x-geodominating set of G is defined as the x-geodomination number of G, denoted by $g_x(G)$. An x-geodominating set of cardinality $g_x(G)$ is called a g_x-set of G. For an edge $e = xy$ in G, a set $S \subseteq V(G)$ is an e-geodominating set of G if each vertex $v \in V(G)$ lies on either an $x - z$ geodesic or an $y - z$ geodesic for some element z in S. The minimum cardinality of an e-geodominating set of G is defined as the e-geodomination number of G, denoted by $g_e(G)$. An e-geodominating set of cardinality $g_e(G)$ is called a g_e-set of G. Some general properties satisfied by e-geodominating sets are studied. We determine bounds for the e-geodomination number and find the same for some special classes of graphs. For positive integers r, d and $n \geq 2$ with $r < d \leq 2r$, there exists a connected graph G with rad $G = r$, diam $G = d$ and $g_{xy}(G) = n$ or $n - 1$ for any edge xy in G. If p, d and n are integers such that $3 \leq d \leq p - 1, 2 \leq n \leq p - 2$ and $p - d - n + 1 \geq 0$, then there exists a graph G of order p, diameter d and $g_{xy}(G) = n$ or $n - 1$ for any edge xy in G.

1 Introduction

By a graph $G = (V, E)$ we mean a finite undirected connected graph without loops or multiple edges. The order and size of G are denoted by p and q

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respectively. For basic graph theoretic terminology we refer to Harary [4]. For vertices x and y in a connected graph G, the *distance* $d(x, y)$ is the length of a shortest $x-y$ path in G. An $x-y$ path of length $d(x, y)$ is called an $x-y$ *geodesic*. A vertex v is said to lie on an $x-y$ geodesic P if v is a vertex of P including the vertices x and y. The *diameter diam* G of a connected graph G is the length of any longest geodesic. For any vertex u of G, the *eccentricity* of u is $e(u) = max\ \{d(u, v) : v \in V\}$. A vertex v of G such that $d(u, v) = e(u)$ is called an eccentric vertex of u. The neighborhood of a vertex v is the set $N(v)$ consisting of all vertices u which are adjacent with v. A vertex v is a *simplicial* vertex if the subgraph induced by its neighborhood $N(v)$ is complete.

The closed interval $I[x, y]$ consists of all vertices lying on some $x-y$ geodesic of G, while for $S \subseteq V$, $I[S] = \bigcup_{x,y \in S} I[x, y]$. A set S of vertices is a *geodetic set* if $I[S] = V$, and the minimum cardinality of a geodetic set is the *geodetic number* $g(G)$. A geodetic set of cardinality $g(G)$ is called a *g-set*. The geodetic number of a graph was introduced in [1,5] and further studied in [2,3]. It was shown in [5] that determining the geodetic number of a graph is an NP-hard problem.

The concept of vertex geodomination number was introduced by Santhakumaran and Titus [7] and further studied in [8,9]. A vertex y in a connected graph G is said to *x-geodominate* a vertex u if u lies on an $x-y$ geodesic. A set S of vertices of G is an *x-geodominating set* if each vertex $v \in V(G)$ is x-geodominated by some element of S. The minimum cardinality of an x-geodominating set of G is defined as the *x-geodomination number* of G and is denoted by $g_x(G)$. An x-geodominating set of cardinality $g_x(G)$ is called a *g_x-set*.

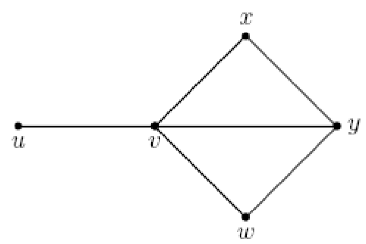

Figure 1.1

Every vertex of an $x-y$ geodesic is x-geodominated by the vertex y. Since, by definition, a g_x-set is minimum, the vertex x and also the internal vertices of an $x-y$ geodesic do not belong to a g_x-set. For the graph G given in

Figure 1.1, $g_u(G) = 3$, $g_v(G) = 4$, $g_w(G) = 2$, $g_x(G) = 2$ and $g_y(G) = 3$ with minimum vertex geodominating sets $\{x, y, w\}$, $\{x, y, u, w\}$, $\{x, u\}$, $\{u, w\}$ and $\{x, u, w\}$ respectively.

It is proved in [7] that for any vertex x in G, g_x-set is unique and $1 \leq g_x(G) \leq p - 1$. An elaborate study of results in vertex geodomination with several interesting applications is given in [7,8]. The following theorems will be used in the sequel.

Theorem 1.1 [4] *Let v be a vertex of a connected graph G. The following statements are equivalent:*

(i) *v is a cut vertex of G.*

(ii) *There exist vertices u and w distinct from v such that v is on every $u - w$ path.*

(iii) *There exists a partition of the set of vertices $V - \{v\}$ into subsets U and W such that for any vertices $u \in U$ and $w \in W$, the vertex v is on every $u - w$ path.*

Theorem 1.2 [8] *For any vertex x in a connected graph G of order $p \geq 2$ and diameter d, $g_x(G) \leq p - d + 1$.*

Theorem 1.3 [8] *For any vertex x in an even cycle C, $g_x(C) = 1$.*

Throughout this paper G denotes a connected graph with at least three vertices.

2 Edge Fixed Geodomination

Definition 2.1 *Let $e = xy$ be any edge of a connected graph G of order at least 3. A set S of vertices of G is an e-geodominating set if every vertex of G lies on either an $x - u$ geodesic or a $y - u$ geodesic in G for some element u in S. The minimum cardinality of an e-geodominating set of G is defined as the e-geodomination number of G and is denoted by $g_e(G)$ or $g_{xy}(G)$. An e-geodominating set of cardinality $g_e(G)$ is called a g_e-set of G.*

Example 2.2 *For the graph G given in Figure 2.1, the minimum edge geodominating sets and the edge geodomination numbers are given in Table 2.1.*

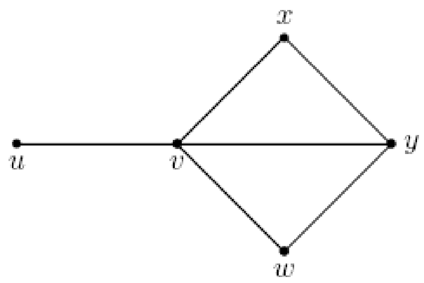

Figure 2.1.

Edge e	Minimum e-geodominating sets	e-geodomination number
xy	$\{z, w\}$	2
yv	$\{z, w\}$	2
vw	$\{z, x\}$	2
uv	$\{z, w, y\}, \{z, w, x\}$	3
zu	$\{y, w\}$	2
xz	$\{w\}$	1
xu	$\{z, w\}$	2

Table 2.1

It is proved in [7] that for any vertex x in G, g_x-set of G with respect to x is unique. However, we observe that in the case of edge geodominating sets, there can be more than one minimum edge geodominating set. For the edge $e = uv$ of the graph G in Figure 2.1, $\{z, w, y\}$ and $\{z, w, x\}$ are two distinct g_e-sets of G.

Theorem 2.3 *For any edge xy in a connected graph G of order at least 3, the vertices x and y do not belong to any minimum xy-geodominating set of G.*

Proof. Suppose that x belongs to a minimum xy-geodominating set, say S of G. Since G is a connected graph with at least three vertices and xy is an edge, it follows from the definition of an xy-geodominating set that S contains a vertex v different from x and y. Since the vertex x lies on every $x-v$ geodesic in G, it follows that $T = S - \{x\}$ is an xy-geodominating set of G, which is a contradiction to S a minimum xy-geodominating set of G. Similarly, y does not belong to any xy-geodominating set of G.

Theorem 2.4 *Let xy be any edge of a connected graph G of order at least 3. Then*

(i) Every simplicial vertex of G other than the vertices x and y (whether x or y is simplicial or not) belongs to every g_{xy}-set.

(ii) No cut vertex of G belongs to any g_{xy}-set.

(iii) If z is an eccentric vertex of both x and y, then z belongs to every xy-geodominating set.

Proof. (i) By Theorem 2.3, the vertices x and y do not belong to any g_{xy}-set. So let $u \neq x, y$ be a simplicial vertex of G. Let S be a g_{xy}-set of G such that $u \notin S$. Then u is an internal vertex of either an $x - v$ geodesic or a $y - v$ geodesic for some v in S. Without loss of generality, let P be an $x - v$ geodesic with u an internal vertex. Then both the neighbors of u on P are not adjacent and hence u is not a simplicial vertex, which is a contradiction.

(ii) Let v be a cut vertex of G. Then by Theorem 1.1, there exists a partition of the set of vertices $V - \{v\}$ into subsets U and W such that for any vertex $u \in U$ and $w \in W$, the vertex v lies on every $u - w$ path. Let S be any g_{xy}-set of G. We consider three cases.

Case 1. Both x and y belong to U. Suppose that $S \cap W = \emptyset$. Let $w_1 \in W$. Since S is an xy-geodominating set, there exists an element z in S such that w_1 lies on either an $x - z$ geodesic or a $y - z$ geodesic in G. Suppose that w_1 lies in an $x - z$ geodesic $P : x = z_0, z_1, ..., w_1, ..., z_n = z$ in G. Then the $x - w_1$ subpath of P and $w_1 - z$ subpath of P both contain v so that P is not a path in G, which is a contradiction. Hence $S \cap W \neq \emptyset$. Let $w_2 \in S \cap W$. Then v is an internal vertex of any $x - w_2$ geodesic and v is also an internal vertex of any $y - w_2$ geodesic. If $v \in S$, then, let $S' = S - \{v\}$. It is clear that every vertex that lies on an $x - v$ geodesic also lies on an $x - w_2$ geodesic, and every vertex that lies on an $y - v$ geodesic also lies on an $y - w_2$ geodesic. Hence it follows that S' is an xy-geodominating set of G, which is a contradiction to S a minimum xy-geodominating set of G. Thus v does not belong to any minimum xy-geodominating set of G.

Case 2. Both x and y belong to W. It is similar to Case 1.

Case 3. Either $x = v$ or $y = v$. By Theorem 2.3, v does not belong to any g_{xy}-set.

(iii) Let z be an eccentric vertex of both x and y so that $d(x, z) = e(x)$ and $d(y, z) = e(y)$. Suppose that z does not belong to a g_{xy}-set, say S. Then there exists a vertex w in S such that z is an internal vertex of either an $x - w$ geodesic or a $y - w$ geodesic. Therefore, either $d(x, z) < d(x, w)$ or

$d(y,z) < d(y,w)$ and hence either $e(x) < d(x,w)$ or $e(y) < d(y,w)$, which is a contradiction.

Note 2.5 *If z is an eccentric vertex of either x or y but not both, then z need not belong to every g_{xy}-set of G. For the cycle $C_4 : x, y, z, w, x$, it is clear that $S_1 = \{z\}$ and $S_2 = \{w\}$ are the two g_{xy}-sets of C_4. Also z is the eccentric vertex of x and not of y. However, z does not belong to S_2.*

Corollary 2.6 *Let T be a tree with number of end vertices k. Then $g_{xy}(T) = k - 1$ or k according as xy is an end edge or cut edge.*

Proof. This follows from Theorem 2.4.

Corollary 2.7 *Let $K_{1,n}$ $(n \geq 2)$ be a star. Then $g_{xy}(K_{1,n}) = n - 1$ for any edge xy in $K_{1,n}$.*

Corollary 2.8 *Let G be the complete graph K_p $(p \geq 3)$. Then $g_{xy}(G) = p - 2$ for any edge xy in G.*

Proposition 2.9 *For any edge xy in a connected graph G of order $p \geq 3$, $1 \leq g_{xy}(G) \leq p - 2$.*

Proof. It is clear from the definition of g_{xy}-set that $g_{xy}(G) \geq 1$. Also, since the vertices x and y do not belong to any g_{xy}-set, it follows that $g_{xy}(G) \leq p - 2$.

Remark 2.10 *The bounds for $g_{xy}(G)$ in Proposition 2.9 are sharp. If G is any cycle, then $g_{xy}(G) = 1$ for any edge xy in G. For any edge xy in a path $P_n (n \geq 3)$, $g_{xy}(P_n) = 1$. For any edge xy in the complete graph $K_p (p \geq 3)$, $g_{xy}(K_p) = p - 2$.*

Now we proceed to characterize graphs for which the lower bound in Proposition 2.9 is attained.

Theorem 2.11 *Let G be a connected graph. For an edge xy in G, $g_{xy}(G) = 1$ if and only if there exists a vertex z in G such that every vertex of G lies on either a diametral path joining x and z or a diametral path joining y and z.*

Proof. Let xy be any edge of G. Let z be a vertex in G such that every vertex of G lies on either a diametral path joining x and z or a diametral path joining y and z. Then $S = \{z\}$ is a g_{xy}-set of G and so $g_{xy}(G) = 1$.

Conversely, let $g_{xy}(G) = 1$ and $S = \{z\}$ be a g_{xy}-set of G. Then every vertex of G lies on either an $x - z$ geodesic or a $y - z$ geodesic. Now we consider three cases.

Case 1. Every vertex of G lies on an $x - z$ geodesic. Let d denote the diameter of G. If $d(x,z) < d$, then there exist vertices u and v on distinct geodesics joining x and z such that $d(u,v) = d$. Thus $d(x,z) < d(u,v)$. Hence we see that

$$d(x,z) = d(x,u) + d(u,z) \tag{1}$$

and $$d(x,z) = d(x,v) + d(v,z) \tag{2}$$

By triangle inequality,

$$d(u,v) \leq d(u,x) + d(x,v) \text{ and } d(u,v) \leq d(u,z) + d(z,v) \tag{3}$$

From (1) and (3), $d(u,z) = d(x,z) - d(x,u)$
$< d(u,v) - d(x,u)$
$\leq d(x,v)$.

Thus $d(u,z) < d(x,v)$. (4)

Now from (2), (3) and (4), we see that $d(u,v) < d(x,v) + d(z,v)$
$= d(x,v) + d(v,z)$
$= d(x,z)$.

Thus $d(u,v) < d(x,z)$, which is a contradiction. Hence $d(x,z) = d$ and each vertex of G lies on a diametral path joining x and y.

Case 2. Every vertex of G lies on a $y - z$ geodesic. It is similar to Case 1.

Case 3. There exist vertices u and v such that u lies on an $x - z$ geodesic but not in any $y - z$ geodesic and v lies on a $y - z$ geodesic but not in any $x - z$ geodesic.

We show that both the $x - z$ geodesic and the $y - z$ geodesic are diametral paths. Suppose that $x - z$ geodesic is not a diametral path. Then $d(x,z) < d = d(u',v')$ for some vertices u' and v' in G.

Case 3a. Suppose that $u', v' \in I[x,z]$. If u' and v' lie on the same $x - z$ geodesic, then it is clear that $d(u',v') < d(x,z)$, which is a contradiction. If u' and v' lie on distinct $x - z$ geodesics, then as in Case 1, $d(u',v') < d(x,z)$, which is a contradiction.

Case 3b. Suppose that $u', v' \in I[y,z]$.

Subcase 3b$_1$. Suppose that u', v' lie on the same $y - z$ geodesic. Then $d(u',v') = d(y,z)$ and so $d(x,z) < d(y,z)$. Then it is clear that $d(x,z) = d(y,z) - 1$. It follows that every vertex of an $x - z$ geodesic lies on a $y - z$ geodesic and so u lies on a $y - z$ geodesic, which is a contradiction.

Subcase 3b$_2$. Suppose that u', v' lie on distinct $y - z$ geodesics. If the $y - z$ geodesics are not diametral paths, then as in Case 3a, we have a contradiction. If the $y - z$ geodesics are diametral paths, then $d(x,z) = d(y,z) - 1$ and it follows that every vertex of an $x - z$ path lies on a $y - z$ geodesic and so u lies on a $y - z$ geodesic, which is a contradiction.

Case 3c. Suppose that $u' \in I[x,z]$ and $u' \notin I[y,z]$, and $v' \in I[y,z]$ and $v' \notin I[x,z]$. It is clear that

$$d(x,z) = d(x,u') + d(u',z) \tag{5}$$

and $$d(y,z) = d(y,v') + d(v',z) \quad (6).$$

By triangle inequality, $d(u', v') \leq d(u', z) + d(z, v')$ and $d(u', v') \leq d(u', x) + d(x, v')$ and $d(u', v') \leq d(u', y) + d(y, v')$ (7).

From (5) and (7), $d(u', z) = d(x, z) - d(x, u')$
$< d(u', v') - d(x, u')$
$\leq d(x, v')$.

Thus $d(u', z) < d(x, v')$ (8).

Now from (6), (7) and (8), we see that $d(u', v') < d(x, v') + d(z, v')$
$= d(x, v') + d(v', z)$
$\leq 1 + d(y, v') + d(v', z)$
$= 1 + d(y, z)$.

Thus $d(u', v') < 1 + d(y, z)$ and so $d(u', v') \leq d(y, z)$. Since $d(u', v') = d$, we have $d(u', v') = d(y, z)$. Then as in Subcase $3b_1$ of Case 3b, u lies on a $y - z$ geodesic, which is a contradiction.

Hence the $x - z$ geodesic is a diametral path. Similarly, the $y - z$ geodesic is a diametral path. Thus the proof is complete.

Theorem 2.12 *For any edge xy in the cube Q_n $(n \geq 3)$, $g_{xy}(Q_n) = 1$.*

Proof. Let $e = xy$ be an edge in Q_n and let $x = (a_1, a_2, ..., a_n)$, where $a_i \in \{0, 1\}$. Let $x' = (a'_1, a'_2, ..., a'_n)$ be another vertex of Q_n such that a'_i is the complement of a_i. Let u be any vertex in Q_n. For convenience, let $u = (a_1, a'_2, a_3, ..., a_n)$. Then u lies on the $x - x'$ geodesic
$P : x = (a_1, a_2, ..., a_n), (a_1, a'_2, a_3, ..., a_n), (a'_1, a'_2, a_3, ..., a_n), (a'_1, a'_2, a'_3, ..., a_n), ..., (a'_1, a'_2, ..., a'_{n-1}, a_n), (a'_1, a'_2, ..., a'_n) = x'$, which is of length n so that it is a diametral path joining x and x'. Hence the result follows from Theorem 2.11.

Theorem 2.13 *(i) For the wheel $W_n = K_1 + C_{n-1}$ $(n \geq 6)$, $g_{xy}(W_n) = n - 5$ or $n - 4$ according as xy is an edge of C_{n-1} or not.*

(ii) For any edge xy in the complete bipartite graph $K_{m,n}$ $(m \leq n)$,

$$g_{xy}(K_{m,n}) = \begin{cases} n - 1 & \text{if } m = 1 \\ 1 & \text{if } m = 2 \\ 2 & \text{if } m \geq 3. \end{cases}$$

Proof. (i) Let $C_{n-1} : u_1, u_2, u_3, ..., u_{n-1}, u_1$ be the cycle of W_n and let z be the vertex K_1. Let xy be any edge in C_{n-1}, say $xy = u_1u_2$. Since $\{u_3, u_4, ..., u_{n-2}\}$ and $\{u_4, u_5, ..., u_{n-1}\}$ are the sets of eccentric vertices of u_1 and u_2 respectively, we have $S = \{u_4, u_5, ..., u_{n-2}\}$ is the set of common eccentric vertices of both u_1 and u_2. It is clear that the vertices u_3, z and u_{n-1} lie on the geodesics $P : u_2, u_3, u_4$; $Q : u_1, z, u_4$; and $R : u_1, u_{n-1}, u_{n-2}$ respectively. Hence by Theorem 2.4(iii), S is the unique g_{xy}-set of W_n so that $g_{xy}(W_n) = n - 5$. Let xy be any edge not in C_{n-1}. Take $xy = u_1z$. Then $\{u_3, u_4, ..., u_{n-2}\}$ is the set of eccentric vertices of u_1 and $V(C_{n-1})$ is the set of eccentric vertices of z so that $S' = \{u_3, u_4, ..., u_{n-2}\}$ is the set of common eccentric vertices of both u_1 and z. Now, by an argument similar to the above, it is easily seen that $g_{xy}(W_n) = n - 4$.

(ii) Let $U = \{u_1, u_2, ..., u_m\}$ and $W = \{w_1, w_2, ..., w_n\}$ be the partite sets of G, where $m \leq n$. If $m = 1$, then by Corollary 2.7, $g_{xy}(K_{1,n)} = n - 1$ for any edge xy in $K_{1,n}$. If $m = 2$, it follows from Theorem 2.11 that $g_e(G) = 1$ for any edge $e = u_iw_j$ $(1 \leq i \leq 2; 1 \leq j \leq n)$. If $m \geq 3$, then it is clear that no singleton subset of V is an xy-geodominating set of G and so $g_{xy}(G) \geq 2$. Without loss of generality, take $e = u_1w_1$. Let $S = \{u_2, w_2\}$. Then every vertex of U lies on a $w_1 - w_2$ geodesic and every vertex of W lies on a $u_1 - u_2$ geodesic. It follows that S is an xy-geodominating set of G and so $g_e(G) = 2$.

Remark 2.14 *Since $W_4 = K_4$, we have $g_{xy}(W_4) = 2$ for any edge xy in W_4. Also, it is easily seen that $g_{xy}(W_5) = 1$ for any edge xy in W_5. Thus Theorem 2.13(i) is not true for $n = 4, 5$.*

Theorem 2.15 *For any edge xy in a connected graph G, every x-geodominating set of G is an xy-geodominating set of G.*

Proof. Let S be an x-geodominating set of G. Then every vertex of G lies on an $x - z$ geodesic for some z in S. It follows that S is an xy-geodominating set of G.

Corollary 2.16 *For any edge $e = xy$ in a connected graph G, $g_{xy}(G) \leq min\{g_x(G), g_y(G)\}$.*

Theorem 2.17 *For every pair a, b of integers with $1 \leq a \leq b$, there is a connected graph G with $g_{xy}(G) = a$ and $g_x(G) = b$ for some edge xy in G.*

Proof. We prove this theorem by considering three cases.

Case 1. Suppose that $a = b = 1$. Then, for any edge xy in an even cycle G, we have $g_{xy}(G) = 1$ (See Remark 2.10) and $g_x(G) = 1$, by Theorem 1.3.

Case 2. Suppose that $a = b \geq 2$. Let $C_4 : x, y, z, u, x$ be a cycle of order 4. Add $a - 1$ new vertices $v_1, v_2, ..., v_{a-1}$ to C_4 and join them to x, thereby producing the graph G of Figure 2.2. Let $S = \{v_1, v_2, ..., v_{a-1}\}$ be the set of all simplicial vertices of G.

First, we show that $g_{xy}(G) = a$ for the edge xy in G. Since S is not an xy-geodominating set, it follows from Theorem 2.4(i) that $g_{xy}(G) \geq a$. On the other hand, $S' = S \cup \{z\}$ is an xy-geodominating set of G and so $g_{xy}(G) = |S'| = a$.

Next, we show that $g_x(G) = b$. By Corollary 2.16, we have $g_x(G) \geq a$. It is clear that S' is an x-geodominating set of G and so $g_x(G) = a$.

Case 3. Suoppose that $a < b$. Let $C_5 : x, y, z, u, v, x$ be a cycle of order

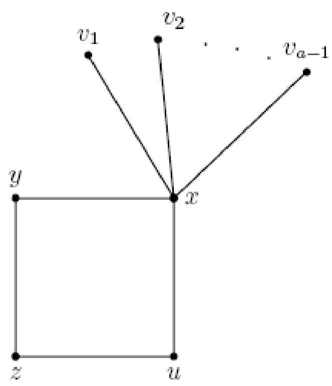

Figure 2.2

5. Add $b-2$ new vertices $v_1, v_2, ..., v_{a-1}, w_1, w_2, ..., w_{b-a-1}$ to C_5 and join $v_i (1 \le i \le a-1)$ to x and join $w_j (1 \le j \le b-a-1)$ to both y and u, thereby producing the graph G of Figure 2.3. Let $S = \{v_1, v_2, ..., v_{a-1}\}$ be the set of all simplicial vertices of G.

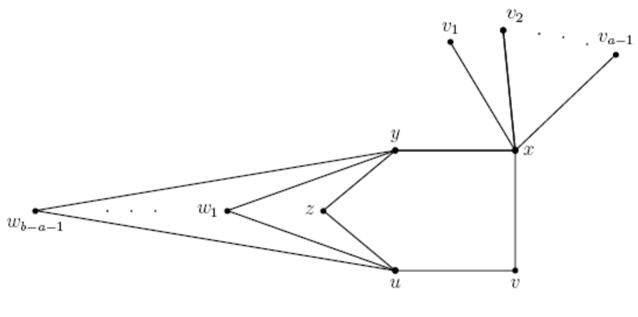

Figure 2.3

First, we show that $g_{xy}(G) = a$ for the edge xy in G. Since S is not an xy-geodominating set, it follows from Theorem 2.4(i) that $g_{xy}(G) \ge a$. On the other hand, $S' = S \bigcup \{u\}$ is an xy-geodominating set of G and so $g_{xy}(G) = |S'| = a$.

Next, we show that $g_x(G) = b$. It is clear that the vertices $v_1, v_2, ..., v_{a-1}, z$, $w_1, w_2, ..., w_{b-a-1}$ must belong to every x-geodominating set and these vertices do not form an x-geodominating set. Hence $g_x(G) \geq b$. On the other hand, the set $S_1 = \{v_1, v_2, ..., v_{a-1}, z, w_1, w_2, ..., w_{b-a-1}, u\}$ is an x-geodominating set of G and so $g_x(G) = b$.

We leave the following problem as an open question.

Problem 2.18 *Characterize graphs G of order $p \geq 3$ for which $g_{xy}(G) = p - 2$, where xy is an edge of G.*

3 The Edge Geodomination Number and Diameter of a Graph

We have seen that if G is a connected graph of order $p \geq 3$, then $1 \leq g_{xy}(G) \leq p - 2$ for any edge xy in G. Also we have for an edge xy in G, $g_{xy}(G) = 1$ if and only if there exists a vertex z such that every vertex of G lies on either a diametral path joining x and z or a diametral path joining y and z. In the following theorem we give an improved upper bound for the edge fixed geodomination number of a graph in terms of its order and diameter.

Theorem 3.1 *If G is a connected graph of order p and diameter d, then $g_{xy}(G) \leq p - d + 1$ for any edge xy in G.*

Proof. This follows from Theorem 1.2 and Corollary 2.16.

Theorem 3.2 *For for any edge xy in a non-trivial tree T, $g_{xy}(T) = p - d$ or $p - d + 1$ if and only if T is a caterpillar.*

Proof. Let T be any non-trivial tree. Let $P : u = v_0, v_1, ..., v_d = v$ be a diametral path. Let k be the number of end vertices of T and l be the number of internal vertices of T other than $v_1, v_2, ..., v_{d-1}$. Then $d - 1 + l + k = p$. By Corollary 2.6, $g_{xy}(T) = k$ or $k - 1$ for any edge xy in T and so $g_{xy}(T) = p - d - l + 1$ or $p - d - l$ for any edge xy in T. Hence $g_{xy}(T) = p - d + 1$ or $p - d$ for any edge xy in T if and only if $l = 0$, if and only if all the internal vertices of T lie on the diametral path P, if and only if T is a caterpillar.

For every connected graph G, $rad\ G \leq diam\ G \leq 2\ rad\ G$. Ostrand [6] showed that every two positive integers a and b with $a \leq b \leq 2a$ are realizable as the radius and diameter, respectively, of some connected graph. Ostrand's theorem can be extended so that the edge fixed geodomination number can be prescribed when $r < d \leq 2r$.

Theorem 3.3 *For positive integers r, d and $n \geq 2$ with $r < d \leq 2r$, there exists a connected graph G with $rad\ G = r$, $diam\ G = d$ and $g_{xy}(G) = n$ or $n - 1$ for any edge xy in G.*

Proof. Let $C_{2r} : v_1, v_2, ..., v_{2r}, v_1$ be a cycle of order $2r$ and let P_{d-r+1} : $u_0, u_1, ..., u_{d-r}$ be a path of order $d - r + 1$. Let H be a graph obtained from C_{2r} and P_{d-r+1} by identifying v_1 in C_{2r} and u_0 in P_{d-r+1}. If $n = 2$, then let $G = H$. Then rad $G = r$ and diam $G = d$. Clearly, $g_{xy}(G) = 1$ or 2 according as $xy \in \{v_r v_{r+1}, v_{r+1} v_{r+2}, u_{d-r-1} u_{d-r}\}$ or
$xy \in \{v_1 u_1, u_1 u_2, ..., u_{d-r-2} u_{d-r-1}, v_1 v_2, v_2 v_3, ..., v_{r-1} v_r, v_{r+2} v_{r+3}, ..., v_{2r} v_1\}$. Thus $g_{xy}(G) = 1$ or 2 for any edge xy in G. If $n \geq 3$, then add $n - 2$ new vertices $w_1, w_2, ..., w_{n-2}$ to H and join each vertex $w_i (1 \leq i \leq n - 2)$ to the vertex u_{d-r-1} and obtain the graph G of Figure 3.1.

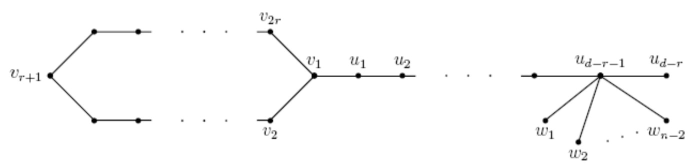

Figure 3.1

Now rad $G = r$, diam $G = d$ and G has $n - 1$ end vertices. Clearly, $g_x(G) = n$ or $n - 1$ according as
$xy \in \{v_1 u_1, u_1 u_2, ..., u_{d-r-2} u_{d-r-1}, v_1 v_2, v_2 v_3, ..., v_{r-1} v_r, v_{r+2} v_{r+3}, ..., v_{2r} v_1\}$ or $xy \in \{v_r v_{r+1}, v_{r+1} v_{r+2}, u_{d-r-1} u_{d-r}, u_{d-r} w_1, u_{d-r} w_2, ..., u_{d-r} w_{n-2}\}$. Thus $g_{xy}(G) = n$ or $n - 1$ for any edge xy in G.

In the following, we construct a graph of prescribed order, diameter and edge fixed geodomination number under suitable conditions.

Theorem 3.4 *If p, d and n are integers such that $3 \leq d \leq p-1$, $2 \leq n \leq p-2$ and $p - d - n + 1 \geq 0$, then there exists a graph G of order p, diameter d and $g_{xy}(G) = n$ or $n - 1$ for any edge xy in G.*

Proof. If $n = 2$, let $P_{d+1} : u_0, u_1, u_2, ..., u_d$ be a path of length d. Add $p - d - 1$ new vertices $w_1, w_2, ..., w_{p-d-1}$ to P_{d+1} and join each vertex to both u_0 and u_2, thereby producing the graph G of Figure 3.2. Then G has order p and diameter d. Clearly, $g_{xy}(G) = 1$ or 2 according as $xy \in \{u_0 u_1, u_0 w_1, u_0 w_2, ..., u_0 w_{p-d-1}, u_{d-1} u_d\}$ or $xy \in \{u_1 u_2, u_2 u_3, ..., u_{d-2} u_{d-1}, u_2 w_1, u_2 w_2, \ldots u_2 w_{p-d-1}\}$.

If $3 \leq n \leq p-2$, then add $p-d-n+1$ new vertices $w_1, w_2, \ldots, w_{p-d-n+1}$ to the path $P_{d+1} : u_0, u_1, u_2, \ldots, u_d$ of length d and join each vertex to both u_0 and u_2, thereby producing the graph H. Then add $n - 2$ new vertices $v_1, v_2, ..., v_{n-2}$ to H and join each vertex $v_i (1 \leq i \leq n - 2)$ to the vertex u_{d-1} and obtain the graph G of Figure 3.3. Then G has order

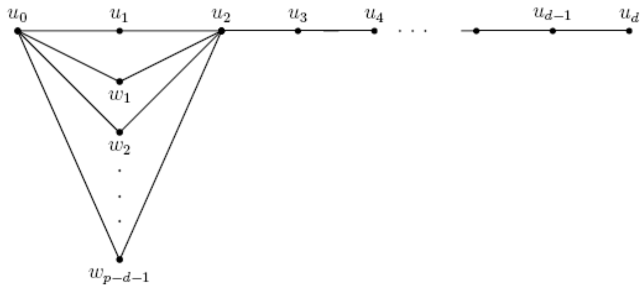

Figure 3.2

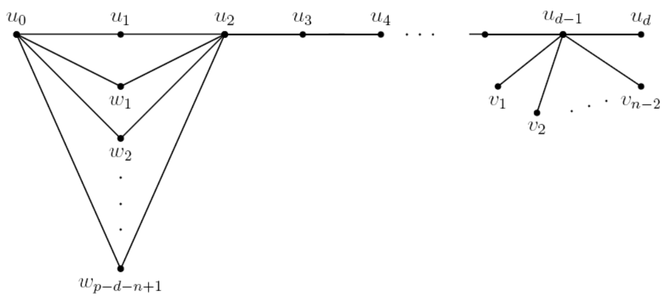

Figure 3.3

p and diameter d. It is easily verified that $g_{xy}(G) = n$ or $n-1$ according as $xy \in \{u_1u_2, u_2u_3, ..., u_{d-2}u_{d-1}, u_2w_1, u_2w_2, ..., u_2w_{p-d-n+1}\}$ or $xy \in \{u_0u_1, u_0w_1, u_0w_2, ..., u_0w_{p-d-n+1}, u_{d-1}u_d, u_{d-1}v_1, u_{d-1}v_2, ..., u_{d-1}v_{n-2}\}$. Then $g_{xy}(G) = n$ or $n-1$ for any edge xy in G.

In view of Theorem 3.4, we leave the following problem as an open question.

Problem 3.5 *If p, d and n are integers such that $3 \leq d \leq p-1$, $2 \leq n \leq p-2$ and $p-d-n+1 \geq 0$, then there exists a graph G of order p, diameter d and $g_{xy}(G) = n$ for every edge xy in G.*

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