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What can be expected from a Boolean derivative?

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Abstract

Several concepts of a Boolean derivative have been investigated in the literature. In this paper we find out whether a Boolean operator can satisfy all of the three basic derivative-like properties: additivity, homogeneity and the Leibniz rule. In a certain sense, the answer is negative.

The attempts to establish Boolean analogues of several concepts and results from Calculus begun in 1917 with a paper by Daniell [6], which sketched a theory of convergence for sequences and series in a Boolean algebra. Some forty years later Reed [11], Huffman [9] and Akers Jr. [1] introduced (partial) derivatives of Boolean functions and pointed out their applicability to switching theory. Ever since then the theory of Boolean derivatives has developed tremendously, both in view of applications and for its own algebraic interest; see e.g. [3], [4], [5], [7], [10], [13], [15], [16], [17], [18], [19].

There are several Boolean analogues of the conventional concept of a derivative; of course, these Boolean derivatives share some, but not all of the properties of their conventional model. A paper by Bazsó and Lábos [2] states that a "good" concept of a derivative should be additive, i.e., $(f+g)' = f' + g'$, homogeneous, i.e., $(kf)' = kf'$, and should satisfy the Leibniz identity, i.e., $(fg)' = fg' + gf'$. Bazsó and Lábos are concerned with algebras of Boolean functions. They remark that the well-known sensitivity function does not

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satisfy the Leibniz identity, but they construct an extension of the original Boolean algebra and they extend the original sensitivity function to the enlarged Boolean algebra in such a way that the extended sensitivity function does satisfy the Leibniz rule.

Having the above in mind, in this paper we address the problem of the existence of a convenient concept of a derivative in an arbitrary Boolean algebra. The paper is organized as follows. In Section 1 we recall a few prerequisites and state precisely the problem of finding the Boolean operators D which satisfy the basic derivative-like properties with respect to two Boolean operations $\oplus$ and $\odot$ which satisfy a few reasonable requirements. In Section 2 we prove there are four couples of functions $\oplus, \odot$ satisfying the requirements, namely I) $x \oplus y = x + y, x \odot y = xy$ (ring sum and conjunction), II) $x \oplus y = x \vee y, x \odot y = xy$, and III), IV) their duals. In Sections 3 and 4 we solve the problem in the cases I) and II), respectively. The last Section is devoted to conclusions.

1 Statement of the problem

We work in a Boolean algebra $(\mathbf{B}, \cdot, \vee, ', 0, 1)$. The meet operation $\cdot$ is also denoted by concatenation, while $+$ stands for the *ring sum* $x + y = xy' \vee x'y$. The algebraic functions (or Grätzer polynomials) of the algebra $\mathbf{B}$ are called *Boolean functions*; they are characterized by the existence and uniqueness of the canonical disjunctive form. Thus e.g. a Boolean function of one variable can be written in the form $f(x) = ax \vee bx'$, where $a = f(1)$, $b = f(0)$, while the Boolean functions of two variables can be expressed as $F(x, y) = \alpha xy \vee \beta xy' \vee \gamma x'y \vee \delta x'y'$, where $\alpha = F(1, 1)$, $\beta = F(1, 0)$, $\gamma = F(0, 1)$, $\delta = F(0, 0)$. In particular the term functions, which in [12] are called *simple Boolean functions*, are those Boolean functions for which the coefficients of the canonical disjunctive form are taken from the set $\{0, 1\} \subseteq \mathbf{B}$. Every function with arguments and values in the two-element Boolean algebra $\{0, 1\}$ is a simple Boolean function. For more details see [12]. We assume the reader has some familiarity with computation in a Boolean algebra, which obeys the same rules as computation with intersection, union and complements of sets, due to the well-known representation theorem for Boolean algebras.

In this paper $\mathbf{B}$ is an arbitrary Boolean algebra. It is well known that *the set* $\mathbf{B}(1)$ *of Boolean functions of one variable* $f : \mathbf{B} \longrightarrow \mathbf{B}$ is itself a Boolean algebra with respect to the operations defined by $(f \vee g)(x) = f(x) \vee g(x), (f \cdot g)(x) = f(x) \cdot g(x), (f')(x) = (f(x))'$. A map $D : \mathbf{B}(1) \longrightarrow \mathbf{B}(1)$ will be called a *Boolean operator* provided there exist two Boolean functions $\varphi, \psi : \mathbf{B}^2 \longrightarrow \mathbf{B}$ such that the following identity holds:

(1) $$(Df)(x) = D(ax \vee bx')(x) = \varphi(a,b)x \vee \psi(a,b)x' \ .$$

The Boolean operator D might be called a "good derivative" of the Boolean algebra $\mathbf{B}(1)$ provided there exist two "convenient" operations $\oplus, \odot : \mathbf{B}^2 \longrightarrow \mathbf{B}$ such that the equalities

(2) $$D(f \oplus g) = Df \oplus Dg \ ,$$

(3) $$D(k \odot f) = k \odot Df \ ,$$

(4) $$D(f \odot g) = (f \odot Dg) \oplus (g \odot Df) \ ,$$

hold for every $f, g \in \mathbf{B}(1)$ and $k \in \mathbf{B}$.

More precisely, the following minimal hypothesis on $\oplus$ and $\odot$ seems natural:

(H) $\oplus$ and $\odot$ are two distinct non-constant commutative simple Boolean functions and $\odot$ distributes over $\oplus$.

The aim of this paper is to determine the tripes $(\oplus, \odot, D)$ satisfying (H) and one or several conditions out of (2), (3), (4).

2 Preliminary results

In this Section we determine all couples $(\oplus, \odot)$ satisfying condition (H) and we express properties (2),(3) (4) in terms of the functions φ, ψ.

Proposition 1 *Two Boolean operations $\oplus . \odot$ are commutative and $\odot$ distributes over $\oplus$ if and only if they are of the form*

(5) $$x \oplus y = Hxy + I(x+y) + Kx'y' \ ,$$

(6) $$x \odot y = Rxy + S(x+y) + Vx'y' \ ,$$

where

(7) $$(R' \vee H')S \vee (V \vee K)S' = 0 \ .$$

PROOF: We start with the canonical disjunctive forms

$$x \oplus y = Hxy \vee Ixy' \vee Jx'y \vee Kx'y' \ ,$$

$$x \odot y = Rxy \vee Sxy' \vee Tx'y \vee Vx'y' \ .$$

A theorem due to E. Schröder [14] (see e.g. [12], Corollary of Theorem 12.6) says that $\odot$ is left- and right-distributive over $\oplus$ if and only if

$$H'(R \vee V)(S \vee T) \vee K(R' \vee V')(S' \vee T')$$

$$\vee(HK \vee H'K' \vee IJ \vee I'J')(R'S \vee R'T \vee S'V \vee T'V) = 0 \; .$$

On the other hand, it is well known and easy to see that the commutativity of $\oplus$ and $\odot$ is equivalent to $J = I$ and $T = S$, hence the above Schröder condition can be written in the equivalent forms

$$H'(R \vee V)S \vee K(R' \vee V')S' \vee R'S \vee S'V = 0 \; ,$$

$$(H'R \vee H'V \vee R')S \vee (KR' \vee KV' \vee V)S' = 0 \; ,$$

and the latter condition coincides with (7). Finally the canonical disjunctive forms of $\oplus$ and $\odot$ can be written in the forms (5) and (6) because $\alpha\beta = 0 \Longleftrightarrow \alpha \vee \beta = \alpha + \beta$. $\square$

Theorem 1 *There are four pairs of functions satisfying hypothesis* (H)*, namely*
(H$_0$) $x \odot y = xy$ *and* $x \oplus y \in \{x + y, x \vee y\}$,
and
(H$_1$) $x \odot y = x \vee y$ *and* $x \oplus y \in \{xy, x + y + 1\}$.

PROOF: In view of Proposition 1, we have to characterize those distinct functions $\oplus, \odot$ which satisfy (5), (6), (7), whose coefficients H, I, K, R, S, V are in $\{0, 1\}$ and which do not reduce to the constant functions 0 or 1.

There are two cases.

1) $S = 0$. Then (7) implies $V = K = 0$, hence (5) and (6) reduce to $x \oplus y = Hxy + I(x + y)$ and $x \odot y = Rxy$. But $\odot$ is not a constant, therefore $R = 1$, while the values 0 or 1 of H and I yield $x \oplus y \in \{xy, x \vee y, x + y + 1, 1\}$. In view of (H) this reduces to (H$_0$).

2) $S = 1$. Then (7) implies $R = H = 1$, hence (5) and (6) reduce to $x \oplus y = xy + I(x + y) + Kx'y'$ and $x \odot y = xy + x + y + Vx'y'$. We are going to use the identities the identities

$$xy + x'y' = (x + y)' = x + y + 1 \text{ and } x \vee y = x + y + xy \; .$$

We first note that $V = 0$, otherwise $V = 1$ would imply $x \odot y = 1$. So $x \odot y = x \vee y$, while $x \oplus y \in \{xy, x \vee y, x + y + 1, 1\}$. In view of (H) this reduces to (H$_1$). $\square$

To express conditions (2)-(4) in terms of the functions φ, ψ, we use the standard notation

$$f(x) = ax \vee bx' = ax + bx', \; g(x) = cx \vee dx' = cx + dx' \qquad (8)$$

for two arbitrary Boolean functions $f, g \in \mathbf{B}(1)$.

Then the definition of Boolean operations in the Boolean algebra $\mathbf{B}(1)$ yields $(f \oplus g)(1) = f(1) \oplus g(1) = a \oplus c$ and $(f \oplus g)(0) = f(0) \oplus g(0) = b \oplus d$, therefore

$$(f \oplus g)(x) = (a \oplus c)x + (b \oplus d)x' \ , \tag{9}$$

and similarly

$$(k \odot f)(x) = (k \odot a)x + (k \odot b)x' \ , \tag{10}$$

$$(f \odot g)(x) = (a \odot c)x + (b \odot d)x' \ . \tag{11}$$

By applying formula (1) to the functions (9), (10) and (11), we obtain

$$[D(f \oplus g)](x) = \varphi(a \oplus c, b \oplus d)x + \psi(a \oplus c, b \oplus d)x' \ , \tag{12}$$

$$[D(k \odot f)](x) = \varphi(k \odot a, k \odot b)x + \psi(k \odot a, k \odot b)x' \ , \tag{13}$$

$$[D(f \odot g)](x) = \varphi(a \odot c, b \odot d)x + \psi(a \odot c, b \odot d)x' \ . \tag{14}$$

On the other hand, taking into account formula (1) and its companion

$$(Dg)(x) = D(cx \vee dx')(x) = \varphi(c, d)x \vee \psi(c, d)x' \ , \tag{11'}$$

we see that formulae (9), (10) and (11) yield

$$(Df \oplus Dg)(x) = [\varphi(a, b) \oplus \varphi(c, d)]x + [\psi(a, b) \oplus \psi(c, d)]x' \ , \tag{15}$$

$$(k \odot Df)(x) = [k \odot \varphi(a, b)]x + [k \odot \psi(c, d)]x' \ , \tag{16}$$

$$(f \odot Dg)(x) = [a \odot \varphi(c, d)]x + [b \odot \psi(c, d)]x' \ , \tag{17}$$

$$(g \odot Df)(x) = [c \odot \varphi(a, b)]x + [d \odot \psi(a, b)]x' \ . \tag{18}$$

It follows from (17) and (18) that

$$\begin{gathered} [(f \odot Dg) \oplus (g \odot Df)](x) = \\ = [(a \odot \varphi(c, d)) \oplus (c \odot \varphi(a, b))]x + [(b \odot \psi(c, d)) \oplus (d \odot \psi(a, b))]x' \ . \end{gathered} \tag{19}$$

Proposition 2 *The following systems* (20), (21) *and* (22) *are equivalent to conditions* (2), (3) *and* (4), *respectively.*

$$\varphi(a \oplus c, b \oplus d) = \varphi(a, b) \oplus \varphi(c, d) \ , \tag{20.1}$$

$$\psi(a \oplus c, b \oplus d) = \psi(a, b) \oplus \psi(c, d) \ , \tag{20.2}$$

$$\varphi(k \odot a, k \odot b) = k \odot \varphi(a, b) \ , \tag{21.1}$$

$$\psi(k \odot a, k \odot b) = k \odot \psi(a, b) \ , \tag{21.2}$$

$$\varphi(a \odot c, b \odot d) = [a \odot \varphi(c,d)] \oplus [c \odot \varphi(a,b)] \ , \tag{22.1}$$

$$\psi(a \odot c, b \odot d) = [b \odot \psi(c,d)] \oplus [d \odot \psi(a,b)] \ . \tag{22.2}$$

PROOF: By (12), (13), (14) and (15), (16), (19). □

To proceed further we introduce the notation

$$\varphi(x,y) = Axy + Bxy' + Cx'y + Ex'y' \ , \tag{23}$$

$$\psi(x,y) = Mxy + Nxy' + Px'y + Qx'y' \ , \tag{24}$$

where the coefficients $A, \dots, Q$ need not be 0,1, but may be any elements of the Boolean algebra **B**. We split the discussion into two cases, corresponding to (H_0) and (H_1) of Theorem 1.

3 The case $x \oplus y = x + y$, $x \odot y = xy$

Conditions (2)-(4) become

$$D(f+g) = Df + Dg \ , \tag{25}$$

$$D(kf) = kDf \ , \tag{26}$$

$$D(fg) = fDg + gDf \ . \tag{27}$$

In order to solve in this case the problem stated in Section 1 we use Proposition 2 with the conditions (20)-(22) corresponding to the present case. We describe an arbitrary Boolean operator D by formulae (1), (23) and (24).

Theorem 2 *Let D be a Boolean operator. Then:*

I) *D satisfies (25) if and only if $A + B + C = M + N + P = E = Q = 0$.*

II) *D satisfies (26) if and only if $E = Q = 0$.*

III) *The only Boolean operator satisfying (25), (26) and (27) is the constant operator $D = 0$.*

PROOF: I) Taking into account that $x' = x + 1$, formula (23) can be written in the form

$$\varphi(x,y) = (A+B+C+E)xy + (B+E)x + (C+E)y + E \ ,$$

therefore in this case condition (20.1) can be written in the equivalent forms

$$(A+B+C+E)(a+c)(b+d) + (B+E)(a+c) + (C+E)(b+d) + E$$

$$= (A + B + C + E)ab + (B + E)a + (C + E)b + E+$$

$$+(A + B + C + E)cd + (B + E)c + (C + E)d + E \ ,$$

$$(A + B + C + E)(ad + bc) + E = 0 \ .$$

Setting $a := b := 0$, we see that $E = 0$, hence $(A + B + C)(ad + bc) = 0$, and setting $a := d := 1, b := 0$, we get $A + B + C = 0$. Clearly condtions $A + B + C = E = 0$ are also sufficient. One proves similarly that (20.2) is equivalent to $M + N + P = Q = 0$.

II) Condition (21.1) can be written in the equivalent forms

$$\varphi(ka, kb) = k\varphi(a.b) \ ,$$

$$Akakb + Bka(k' \vee b') + C(k' \vee a')kb + E(ka + 1)(kab + 1)$$

$$= kAab + kBab' + kCa'b + kE(a + 1)(b + 1) \ ,$$

$$E(kab + ka + kb + 1 + kab + ka + kb + k) = 0 \ ,$$

$$E(1 + k) = 0 \ ,$$

which holds for any k iff $E = 0$. Similarly, the corresponding condition (21.2) holds iff $Q = 0$.

III) The trivial operator $D = 0$ satisfies (25)-(27). Conversely, suppose D fulfils (25), (26) and (27). Then $E = Q = 0$ and $A + B + C = M + N + P = 0$ by I) and II). Besides, φ satisfies (22.1), which becomes

$$\varphi(ac, bd) = a\varphi(c, d) + c\varphi(a, b) \ . \tag{28}$$

Setting $a := d := 0, b := c := 1$, we obtain $\varphi(0, 0) = \varphi(0, 1)$, i.e., $E = C$. So $C = 0$, which implies $A + B = 0$, that is, $A = B$. It follows that $\varphi(x, y) = Axy + Axy' = Ax$. Therefore condition (28) can be written $Aac = aAc + cAa$, that is, $Aac = 0$, which is equivalent to $A = 0$. One proves similarly that $M = N = P = 0$. □

4 The case $x \oplus y = x \vee y$, $x \odot y = xy$

Conditions (2)-(4) become

$$D(f \vee g) = Df \vee Dg \ , \tag{29}$$

$$D(kf) = kDf \ , \tag{30}$$

$$D(fg) = fDg \vee gDf \ . \tag{31}$$

We use Proposition 2 with the conditions (20)-(22) corresponding to the present case. Again, an arbitrary Boolean operator D is described by formulae (1), (23) and (24).

Theorem 3 *Let D be a Boolean operator. Then:*

I) *D satisfies* (30) *if and only if $E = Q = 0$.*

II) *D satisfies* (29) *and* (30) *if and only if $E = Q = 0$, $A = B \vee C$ and $M = N \vee P$. These conditions are equivalent to $\varphi(x, y) = Bx \vee Cy$ and $\psi(x, y) = Nx \vee Py$.*

III) *D satisfies* (29), (30) *and* (31) *if and only if $C = E = N = Q = 0$ and $A = B$ and $M = P$. These conditions are equivalent to $\varphi(x, y) = Bx$ and $\psi(x, y) = Py$, that is,*

$$(Df)(x) = D(ax \vee bx')(x) = Bax \vee Pbx' \; . \tag{32}$$

PROOF: I) By Theorem 2.II), as that result does not depend on the operation $\oplus$.

II) Suppose D satisfies (29) and (30). Then $E = Q = 0$ by I), while condition (20.1) becomes $\varphi(a \vee c, b \vee d) = \varphi(a, b) \vee \varphi(c, d)$. Setting $a := d := 1, b := c := 0$, we obtain $\varphi(1, 1) = \varphi(1, 0) \vee \varphi(0, 1)$, i.e., $A = B \vee C$. On proves similarly that $M = N \vee P$.

Conversely, suppose $E = Q = 0$, $A = B \vee C$ and $M = N \vee P$. Then (30) holds by I), while

$$\varphi(x, y) = (B \vee C)xy \vee Bxy' \vee Cx'y = Bx \vee Cy \; ,$$

$$\varphi(a \vee c, b \vee d) = B(a \vee c) \vee C(b \vee d) = \varphi(a, b) \vee \varphi(c, d) \; ,$$

that is, (20.1)holds. One proves similarly (20.2), therefore D satisfies (29).

III) Suppose D satisfies (29), (30) and (31). Then $A = B \vee C$, $M = N \vee P$ and $E = Q = 0$ by II), while (22.1) becomes $\varphi(ac, bd) = a\varphi(c, d) \vee c\varphi(a, b)$. Setting $a := d := 0, b := c := 1$, we obtain $\varphi(0, 0) = \varphi(0, 1)$, that is, $E = C$. So $C = 0$ and $A = B$. One proves similarly, using (22.2), that $\varphi(0, 0) = \varphi(1, 0)$, that is, $Q = N$, hence $N = 0$ and $M = P$. We have thus obtained (32).

Conversely, suppose $C = E = N = Q = 0$, $A = B$ and $M = P$, or equivalently, D is of the form (32). Then one can check directly that D satisfies (29), (30) and (31). □

Remark 1 The operators (32) satisfy condition (31) in the stronger form $D(fg) = fDg = gDf$.

Remark 2 While most Boolean derivatives occurring in the literature are defined in terms of the ring sum $+$, Fadini [8] defines $Df = f(0) \vee f(1)$. The Fadini derivative falls within case II) of Theorem 3, with $B = C = N = P = 1$.

5 Conclusions

Let $\mathbf{B}(1)$ be the set of Boolean (i.e., algebraic) functions $f : \mathbf{B} \longrightarrow \mathbf{B}$ over an arbitrary Boolean algebra $(\mathbf{B}, \vee, \cdot, ', 0, 1)$. We have investigated the possibility of the existence of a Boolean operator $D : \mathbf{B}(1) \longrightarrow \mathbf{B}(1)$ satisfying the properties

$$D(f \oplus g) = Df \oplus Dg \ , \tag{2}$$

$$D(k \odot f) = k \odot Df \ , \tag{3}$$

$$D(f \odot g) = (f \odot Dg) \oplus (g \odot Df) \ , \tag{4}$$

under mild conditions on the term functions $\oplus, \odot : \mathbf{B}^2 \longrightarrow \mathbf{B}$.

There are four such pairs $(\oplus, \odot)$, namely $(x + y, xy), (x \vee y, xy), (x + y + 1, x \vee y)$ and $(xy, x \vee y)$, where $x + y = xy' \vee x'y$ is the ring sum and $x + y + 1 = (x + y)' = (x \vee y')(x' \vee y)$. Having in view duality, we have dealt only with the first two cases. In the first case we hace constructed all the Boolean operators D which satisfy (2)/ which satisfy (3)/ which satisfy (2), (3) and (4). In the second case we have obtained characterizations of the Boolean operators D which satisfy (3)/ which satisfy (2) and (3)/ which satisfy (2), (3) and (4).

Our problem comes from switching theory, where a few concepts of Boolean derivative have been intensively studied and which are pretty good analogues of the conventional concept of derivative for the Boolean ring $(\mathbf{B}, \oplus, \cdot, 0, 1)$, but fail to satisfy the Leibniz rule (4). This was the starting point of a paper by Bazsó and Lábos [2].

Our paper implies that *there is no "good" Boolean derivative satisfying all of the properties* (2), (3) *and* (4). For in the above first case

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