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Approximating common fixed points of Presić-Kannan type operators by a multi-step iterative method

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Abstract

The existence of coincidence points and common fixed points for operators satisfying a Presić-Kannan type contraction condition in a metric spaces setting is proved. A multi-step iterative method for constructing the common fixed points is also provided.

1 Introduction

In 1965 S.B. Presić extended Banach's contraction mapping principle (see [2]) to operators defined on product spaces. It is easy to see that by taking $k = 1$, Theorem 1.1 below reduces to Banach's theorem.

Theorem 1.1 (S.B. Presić [12], 1965) *Let (X, d) be a complete metric space, k a positive integer, $\alpha_1, \alpha_2, \ldots, \alpha_k \in \mathbb{R}_+$, $\sum\limits_{i=1}^{k} \alpha_i = \alpha < 1$ and $f : X^k \to X$ an operator satisfying*

$$d(f(x_0, \ldots, x_{k-1}), f(x_1, \ldots, x_k)) \leq \alpha_1 d(x_0, x_1) + \cdots + \alpha_k d(x_{k-1}, x_k), \quad \text{(P)}$$

for all $x_0, \ldots, x_k \in X$.

Then:

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1) *f has a unique fixed point x^*, that is, there exists a unique $x^* \in X$ such that $f(x^*, \ldots, x^*) = x^*$;*

2) *the sequence $\{x_n\}_{n\geq 0}$ defined by*

$$x_{n+1} = f(x_{n-k+1}, \ldots, x_n)\,, \quad n = k-1, k, k+1, \ldots \tag{1.1}$$

converges to x^*, for any $x_0, \ldots, x_{k-1} \in X$.

On the other hand, in 1968 R. Kannan [9] (see also [3], [4], [5], [15], for some recent extensions of this result) proved a fixed point result for operators $f : X \to X$ satisfying the following contraction condition:

$$d(f(x), f(y)) \leq a\,[d(x, f(x) + d(y, f(y))]\,, \tag{1.2}$$

for any $x, y \in X$, where $a \in [0, \frac{1}{2})$ is constant.

In a similar manner to that used by S.B. Presić [12] when extending Banach contractions to product spaces, by I.A. Rus in [13] when doing the same for $\varphi-$contractions or by L.Ćirić and S. Presić in the recent [6], we proved a generalization of Kannan's theorem in [10] by showing that an operator $f : X^k \to X$ satisfying

$$d(f(x_0, \ldots, x_{k-1}), f(x_1, \ldots, x_k)) \leq a\sum_{i=0}^{k} d(x_i, f(x_i, \ldots, x_i)), \tag{1.3}$$

for any $x_0, \ldots, x_k \in X$, where $0 \leq ak(k+1) < 1$, has a unique fixed point $x^* \in X$. This fixed point can be approximated by means of the $k-$step iterative method $\{x_n\}_{n\geq 0}$, defined by (1.1), for any $x_0, \ldots, x_{k-1} \in X$.

In the very recent paper [1], M. Abbas and G. Jungck extended Kannan's theorem to a common fixed point result in cone metric spaces, considering the concept of *weakly compatible* mappings introduced by G. Jungck in [8].

In the present paper, by extending the concept of weakly compatible mappings to operators defined on Cartesian product, we obtain some results regarding the existence and uniqueness of coincidence/common fixed points for operators satisfying a Presić- Kannan type condition and also provide an iterative method for obtaining them. A similar approach of the contraction condition introduced in [13], which gives a Presić type extension for φ-contractions, can be found in our recent paper [11].

2 Preliminaries

We begin by recalling some concepts used in [1], [7], [8] and several related papers.

Definition 2.1 ([7]) *Let X be nonempty set and $f, g : X \to X$ two operators.*
An element $p \in X$ is called a ***coincidence point*** *of f and g if*

$$f(p) = g(p).$$

In this case $s = f(p) = g(p)$ is a ***coincidence value*** *of f and g.*
An element $p \in X$ is called a ***common fixed point*** *of f and g if*

$$f(p) = g(p) = p.$$

Remark 2.1 We shall denote by

$$C(f, g) = \{p \in X \,|\, f(p) = g(p)\}$$

the set of all coincidence points of f and g.
Obviously, the following hold:

a) $F_f \cap F_g \subset C(f, g)$;

b) $F_f \cap C(f, g) = F_g \cap C(f, g) = F_f \cap F_g$.

Definition 2.2 ([8]) *Let X be a nonempty set and $f, g : X \to X$. The operators f and g are said to be* ***weakly compatible*** *if they commute at their coincidence points, namely if*

$$f(g(p)) = g(f(p)),$$

for any coincidence point p of f and g.

Lemma 2.1 *Let X be a nonempty set and $f, g : X \to X$ two operators. If f and g are weakly compatible, then $C(f, g)$ is invariant for both f and g.*

Proof. Let $p \in C(f, g)$. We shall prove that $f(p), g(p) \in C(f, g)$, as well. By definition,

$$f(p) = g(p) = q \in X. \tag{2.1}$$

As f and g are weakly compatible, we have:

$$f(g(p)) = g(f(p)),$$

which by (2.1) yields

$$f(q) = g(q),$$

so $q = f(p) = g(p) \in C(f, g)$. Thus, $C(f, g)$ is an invariant set for both f and g.

□

Using this Lemma, the proof of the following proposition is immediate.

Proposition 2.1 ([1]) *Let X be a nonempty set and $f, g : X \to X$ two weakly compatible operators.*

If they have a unique coincidence value $x^* = f(p) = g(p)$, for some $p \in X$, then x^* is their unique common fixed point.

Remark 2.2 For any operator $f : X^k \to X$, k a positive integer, we can define its **associate operator** $F : X \to X$ by

$$F(x) = f(x, \ldots, x), x \in X. \tag{2.2}$$

Obviously, $x \in X$ is a fixed point of $f : X^k \to X$, i.e., $x = f(x, \ldots, x)$, if and only if it is a fixed point of its associate operator F, in the sense of the classical definition. For details see for example [14].

Based on this remark, we can extend the previous definitions for the case $f : X^k \to X$, k a positive integer.

Definition 2.3 *Let X be a nonempty set, k a positive integer and $f : X^k \to X$, $g : X \to X$ two operators.*

*An element $p \in X$ is called a **coincidence point** of f and g if it is a coincidence point of F and g, where F is given by (2.2).*

*Similarly, $s \in X$ is a **coincidence value** of f and g if it is a coincidence value of F and g.*

*An element $p \in X$ is a **common fixed point** of f and g if it is a common fixed point of F and g.*

Definition 2.4 *Let X be a nonempty set, k a positive integer and $f : X^k \to X$, $g : X \to X$. The operators f and g are said to be **weakly compatible** if F and g are weakly compatible.*

The following result is a generalization of Proposition 1.4 in [1], included above as Proposition 2.1.

Proposition 2.2 *Let X be a nonempty set, k a positive integer and $f : X^k \to X$, $g : X \to X$ two weakly compatible operators.*

If f and g have a unique coincidence value $x^* = f(p, \ldots, p) = g(p)$, then x^* is the unique common fixed point of f and g.

Proof. As f and g are weakly compatible, F and g are also weakly compatible. The proof follows by Proposition 2.1.

□

In order to prove our main result, we also need the following lemma, due to S. Presić [12].

Lemma 2.2 ([12]) *Let k be a positive integer and $\alpha_1, \alpha_2, \ldots, \alpha_k \in \mathbb{R}_+$ such that $\sum_{i=1}^{k} \alpha_i = \alpha < 1$. If $\{\Delta_n\}_{n\geq 1}$ is a sequence of positive numbers satisfying*

$$\Delta_{n+k} \leq \alpha_1 \Delta_n + \alpha_2 \Delta_{n+1} + \ldots + \alpha_k \Delta_{n+k-1}, \ n \geq 1,$$

then there exist $L > 0$ and $\theta \in (0,1)$ such that

$$\Delta_n \leq L \cdot \theta^n, \ \text{for all } n \geq 1.$$

3 The main result

The main result of this paper is the following theorem.

Theorem 3.1 *Let (X, d) be a metric space and k a positive integer. Let $f : X^k \to X$, $g : X \to X$ be two operators for which there exists a complete metric subspace $Y \subseteq X$ such that $f(X^k) \subseteq Y \subseteq g(X)$ and*

$$d(f(x_0, \ldots, x_{k-1}), f(x_1, \ldots, x_k)) \leq a \sum_{i=0}^{k} d(g(x_i), f(x_i, \ldots, x_i)), \qquad \text{(PK-C)}$$

for any $x_0, \ldots, x_k \in X$, where the real constant a fulfills $0 \leq ak(k+1) < 1$. Then:

1) *f and g have a unique coincidence value, say x^*, in X;*

2) *the sequence $\{g(z_n)\}_{n\geq 0}$ defined by $z_0 \in X$ and*

$$g(z_n) = f(z_{n-1}, \ldots, z_{n-1}), n \geq 1, \qquad (3.1)$$

converges to x^*;

3) *the sequence $\{g(x_n)\}_{n\geq 0}$ defined by $x_0, \ldots, x_{k-1} \in X$ and*

$$g(x_n) = f(x_{n-k}, \ldots, x_{n-1}), n \geq k, \qquad (3.2)$$

converges to x^* as well, with a rate estimated by

$$d(g(x_n), x^*) \leq C\theta^n, \qquad (3.3)$$

where C is a positive constant and $\theta \in (0,1)$;

4) if in addition f and g are weakly compatible, then x^* is their unique common fixed point.

Proof. 1),2) Let $z_0 \in X$. Then $f(z_0, \dots, z_0) \in f(X^k) \subset g(X)$, so there exists $z_1 \in X$ such that

$$f(z_0, \dots, z_0) = g(z_1).$$

Further on, $f(z_1, \dots, z_1) \in f(X^k) \subset g(X)$, so there exists $z_2 \in X$ such that

$$f(z_1, \dots, z_1) = g(z_2).$$

In this manner we construct a sequence $\{g(z_n)\}_{n \geq 0}$ with $z_0 \in X$ and

$$g(z_n) = f(z_{n-1}, \dots, z_{n-1}), n \geq 1. \tag{3.4}$$

Due to the manner $\{g(z_n)\}_{n \geq 0}$ was constructed, it is easy to remark that

$$\{g(z_n)\}_{n \geq 0} \subseteq f(X^k) \subseteq Y \subseteq g(X). \tag{3.5}$$

We can estimate now:

$$\begin{aligned} d(g(z_n), g(z_{n+1})) &= d(f(z_{n-1}, \dots, z_{n-1}), f(z_n, \dots, z_n)) \leq \\ &\leq d(f(z_{n-1}, \dots, z_{n-1}), f(z_{n-1}, \dots, z_{n-1}, z_n)) + \\ &+ \dots + d(f(z_{n-1}, z_n, \dots, z_n), f(z_n, \dots, z_n)). \end{aligned}$$

By (PK-C), this implies:

$$\begin{aligned} &d(g(z_n), g(z_{n+1})) \leq \\ &\leq a\left[kd(g(z_{n-1}), f(z_{n-1}, \dots, z_{n-1})) + d(g(z_n), f(z_n, \dots, z_n))\right] + \\ &+a\left[(k-1)d(g(z_{n-1}), f(z_{n-1}, \dots, z_{n-1})) + 2d(g(z_n), f(z_n, \dots, z_n))\right] + \\ &+ \dots + \\ &+a\left[d(g(z_{n-1}), f(z_{n-1}, \dots, z_{n-1})) + kd(g(z_n), f(z_n, \dots, z_n))\right] = \\ &= a\frac{k(k+1)}{2}\left[d(g(z_{n-1}), f(z_{n-1}, \dots, z_{n-1})) + d(g(z_n), f(z_n, \dots, z_n))\right] = \\ &= a\frac{k(k+1)}{2}\left[d(g(z_{n-1}), g(z_n)) + d(g(z_n), g(z_{n+1}))\right]. \end{aligned}$$

By denoting $A = \dfrac{ak(k+1)}{2} \in [0, \frac{1}{2})$ and $B = \dfrac{A}{1-A} \in [0, 1)$, the previous inequality implies:

$$d(g(z_n), g(z_{n+1})) \leq Bd(g(z_{n-1}), g(z_n)), \tag{3.6}$$

which by induction yields

$$d(g(z_n), g(z_{n+1})) \leq B^n d(g(z_0), g(z_1)), n \geq 0. \tag{3.7}$$

Since the series $\sum_{n=0}^{\infty} B^n$ converges, it follows by the well known Weierstrass criterion that $\{g(z_n)\}_{n\geq 0}$ is a Cauchy sequence included, by (3.5), in the complete subspace Y. Thus, there exists $x^* \in Y$ such that $\lim_{n\to\infty} g(z_n) = x^*$ and, since $Y \subseteq g(X)$, there exists $p \in X$ such that

$$g(p) = x^* = \lim_{n\to\infty} g(z_n).$$

Next we shall prove that $f(p, \ldots, p) = x^*$ as well. In this respect we estimate:

$$\begin{aligned}
&d(g(z_n), f(p, \ldots, p)) = d(f(z_{n-1}, \ldots, z_{n-1}), f(p, \ldots, p)) \leq \\
&\quad \leq d(f(z_{n-1}, \ldots, z_{n-1}), f(z_{n-1}, \ldots, z_{n-1}, p)) + \cdots + \\
&\quad\quad + d(f(z_{n-1}, p, \ldots, p), f(p, \ldots, p)).
\end{aligned}$$

By (PK-C) this yields

$$\begin{aligned}
&d(g(z_n), f(p, \ldots, p)) \leq \\
&\quad \leq a\left[kd(g(z_{n-1}), f(z_{n-1}, \ldots, z_{n-1})) + d(g(p), f(p, \ldots, p))\right] + \\
&\quad\quad + \cdots + \\
&\quad\quad + a\left[d(g(z_{n-1}), f(z_{n-1}, \ldots, z_{n-1})) + kd(g(p), f(p, \ldots, p))\right] = \\
&\quad = A\left[d(g(z_{n-1}), g(z_n)) + d(g(p), f(p, \ldots, p))\right],
\end{aligned}$$

which implies

$$\begin{aligned}
&d(g(z_n), f(p, \ldots, p)) \leq \\
&\quad \leq A\left[d(g(z_{n-1}), g(z_n)) + d(g(p), g(z_n)) + d(g(z_n), f(p, \ldots, p))\right].
\end{aligned}$$

From here we obtain that

$$d(g(z_n), f(p, \ldots, p)) \leq Bd(g(z_{n-1}), g(z_n)) + Bd(g(p), g(z_n))$$

or, by (3.7),

$$d(g(z_n), f(p, \ldots, p)) \leq B^n d(g(z_0), g(z_1)) + Bd(g(p), g(z_n)). \tag{3.8}$$

We already know that $B \in [0, 1)$ and that $g(z_n) \to x^* = g(p)$ as $n \to \infty$. Thus, by (3.8) it is immediate that $d(g(z_n), f(p, \ldots, p)) \to 0$ as $n \to \infty$, so indeed

$$f(p, \ldots, p) = x^* = g(p),$$

that is, p is a coincidence point for f and g, while x^* is a coincidence value for them.
In order to prove the uniqueness of x^* we suppose there would be some $q \in X$ such that

$$f(q, \dots, q) = g(q) \neq x^*. \tag{3.9}$$

Then for the coincidence points p and q we have:

$$\begin{aligned} d(g(p), g(q)) &= d(f(p, \dots, p), f(q, \dots, q)) \leq \\ &\leq d(f(p, \dots, p), f(p, \dots, p, q)) + \dots + \\ &\quad + d(f(p, q, \dots, q), f(q, \dots, q)), \end{aligned}$$

which by (PK-C) implies

$$d(g(p), g(q)) \leq A \left[d(g(p), f(p, \dots, p)) + d(g(q), f(q, \dots, q)) \right].$$

This obviously leads to $d(g(p), g(q)) \leq 0$, which contradicts (3.9), so x^* is the unique coincidence value for f and g and it can be approximated by means of the sequence $\{g(z_n)\}_{n \geq 0}$ given by (3.1).

3) Now there is still to be proved that the $k-$step iteration method $\{g(x_n)\}_{n \geq 0}$ given by (3.2) converges to the unique coincidence value x^* as well.
In this respect we estimate

$$\begin{aligned} d(g(x_n), g(p)) &= d(f(x_{n-k}, \dots, x_{n-1}), f(p, \dots, p)) \leq \\ &\leq d(f(x_{n-k}, \dots, x_{n-1}), f(x_{n-k+1}, \dots, x_{n-1}, p)) + \dots + \\ &\quad + d(f(x_{n-1}, p, \dots, p), f(p, \dots, p)), \end{aligned}$$

which by (PK-C) and knowing that $d(g(p), f(p, \dots, p)) = 0$ yields

$$\begin{aligned} & d(g(x_n), g(p)) \leq \\ & \leq a \left[d(g(x_{n-k}), f(x_{n-k}, \dots, x_{n-k})) + \right. \\ & \qquad \left. + \dots + d(g(x_{n-1}), f(x_{n-1}, \dots, x_{n-1})) + 0 \right] + \\ & + a \left[d(g(x_{n-k+1}), f(x_{n-k+1}, \dots, x_{n-k+1})) + \right. \\ & \qquad \left. + \dots + d(g(x_{n-1}), f(x_{n-1}, \dots, x_{n-1})) + 0 + 0 \right] + \\ & + \dots + \\ & + a[d(g(x_{n-1}), f(x_{n-1}, \dots, x_{n-1})) + \underbrace{0 + \dots + 0}_{k \text{ times}}]. \end{aligned}$$

Therefore

$$\begin{aligned} d(g(x_n), g(p)) &\leq a d(g(x_{n-k}), f(x_{n-k}, \dots, x_{n-k})) + \\ &\quad + 2a \cdot d(g(x_{n-k+1}), f(x_{n-k+1}, \dots, x_{n-k+1})) + \dots + \\ &\quad + ka \cdot d(g(x_{n-1}), f(x_{n-1}, \dots, x_{n-1})). \end{aligned} \tag{3.10}$$

As $g(p) = f(p, \ldots, p) = x^*$, for each $j \in \mathbb{N}$ we have that

$$d(g(x_j), f(x_j, \ldots, x_j)) \leq d(g(x_j), g(p)) + d(f(p, \ldots, p), f(x_j, \ldots, x_j)). \quad (3.11)$$

Now using the same technique as several times before in this proof, we get that for each $j \in \mathbb{N}$

$$d(f(p, \ldots, p), f(x_j, \ldots, x_j)) \leq \\ \leq A \left[d(g(p), f(p, \ldots, p)) + d(g(x_j), f(x_j, \ldots, x_j)) \right],$$

that is,

$$d(f(p, \ldots, p), f(x_j, \ldots, x_j)) \leq A d(g(x_j), f(x_j, \ldots, x_j)),$$

so (3.11) becomes

$$d(g(x_j), f(x_j, \ldots, x_j)) \leq \frac{1}{1-A} d(g(x_j), g(p)), j \in \mathbb{N}. \quad (3.12)$$

Getting back to the above relation (3.10), by (3.12) it is immediate that:

$$d(g(x_n), g(p)) \leq \frac{a}{1-A} d(g(x_{n-k}), g(p)) + \frac{2a}{1-A} d(g(x_{n-k+1}), g(p)) + \\ + \cdots + \frac{ka}{1-A} d(g(x_{n-1}), g(p)). \quad (3.13)$$

Denoting $\Delta_n = d(g(x_n), x^*)$, the sequence $\{\Delta_n\}_{n \geq 0}$ will satisfy the conditions in Lemma 2.2 due to Presić:

$$\Delta_n \leq \frac{a}{1-A} \Delta_{n-k} + \frac{2a}{1-A} \Delta_{n-k+1} + \cdots + \frac{ka}{1-A} \Delta_{n-1}, n \geq 1,$$

as well as

$$\sum_{i=1}^{k} \frac{ia}{1-A} = \frac{A}{1-A} < 1.$$

Then by the aforementioned lemma there exist $L > 0$ and $\theta \in (0, 1)$ such that $\Delta_n \leq L\theta^n, n \geq 0$, which actually means that

$$d(g(x_n), g(p)) \leq L\theta^n, n \geq 0. \quad (3.14)$$

It is now immediate that

$$d(g(x_n), x^*) \to 0, \text{ as } n \to \infty.$$

This proves the convergence of the $k-$step iterative method $\{g(x_n)\}_{n \geq 0}$ given by (3.2) to the unique coincidence value x^* of the operators f and g. Its rate

of convergence is given by the estimation (3.3) which can be easily deduced from relation (3.13) by repeatedly using (3.14), where $C = \frac{aL}{1-A}\sum_{i=1}^{k} i\theta^{i-k}$.
4) Supposing f and g are weakly compatible, by Proposition 2.2 it follows that they have a unique common fixed point, which is exactly their unique coincidence value x^*.

Now the proof is complete.

□

Remark 3.1 For $k = 1$, $g = 1_X$ and $Y = X$, Theorem 3.1 reduces to the result of Kannan [9]. For $g = 1_X$ and $Y = X$ our result in [10] is obtained.

Remark 3.2 The particular case for metric spaces of Theorem 2.2 due to M. Abbas and G. Jungck [1], originally proved in cone metric spaces, can be obtained from the above Theorem 3.1 if $k = 1$ and $Y = g(X)$.

We mention that in [1] $g(X)$ is required to be a complete metric space, a condition which turns to be too restrictive in applications. We replaced it by the more practical and slightly relaxed "there exists a complete metric subspace $Y \subseteq X$ such that $f(X^k) \subseteq Y \subseteq g(X)$", which also implies that $f(X^k) \subseteq g(X)$ as in [1].

In the following we present a very simple example of a pair f and g that satisfies the conditions in Theorem 3.1 above, while f does not satisfy condition (P) due to S. Presić.

Example 3.1 Let $X = [0, 1]$ with the usual metric, $k = 2$ and the operators $f : [0, 1] \times [0, 1] \to [0, 1]$ and $g : [0, 1] \to [0, 1]$ defined by

$$f(x, y) = \begin{cases} \frac{1}{6}, & x < \frac{4}{5}, \quad y \in [0, 1] \\ \\ \frac{1}{20}, & x \geq \frac{4}{5}, \quad y \in [0, 1] \end{cases}$$

and

$$g(x) = \begin{cases} x, & x < \frac{4}{5} \\ \\ 1, & x \geq \frac{4}{5}, \end{cases}$$

respectively. Then:

1) f and g satisfy the conditions in Theorem 3.1;

2) f does not satisfy condition (P) from Theorem 1.1.

Proof. 1) Let us check first that f and g satisfy condition (PK-C) from the above Theorem 3.1. In our particular case $k = 2$, so the above condition (PK-C) becomes

$$\begin{aligned}|f(x_0,x_1)-f(x_1,x_2)| \le a\,[&|g(x_0)-f(x_0,x_0)|+\\ &+|g(x_1)-f(x_1,x_1)|+|g(x_2)-f(x_2,x_2)|]\,,\end{aligned} \tag{3.15}$$

for any $x_0, x_1, x_2 \in [0,1]$, where $a \in (0, \frac{1}{6})$ is constant.
Theoretically we should analyze 8 cases, as each of x_0, x_1 and x_2 can be either $< \frac{4}{5}$ or $\ge \frac{4}{5}$, but considering the definitions of f and g we only have to discuss:

I. $x_0, x_1 < \frac{4}{5}$ or $x_0, x_1 \ge \frac{4}{5}$, while $x_2 \in [0,1]$.
In this case $f(x_0,x_1) = f(x_1,x_2)$ and the left hand side of (3.15) will be equal to 0, so (3.15) holds for any $a \in (0, \frac{1}{6})$.

II. $x_0 < \frac{4}{5}$, $x_1 \ge \frac{4}{5}$ and $x_2 < \frac{4}{5}$.
Then $f(x_0,x_1) = \frac{1}{6}$, $f(x_1,x_2) = \frac{1}{20}$, $f(x_0,x_0) = \frac{1}{6}$, $f(x_1,x_1) = \frac{1}{20}$, $f(x_2,x_2) = \frac{1}{6}$, $g(x_0) = x_0$, $g(x_1) = 1$ and $g(x_2) = x_2$, and (3.15) becomes

$$\left|\frac{1}{6}-\frac{1}{20}\right| \le a\left[\left|x_0-\frac{1}{6}\right|+\left|1-\frac{1}{20}\right|+\left|x_2-\frac{1}{6}\right|\right]. \tag{3.16}$$

As $\left|x_0-\frac{1}{6}\right| \ge 0$ and $\left|x_2-\frac{1}{6}\right| \ge 0$, the minimum value of the right hand side in (3.16) will be $a\frac{19}{20}$. Therefore a necessary condition for (3.16) to hold is $\frac{7}{60} \le a\frac{19}{20}$, which finally yields $a \ge \dfrac{7}{57}$.

III. $x_0 \ge \frac{4}{5}$, $x_1, x_2 < \frac{4}{5}$.
Similarly to case II it follows that $a \ge \dfrac{7}{57}$.

IV. $x_0 \ge \frac{4}{5}$, $x_1 < \frac{4}{5}$ and $x_2 \ge \frac{4}{5}$.
Then $f(x_0,x_1) = \frac{1}{20}$, $f(x_1,x_2) = \frac{1}{6}$, $f(x_0,x_0) = \frac{1}{20}$, $f(x_1,x_1) = \frac{1}{6}$, $f(x_2,x_2) = \frac{1}{20}$, $g(x_0) = 1$, $g(x_1) = x_1$ and $g(x_2) = 1$, and (3.15) becomes

$$\left|\frac{1}{20}-\frac{1}{6}\right| \le a\left[\left|1-\frac{1}{20}\right|+\left|x_1-\frac{1}{6}\right|+\left|1-\frac{1}{20}\right|\right]. \tag{3.17}$$

As $\left|x_1-\frac{1}{6}\right| \ge 0$, the minimum value of the right hand side in (3.17) will be $a\frac{19}{10}$. Therefore a necessary condition for (3.17) to hold is $\frac{7}{60} \le a\frac{19}{10}$, which finally yields $a \ge \dfrac{7}{114}$.

V. $x_0 < \frac{4}{5}$, $x_1, x_2 \geq \frac{4}{5}$.

Similarly to case IV it follows that $a \geq \dfrac{7}{114}$.

Finally we conclude that neccessarily $a \in \left[\frac{7}{57}, \frac{1}{6}\right)$, so f and g satisfy condition (PK-C), for example with constant $a = \dfrac{7}{57} \in (0, \dfrac{1}{6})$.

Since $f([0,1] \times [0,1]) = \left\{\dfrac{1}{20}; \dfrac{1}{6}\right\}$ and $g([0,1]) = [0,1]$, there exists for example the complete metric subspace $Y = \left[0, \frac{1}{2}\right] \subset [0,1]$ such that $f([0,1] \times [0,1]) \subset Y \subset g([0,1])$.

Then according to Theorem 3.1 f and g have a unique coincidence value in $[0,1]$, which can be approximated either by means of the sequence $\{g(z_n)\}_{n\geq 0}$ defined by

$$g(z_n) = f(z_{n-1}, z_{n-1}), n \geq 1,$$

starting from any $z_0 \in [0,1]$, or by means of the 2-step iterative method $\{g(x_n)\}_{n\geq 0}$ defined by

$$g(x_n) = f(x_{n-2}, x_{n-1}), n \geq 2,$$

for any initial values $x_0, x_1 \in [0,1]$.
Indeed, as one can easily check, $F_f = \left\{\dfrac{1}{6}\right\}$, $F_g = \left[0, \dfrac{4}{5}\right) \cup \{1\}$ and the set of coincidence values is given by $C(f,g) = \left\{\dfrac{1}{6}\right\}$.

Moreover, f and g are weakly compatible, as $f(g(\frac{1}{6})) = g(f(\frac{1}{6})) = \frac{1}{6}$, so, by Theorem 3.1, $\dfrac{1}{6}$ is also their unique common fixed point. Indeed, it is easy to see that $F_f \cap F_g = \left\{\dfrac{1}{6}\right\}$.

2) Now we shall prove that f is not a Presić operator. In our particular case inequality (P) becomes:

$$|f(x_0, x_1) - f(x_1, x_2)| \leq \alpha_1 |x_0 - x_1| + \alpha_2 |x_1 - x_2|, \tag{3.18}$$

where $\alpha_1, \alpha_2 \in \mathbb{R}_+$, $\alpha_1 + \alpha_2 < 1$.
We will show that for certain points in $[0,1]$ inequality (3.18) is not satisfied. For example, let $x_0 = \dfrac{4}{5}$ and $x_1 = x_2 = \dfrac{2}{5}$. Then $f(x_0, x_1) = \dfrac{1}{20}$, while

$f(x_1, x_2) = \dfrac{1}{6}$. Inequality (3.18) becomes:

$$\left|\frac{1}{20} - \frac{1}{6}\right| \leq \alpha_1 \left|\frac{4}{5} - \frac{2}{5}\right| + \alpha_2 \left|\frac{2}{5} - \frac{2}{5}\right|,$$

which is equivalent to

$$\frac{7}{60} \leq \alpha_1 \frac{2}{5}. \tag{3.19}$$

But $\alpha_1 < 1$, so it is obvious that (3.19) will never hold. Thus, f is not a Presić operator, so our Theorem 3.1 effectively extends Theorem 1.1 of S. Presić.

□

4 An extension of the main result

Theorem 3.1 offers information about coincidence and common fixed points of two operators, one of them defined on the Cartesian product X^k, $f : X^k \to X$, where k is a positive integer, and the second one a self-operator on X, $g : X \to X$. As the great majority of the common fixed point results in literature deal with the case when both f and g are self-operators on X, our aim in this section is to establish a common fixed point theorem for the more general case $f : X^k \to X$ and $g : X^l \to X$, with k and l positive integers. In this respect we shall begin with some definitions which extend the corresponding ones in the previous section, and which can also be found in our recent paper [11].

Definition 4.1 *Let X be a metric space, k, l positive integers and $f : X^k \to X$, $g : X^l \to X$ two operators.*

An element $p \in X$ is called a ***coincidence point*** *of f and g if it is a coincidence point of F and G, where $F, G : X \to X$ are the associate operators of f and g, respectively, see Remark 2.2.*

An element $s \in X$ is called a ***coincidence value*** *of f and g if it is a coincidence value of F and G.*

An element $p \in X$ is called a ***common fixed point*** *of f and g if it is a common fixed point of F and G.*

Definition 4.2 *Let (X, d) be a metric space, k, l positive integers and $f : X^k \to X$, $g : X^l \to X$. The operators f and g are said to be* ***weakly compatible*** *if F and G are weakly compatible.*

In these terms we state now the following result, which extends the above Theorem 3.1.

Theorem 4.1 *Let (X,d) be a metric space, k and l positive integers, $f : X^k \to X$ and $g : X^l \to X$ two operators such that f and G fulfill the conditions in Theorem 3.1, where $G : X \to X$ is the associated operator of g.*

Then:

1) *f and g have a unique coincidence value, say x^*, in X;*

2) *the sequence $\{G(z_n)\}_{n\geq 0}$ defined by $z_0 \in X$ and*

$$G(z_n) = f(z_{n-1}, \dots, z_{n-1}), n \geq 1, \tag{4.1}$$

converges to x^*;

3) *the sequence $\{G(x_n)\}_{n\geq 0}$ defined by $x_0, \dots, x_{k-1} \in X$ and*

$$G(x_n) = f(x_{n-k}, \dots, x_{n-1}), n \geq k, \tag{4.2}$$

converges to x^* as well, with a rate estimated by

$$d(G(x_n), x^*) \leq C\theta^n, \tag{4.3}$$

where C is a positive constant and $\theta \in (0,1)$;

4) *if in addition f and g are weakly compatible, then x^* is their unique common fixed point.*

Proof. Having in view the definitions given in this section, all the conclusions follow by applying Theorem 3.1 for $f : X^k \to X$ and $G : X \to X$.

□

Remark 4.1 If we take $l = 1$, then by Theorem 4.1 we get Theorem 3.1 in this paper, while for $l = 1$, $g = 1_X$ and $Y = X$ the fixed point theorem in [10] is obtained. Moreover, if we take $k = 1$, $l = 1$, $g = 1_X$ and $Y = X$, by Theorem 4.1 we obtain the well known Kannan fixed point theorem [9], which could be similarly stated in a cone metric space setting, as in [1].

We shall end with the following example which illustrates Theorem 4.1.

Example 4.1 Let $X = [0,1]$, $k = 2$, $l = 3$, $f : [0,1] \times [0,1] \to [0,1]$ as in Example 3.1 and $h : [0,1] \times [0,1] \times [0,1] \to [0,1]$ defined by:

$$h(x,y,z) = \begin{cases} \frac{1}{2}(y+z), & (x,y,z) \in D_1 \\ xyz, & (x,y,z) \in D_2 \\ 1-(x-y)^2, & (x,y,z) \in D_3, \end{cases}$$

where

$$D_1 = [0,1] \times [0, \frac{4}{5}) \times [0, \frac{4}{5}),$$
$$D_2 = [0, \frac{4}{5}) \times [\frac{4}{5}, 1] \times [\frac{4}{5}, 1] \cup [0,1] \times [\frac{4}{5}, 1] \times [0, \frac{4}{5}) \cup [0,1] \times [0, \frac{4}{5}) \times [\frac{4}{5}, 1],$$
$$D_3 = [\frac{4}{5}, 1] \times [\frac{4}{5}, 1] \times [\frac{4}{5}, 1].$$

Then f and h have a unique common fixed point in $[0,1]$.

Proof. We remark that the associated operator of h is $H : [0,1] \to [0,1]$ defined by:

$$H(x) = h(x,x,x) = \begin{cases} x, & x < \frac{4}{5} \\ \\ 1, & x \geq \frac{4}{5}. \end{cases}$$

By Example 3.1, f and H fulfill the conditions in Theorem 3.1, and the rest follows by Theorem 4.1 above.

□

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