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A Bernstein-Stancu type operator which preserves e_2

Ingrid OANCEA

Abstract

In this paper we construct a Bernstein-Stancu type operator following a J.P.King model.

1 Introduction

Most of linear and positive operators on $C[a, b]$ preserve e_0 and e_1 :

$$\begin{aligned} L_n(e_0)(x) &= e_0(x) \\ L_n(e_1)(x) &= e_1(x) \end{aligned}$$

for each $n = 0, 1, 2, \ldots$ and $x \in [a, b]$.

J.P. King defined in [3] an interesting class of operators which preserve e_2. Let $(s_n(x))_{n\in\mathbb{N}}$ be a sequence of continuous functions on $[0, 1]$ so that $0 \le s_n(x) \le 1$. For any $f \in C[0, 1]$ and $x \in [0, 1]$ let $V_n : C[0, 1] \to C[0, 1]$ be defined by

$$(V_n f)(x) = \sum_{k=0}^{n} \binom{n}{k} s_n^k(x)(1 - s_n(x))^{n-k} f\left(\frac{k}{n}\right). \tag{1}$$

For $s_n(x) = x$, $n \in \mathbb{N}$ operators V_n become Bernstein operators. The values of the operators V_n on test functions $e_j = x^j$, $j = 0, 1, 2$ are given by

$$(V_n e_0)(x) = 1$$

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$$(V_n e_1)(x) = s_n(x)$$
$$(V_n e_2)(x) = \frac{1}{n} s_n(x) + \frac{n-1}{n} s_n^2(x).$$

Using Bohman-Korovkin theorem ([1], [4]) it follows immediately that $\lim_{n\to\infty} V_n f = f$ uniformly on $[0,1]$ if and only if $\lim_{n\to\infty} s_n(x) = x$ uniformly on $[0,1]$.

In order to preserve e_2, the s_n sequence has to be as it follows:

$$\begin{cases} s_1(x) = x^2 \\ s_n(x) = -\frac{1}{2(n-1)} + \sqrt{\frac{n}{n-1}x^2 + \frac{1}{4(n-1)^2}}, n = 2, 3, \ldots \end{cases}$$

2 Main results

D.D. Stancu (see [5], [6]) defined for two positive numbers $0 \le \alpha \le \beta$ independent of n and for any function $f \in C[0,1]$ the operator,

$$(P_n^{(\alpha,\beta)} f)(x) = \sum_{k=0}^{n} p_{n,k}(x) f\left(\frac{k+\alpha}{n+\beta}\right). \tag{2}$$

The Bernstein Stancu operator uses the equidistant knots $a_0 = \frac{\alpha}{n+\beta}$, $a_1 = x_0 + h, \ldots, a_n = x_0 + nh$ where $h = \frac{1}{n+\beta}$ and because $\left(P_n^{(\alpha,\beta)} f\right)(0) = f\left(\frac{\alpha}{n+\beta}\right)$ and $\left(P_n^{(\alpha,\beta)} f\right)(1) = f\left(\frac{n+\alpha}{n+\beta}\right)$, interpolates function f in $x = 0$ if $\alpha = 0$ and in $x = 1$ if $\alpha = \beta$.

Values on test function are given by:

$$\left(P_n^{(\alpha,\beta)} e_0\right)(x) = 1 \tag{3}$$

$$\left(P_n^{(\alpha,\beta)} e_1\right)(x) = x + \frac{\alpha - \beta x}{n+\beta} \tag{4}$$

$$\left(P_n^{(\alpha,\beta)} e_2\right)(x) = x^2 + \frac{nx(1-x) + (\alpha - \beta x)(2nx + \beta x + \alpha)}{(n+\beta)^2} \tag{5}$$

so we can state that for any $f \in C[0,1]$ the sequence $\left((P_n^{(\alpha,\beta)} f)(x)\right)_{n\in\mathbb{N}}$ converges uniformly to $f(x)$ on $[0,1]$.

We define now the operators $V_n^{(\alpha,\beta)} : C[0,1] \to C[0,1]$ by

$$\left(V_n^{(\alpha,\beta)} f\right)(x) = \sum_{k=0}^{n} \binom{n}{k} r_n^k(x)(1 - r_n(x))^{n-k} f\left(\frac{k+\alpha}{n+\beta}\right), \tag{6}$$

for any function $f \in C[0,1]$ and $x \in [0,1]$.

It's obvious that for $r_n(x) = x$, $n \in \mathbb{N}$ the Bernstein-Stancu operators are obtained, and for $a = \beta = 0$ the V_n operators defined by (1) are obtained.

The $V_n^{(\alpha,\beta)}$ operators are linear and positive.

Using the relations (3)-(5) the following theorem can be easily proved:

Theorem 2.1 *The operators $V_n^{(\alpha,\beta)}$ have the following properties:*

1.

$$\left(V_n^{(\alpha,\beta)} e_0\right)(x) = 1 \tag{7}$$

$$\left(V_n^{(\alpha,\beta)} e_1\right)(x) = \frac{n}{n+\beta} r_n(x) + \frac{\alpha}{n+\beta} \tag{8}$$

$$\left(V_n^{(\alpha,\beta)} e_2\right)(x) = \frac{1}{(n+\beta)^2}\left(n(n-1)r_n^2(x) + n(1+2\alpha)r_n(x) + \alpha^2\right) \tag{9}$$

2. For any function $f \in C[0,1]$ şi $x \in [0,1]$ we have

$$\lim_{n\to\infty} V_n^{(\alpha,\beta)} f = f$$

uniformly on $[0,1]$ if and only if

$$\lim_{n\to\infty} r_n(x) = x$$

uniformly on $[0,1]$.

Next we impose the condition $V_n^{(\alpha,\beta)} e_2 = e_2$, that is

$$\frac{1}{(n+\beta)^2}\left(n(n-1)r_n^2(x) + n(1+2\alpha)r_n(x) + \alpha^2\right) = x^2$$

or

$$n(n-1)r_n^2(x) + n(1+2\alpha)r_n(x) + \alpha^2 - x^2(n+\beta)^2 = 0.$$

If we denote

$$\begin{aligned} a &= n(n-1) \\ b &= n(1+2\alpha) \\ c &= \alpha^2 - (n+\beta)^2 x^2 \end{aligned}$$

then the discriminant is given by

$$\Delta = n^2(1+2\alpha)^2 - 4n(n-1)\left(\alpha^2 - (n+\beta)^2 x^2\right) =$$

$$= n^2 + 4n\alpha(n+\alpha) + 4n(n-1)(n+\beta)^2 x^2 \geq 0$$

for any $x \in [0,1]$. For $n \neq 1$ the solutions of the equation are

$$(r_n(x))_{1,2} = \frac{-n(1+2\alpha) \pm \sqrt{n^2(1+2\alpha)^2 - 4n(n-1)\left(\alpha^2 - (n+\beta)^2x^2\right)}}{2n(n-1)}.$$

We choose

$$r_n^*(x) = \frac{-n(1+2\alpha) + \sqrt{n^2(1+2\alpha)^2 - 4n(n-1)\left(\alpha^2 - (n+\beta)^2x^2\right)}}{2n(n-1)}, \quad n > 1 \tag{10}$$

and

$$r_1^*(x) = x^2. \tag{11}$$

Lemma 2.2 *For any* $x \in \left[\frac{\alpha}{n+\beta}, \frac{n+\alpha}{n+\beta}\right]$ *the following inequality holds* $0 \leq r_n^*(x) \leq 1$.

Proof. Because $r_n^*(x) = \frac{-b+\sqrt{b^2-4ac}}{2a}$ the inequality $0 \leq r_n^*(x) \leq 1$ becomes

$$0 \leq \frac{-b + \sqrt{b^2 - 4ac}}{2a} \leq 1$$

Since $a > 0$ we get

$$0 \leq -b + \sqrt{b^2 - 4ac} \leq 2a$$

$$0 \leq b \leq \sqrt{b^2 - 4ac} \leq 2a + b$$

which leads to

$$b^2 \leq b^2 - 4ac \leq 4a^2 + 4ab + b^2$$

$$0 \leq -ac \leq a^2 + ab.$$

It results that we have to find $x \in [0,1]$ such that

$$\begin{cases} c \leq 0 \\ a + b + c \geq 0 \end{cases}.$$

Replacing a, b, c we obtain $c \leq 0$ if $\alpha^2 - (n+\beta)^2x^2 \leq 0$, that is $x \in \left[\frac{\alpha}{n+\beta}, 1\right]$, and $a + b + c \geq 0$ becomes

$$n(n-1) + n(1+2\alpha) + \alpha^2 - (n+\beta)^2x^2 \geq 0$$

$$x^2 \leq \left(\frac{n+\alpha}{n+\beta}\right)^2,$$

therefore $x \in \left[0, \frac{n+\alpha}{n+\beta}\right]$ which eventually gives us that $x \in \left[\frac{\alpha}{n+\beta}, \frac{n+\alpha}{n+\beta}\right]$.

If we denote $I_n = \left[\frac{\alpha}{n+\beta}, \frac{n+\alpha}{n+\beta}\right]$ from the inequalities

$$\frac{\alpha}{n+\beta+1} \leq \frac{\alpha}{n+\beta} \leq \frac{n+\alpha}{n+\beta} \leq \frac{n+\alpha+1}{n+\beta+1}$$

it follows that $I_n \subset I_{n+1}$, $n \in \mathbb{N}$; moreover for $n \to \infty$ the interval I_n becomes $[0, 1]$.

One can notice that $\lim\limits_{n\to\infty} r_n^*(x) = x$, so we have the following

Theorem 2.3 *The operators $V_n^{(\alpha,\beta)}$ given by 6 with the sequence $(r_n^*(x))_{n\in\mathbb{N}}$ defined by 10, 11 have the following properties:*

1. *they are linear and positive on $C[0,1]$*
2. $\left(V_n^{(\alpha,\beta)} e_2\right)(x) = e_2(x)$, $n \in \mathbb{N}^*$ *for any* $x \in \left[\frac{\alpha}{n+\beta}, \frac{n+\alpha}{n+\beta}\right]$
3. $\lim\limits_{n\to\infty} V_n^{(\alpha,\beta)} f = f$ *for any* $f \in C[0,1]$, $x \in \left[\frac{\alpha}{n+\beta}, \frac{n+\alpha}{n+\beta}\right]$.

If L is a linear and positive operator on $C[a,b]$, then for any continuous function $f \in C[a,b]$ and $x \in [a,b]$ we have the evaluation (see [2], pg. 30)

$$|(Lf)(x) - f(x)| \leq |f(x)|\,|(Le_0)(x) - 1| + \left((Le_0)(x) + \frac{(L\varphi_x)(x)}{\delta}\right)\omega(f,\delta) \leq$$

$$\leq |f(x)|\,|(Le_0)(x) - 1| + \left((Le_0)(x) + \frac{\sqrt{(Le_0)(x)(L\varphi_x^2)(x)}}{\delta}\right)\omega(f,\delta) \qquad (12)$$

$(\forall)\, x \in I$, $(\forall)\, \delta > 0$, where $\varphi_x = e_1 - xe_0$

If the operator L satisfies the conditions $Le_0 = e_0$ şi $Le_2 = e_2$ then the evaluation (12) can be written as:

$$|(Lf)(x) - f(x)| \leq \left(1 + \frac{\sqrt{(L\varphi_x^2)(x)}}{\delta}\right)\omega(f,\delta)$$

and since

$$(L\varphi_x^2)(x) = L\left((e_1 - xe_0)^2, x\right) = (Le_2)(x) - 2x\,(Le_1)(x) + x^2\,(Le_0)(x) =$$

$$= 2x^2 - 2x\,(Le_1)(x) = 2x\,(x - (Le_1)(x)), \qquad (13)$$

we can also write that

$$|(Lf)(x) - f(x)| \leq \left(1 + \frac{\sqrt{2x\,(x - (Le_1)(x))}}{\delta}\right)\omega(f,\delta)$$

for any $f \in C[a,b]$ and $x \in [a,b]$.

Since the operator L is positive and $\varphi_x^2 \geq 0$ we get that $L\varphi_x^2 \geq 0$ which is equivalent with $2x\left(x-\left(Le_1\right)(x)\right)$. It follows that for any $x \in [a,b], a \geq 0$ the inequality

$$\left(Le_1\right)(x) \leq x.$$

holds true.

Taking $[a,b] = I_n$ and $L = V_n^{(\alpha,\beta)}$ as a particular case we obtain:

Lemma 2.4 *For any $x \in I_n$ if $r_n(x) = r_n^*(x)$ we have*

$$\left(V_n^{(\alpha,\beta)} e_1\right)(x) \leq x.$$

We got that for any $x \in I_n$ we have $\left(V_n^{(\alpha,\beta)} e_0\right)(x) = e_0(x)$, $\left(V_n^{(\alpha,\beta)} e_2\right)(x) = e_2(x)$ şi $\left(V_n^{(\alpha,\beta)} e_1\right)(x) \leq x$; therefore the following evaluation stands:

$$\left|(V_n^{(\alpha,\beta)} f)(x) - f(x)\right| \leq \left(1 + \frac{\sqrt{2x\left(x - \left(V_n^{(\alpha,\beta)} e_1\right)(x)\right)}}{\delta}\right)\omega(f,\delta).$$

The order of approximation is at least as good as in case of approximation by Bernstein-Stancu polynomials for those $x \in I_n$ for which the following inequality is true

$$(V_n^{(\alpha,\beta)} \varphi_x^2)(x) \leq (P_n^{(\alpha,\beta)} \varphi_x^2)(x). \tag{14}$$

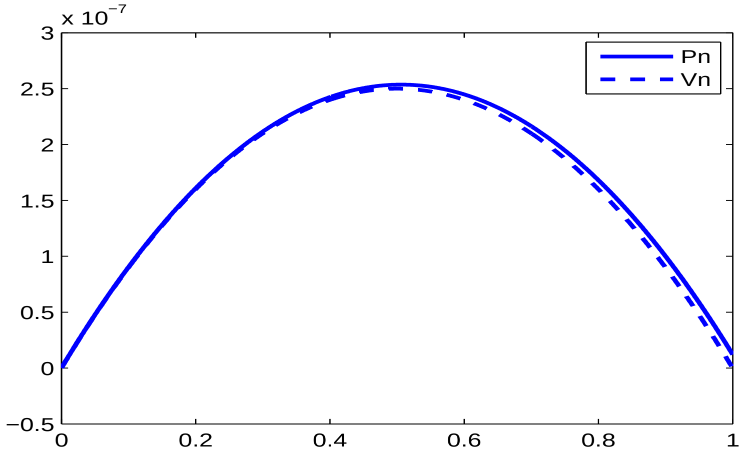
For $n > \beta^2$ the second order moment of Stancu operator is given by

$$(P_n^{(\alpha,\beta)} \varphi_x^2)(x) = (P_n^{(\alpha,\beta)} (e_1 - xe_0)^2)(x) = \frac{nx(1-x) + (\beta x - \alpha)^2}{(n+\beta)^2}. \tag{15}$$

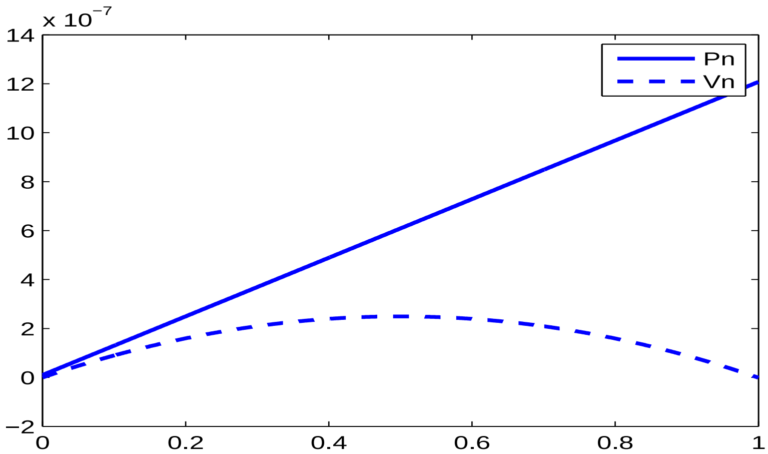
Taking into account the expressions of the moments for the two operators from relations (13) and (15), we can rewrite the inequality (14) as:

$$2x\left(x - \frac{n}{n+\beta} r_n^*(x) + \frac{\alpha}{n+\beta}\right) \leq \frac{nx(1-x) + (\beta x - \alpha)^2}{(n+\beta)^2}. \tag{16}$$

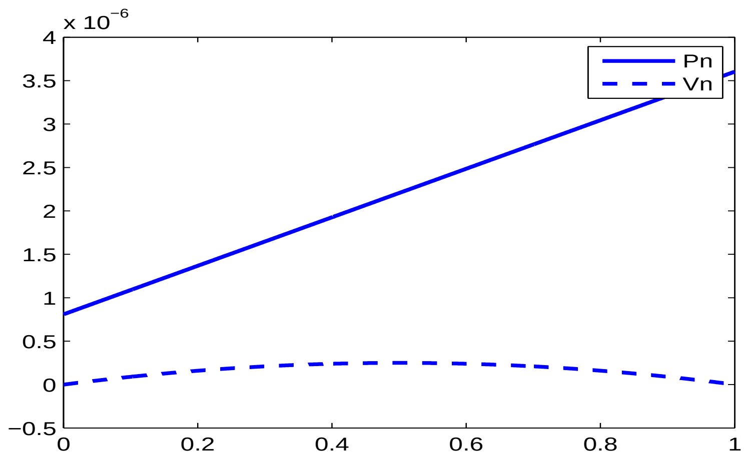
We present the graphics of the two members of inequality for some particular cases:



$n = 1.000.000$; $\alpha = 10$; $\beta = 100$



$n = 1.000.000$; $\alpha = 100$; $\beta = 1.000$



$n = 1.000.000;\ \alpha = 900;\ \beta = 1.000$

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Valahia University of Târgovişte,
Department of Mathematics,
Bd. Unirii No 18, 130082, Târgovişte,
Romania
ingrid.oancea@gmail.com