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Some isotopy-isomorphy conditions for m-inverse quasigroups and loops

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Abstract

This work presents a special type of middle isotopism under which m-inverse quasigroups are isotopic invariant. Two distinct isotopy-isomorphy conditions for m-inverse loops are established. Only one of them characterizes isotopy-isomorphy in m-inverse loops while the other is just a sufficient condition for isotopy-isomorphy for specially middle isotopic m-inverse quasigroup.

1 Introduction

Let L be a non-empty set. Define a binary operation $(\cdot)$ on L : If $x \cdot y \in L$ for all $x, y \in L$, $(L, \cdot)$ is called a groupoid. If the system of equations ;

$$a \cdot x = b \qquad \text{and} \qquad y \cdot a = b$$

have unique solutions for x and y respectively, then $(L, \cdot)$ is called a quasigroup. For each $x \in L$, the elements $x^\rho = xJ_\rho, x^\lambda = xJ_\lambda \in L$ such that $xx^\rho = e$ and $x^\lambda x = e$ are called the right, left inverses of x respectively. Now, if there exists a unique element $e \in L$ called the identity element such that for all $x \in L$, $x \cdot e = e \cdot x = x$, $(L, \cdot)$ is called a loop.

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Karklin's and Karklin' [10] introduced m-inverse loops. A loop is an m-inverse loop(m-IL) if and only if it obeys any of the equivalent conditions

$$(xy)J_\rho^m \cdot xJ_\rho^{m+1} = yJ_\rho^m \qquad \text{and} \qquad xJ_\lambda^{m+1} \cdot (yx)J_\lambda^m = yJ_\lambda^m.$$

Keedwell and Shcherbacov [12] originally defined an m-inverse quasigroup(m-IQ) as a quasigroup that obeys the identity $(xy)J^m \cdot xJ^{m+1} = yJ^m$, where J is a permutation. For the sake of this present study, we shall take $J = J_\rho$ and so m-IQs obey the equivalent identities that define m-ILs.

m-IQs and m-ILs are generalizations of WIPLs and CIPLs, which corresponds to $m = -1$ and $m = 0$ respectively. After the study of m-inverse loops by Keedwell and Shcherbacov [12], they have also generalized them to quasigroups called (r, s, t)-inverse quasigroups in [13] and [14]. Keedwell and Shcherbacov [12] investigated the existence of m-inverse quasigroups and loops with long inverse cycle such that $m \geq 1$.They have been able to establish that the direct product of two m-inverse quasigroups is an m-inverse quasigroup.

Consider $(G, \cdot)$ and $(H, \circ)$ two distinct groupoids (quasigroups, loops). Let A, B and C be three distinct non-equal bijective mappings, that map G onto H. The triple $\alpha = (A, B, C)$ is called an *isotopism* of $(G, \cdot)$ onto $(H, \circ)$ if

$$xA \circ yB = (x \cdot y)C \ \forall \ x, y \in G.$$

- If $\alpha = (A, B, B)$, then the triple is called a *left isotopism* and the groupoids(quasigroups, loops) are called *left isotopes.*
- If $\alpha = (A, B, A)$, then the triple is called a *right isotopism* and the groupoids(quasigroups, loops) are called *right isotopes.*
- If $\alpha = (A, A, B)$, then the triple is called a *middle isotopism* and the groupoids are called *middle isotopes.*

If $(G, \cdot) = (H, \circ)$, then the triple $\alpha = (A, B, C)$ of bijections on $(G, \cdot)$ is called an *autotopism* of the groupoid(quasigroup, loop) $(G, \cdot)$. Such triples form a group $AUT(G, \cdot)$ called the *autotopism group* of $(G, \cdot)$. Furthermore, if $A = B = C$, then A is called an *automorphism of the groupoid* (quasigroup, loop) $(G, \cdot)$. Such bijections form a group $AUM(G, \cdot)$ called the *automorphism group* of $(G, \cdot)$.

As it was observed by Osborn [15], a loop is a WIPL and an AIPL if and only if it is a CIPL. The past efforts of Artzy [1, 4, 3, 2], Belousov and Tzurkan [5] and recent studies of Keedwell [11], Keedwell and Shcherbacov [12, 13, 14] are of great significance in the study of WIPLs, AIPLs, CIPQs and CIPLs, their generalizations(i.e m-inverse loops and quasigroups, (r,s,t)-inverse quasigroups) and applications to cryptography.

The universality of WIPLs and CIPLs have been addressed by Osborn [15] and Artzy [2] respectively. Artzy showed that isotopic CIPLs are isomorphic. In 1970, Basarab [7] continued the work of Osborn since 1961 on universal WIPLs by studying isotopes of WIPLs that are also WIPLs after he had studied a class of WIPLs([6]) in 1967. Osborn [15], while investigating the universality of WIPLs, discovered that a universal WIPL $(G, \cdot)$ obeys the identity

$$yx \cdot (zE_y \cdot y) = (y \cdot xz) \cdot y \ \forall \ x, y, z \in G \tag{1}$$

where $E_y = L_y L_{y^\lambda} = R_{y^\rho}^{-1} R_y^{-1} = L_y R_y L_y^{-1} R_y^{-1}$.

Eight years after Osborn's [15] 1960 work on WIPL, in 1968, Huthnance Jr. [9] studied the theory of generalized Moufang loops. He named a loop that obeys (1) *a generalized Moufang loop* and later on in the same thesis, he called them M-loops. On the other hand, he called a universal WIPL an Osborn loop and the same definition was adopted by Chiboka [8].

Moreover, it can be seen that neither WIPLs nor CIPLs have been shown to be isotopic invariant. In fact, it is yet to be shown that there exists a special type of isotopism(e.g left, right or middle isotopism) under which the WIPs or CIPs are isotopic invariant. Aside this, there has never been any investigation into the isotopy of m-inverse quasigroups and loops.

The aim of the present study is to present a special type of middle isotopism under which m-inverse quasigroups are isotopic invariant. Two distinct isotopy-isomorphy conditions for m-inverse loops are established. Only one of them characterizes isotopy-isomorphy in m-inverse loops while the other is just a sufficient condition for isotopy-isomorphy for specially middle isotopic m-inverse quasigroup.

2 Preliminaries

Definition 2.1 *Let L be a quasigroup and $m \in \mathbb{Z}$. A mapping $\alpha \in SYM(L)$ where $SYM(L)$ is the group of all bijections on L which obeys the identity $x^{\rho^m} = [(x\alpha)^{\rho^m}]\alpha$ is called an n-weak right inverse permutation. Their set is represented by $S_{(\rho,m)}(L)$. Here, $x^{\rho^m} = xJ_\rho^m$ and $x^{\lambda^m} = xJ_\lambda^m$.*

Similarly, if α obeys the identity $x^{\lambda^m} = [(x\alpha)^{\lambda^m}]\alpha$ it is called an m-weak left inverse permutation. Their set is denoted by $S_{(\lambda,m)}(L)$

If α satisfies both, it is called a weak inverse permutation. Their set is denoted by $S'_m(L)$.

It can be shown that $\alpha \in SYM(L)$ is an m-weak right inverse if and only if it is an m-weak left inverse permutation. So, $S'_m(L) = S_{(\rho,m)}(L) = S_{(\lambda,m)}(L)$. And thus, α is called an m-weak inverse permutation.

Remark 2.1 *Every permutation of order 2 that preserves the right(left) inverse of each element in an m-inverse quasigroup is an m-weak right(left) inverse permutation.*

Throughout, we shall employ the use of the bijections $J_\rho : x \mapsto x^\rho$, $J_\lambda : x \mapsto x^\lambda$, $L_x : y \mapsto xy$ and $R_x : y \mapsto yx$ for a loop and the bijections $J'_\rho : x \mapsto x^{\rho'}$, $J'_\lambda : x \mapsto x^{\lambda'}$, $L'_x : y \mapsto xy$ and $R'_x : y \mapsto yx$ for its loop isotope. If the identity element of a loop is e, then that of the isotope shall be denoted by e'.

Lemma 2.1 *In a quasigroup, the set of weak inverse permutations that commute forms an abelian group.*

Definition 2.2 *($\mathcal{T}$-condition)*

Let $(G,\cdot)$ and $(H,\circ)$ be two distinct quasigroups that are isotopic under the triple (A,B,C). $(G,\cdot)$ obeys the $\mathcal{T}_{(1,m)}$ condition if $A=B$. $(G,\cdot)$ obeys the $\mathcal{T}_{(2,m)}$ condition if $J'^m_\rho = C^{-1}J^m_\rho A = B^{-1}J^m_\rho C$. $(G,\cdot)$ obeys the $\mathcal{T}_{(3,m)}$ condition if $J'^m_\lambda = C^{-1}J^m_\lambda B = A^{-1}J^m_\lambda C$. So, $(G,\cdot)$ obeys the $\mathcal{T}_m$ condition if it obeys $\mathcal{T}_{(1,m)}$ and $\mathcal{T}_{(2,m)}$ conditions or $\mathcal{T}_{(1,m)}$ and $\mathcal{T}_{(3,m)}$ conditions since $\mathcal{T}_{(2,m)} \equiv \mathcal{T}_{(3,m)}$.

It must here by be noted that the $\mathcal{T}_m$-conditions refer to a pair of isotopic loops at a time. This statement might be omitted at times. That is whenever we say a loop $(G,\cdot)$ has the $\mathcal{T}_m$-condition, then this is relative to some isotope $(H,\circ)$ of $(G,\cdot)$

Lemma 2.2 *Let L be a quasigroup. The following properties are equivalent.*

1. *L is a m-inverse quasigroup.*
2. *$R_x J^m_\lambda L_{xJ^{m+1}_\lambda} = J^m_\lambda \ \forall\ x \in L$.*
3. *$L_x J^m_\rho R_{xJ^{m+1}_\rho} = J^m_\rho \ \forall\ x \in L$.*

3 Main Results

Theorem 3.1 *Let $(G,\cdot)$ and $(H,\circ)$ be two distinct quasigroups that are isotopic under the triple (A,B,C).*

1. *If the pair of $(G,\cdot)$ and $(H,\circ)$ obeys the $\mathcal{T}_m$ condition, then $(G,\cdot)$ is an m-inverse quasigroup if and only if $(H,\circ)$ is an m-inverse quasigroup.*

2. *If $(G,\cdot)$ and $(H,\circ)$ are m-inverse quasigroups, then* $J_\rho^m R_{xJ_\rho^{m+1}} J_\lambda^m B = CJ_\rho'^m R'_{xAJ_\rho'^{m+1}} J_\lambda'^m$ *and* $J_\lambda^m L_{xJ_\lambda^{m+1}} J_\rho^m A = CJ_\lambda'^m L'_{xBJ_\lambda'^{m+1}} J_\rho'^m$, *for all* $x \in G$.

Proof

1. $(A,B,C) : G \to H$ is an isotopism $\Leftrightarrow xA \circ yB = (x\cdot y)C \Leftrightarrow yBL'_{xA} = yL_xC \Leftrightarrow BL'_{xA} = L_xC \Leftrightarrow L'_{xA} = B^{-1}L_xC \Leftrightarrow$
$$L_x = BL'_{xA}C^{-1} \tag{2}$$
Also, $(A,B,C) : G \to H$ is an isotopism $\Leftrightarrow xAR'_{yB} = xR_yC \Leftrightarrow AR'_{yB} = R_yC \Leftrightarrow R'_{yB} = A^{-1}R_yC \Leftrightarrow$
$$R_y = AR'_{yB}C^{-1} \tag{3}$$
Let G be an m-inverse quasigroup. Applying (2) and (3) to Lemma 2.2 separately, we have : $L_xJ_\rho^m R_{xJ_\rho^{m+1}} = J_\rho^m$, $R_xJ_\lambda^m L_{xJ_\lambda^{m+1}} = J_\lambda^m \Rightarrow (AR'_{xB}C^{-1})J_\lambda^m(BL'_{xJ_\lambda^{m+1}A}C^{-1}) = J_\lambda^m$, $(BL'_{xA}C^{-1})J_\rho^m(AR'_{xJ_\rho^{m+1}B}C^{-1}) = J_\rho^m \Leftrightarrow AR'_{xB}(C^{-1}J_\lambda^m B)L'_{xJ_\lambda^{m+1}A}C^{-1} = J_\lambda^m$, $BL'_{xA}(C^{-1}J_\rho^m A)R'_{xJ_\rho^{m+1}B}C^{-1} = J_\rho^m \Leftrightarrow$
$$R'_{xB}(C^{-1}J_\lambda^m B)L'_{xJ_\lambda^{m+1}A} = A^{-1}J_\lambda^m C,\ L'_{xA}(C^{-1}J_\rho^m A)R'_{xJ_\rho^{m+1}B} = B^{-1}J_\rho^m C. \tag{4}$$
Let $J_\lambda'^m = C^{-1}J_\lambda^m B = A^{-1}J_\lambda^m C$, $J_\rho'^m = C^{-1}J_\rho^m A = B^{-1}J_\rho^m C$. Then, $J'_\lambda = C^{-1}J_\lambda B$, $J'_\rho = C^{-1}J_\rho A$. So, $J_\lambda'^{m+1} = (A^{-1}J_\lambda^m C)(C^{-1}J_\lambda B) = A^{-1}J_\lambda^{m+1}B$, $J_\rho'^{m+1} = (B^{-1}J_\rho^m C)(C^{-1}J_\rho^A) = B^{-1}J_\rho^{m+1}A$.
Then, from (4) and using the $\mathcal{T}_m$-condition, we have
$$R'_{xB}J_\lambda'^m L'_{xJ_\lambda^{m+1}A} = J_\lambda'^m = R'_{xB}J_\lambda'^m L'_{xAJ_\lambda'^{m+1}B^{-1}A} = R'_{xA}J_\lambda'^m L'_{xAJ_\lambda'^{m+1}}, \tag{5}$$
$$L'_{xA}J_\rho'^m R'_{xJ_\rho^{m+1}B} = J_\rho'^m = L'_{xA}J_\rho'^m R'_{xBJ_\rho'^{m+1}A^{-1}B} = L'_{xB}J_\rho'^m R'_{xBJ_\rho'^{m+1}} \tag{6}$$
Thus, by Lemma 2.2, (5) and (6), H is an m-inverse quasigroup. This completes the proof of the forward part. To prove the converse, carry out the same procedure, assuming the $\mathcal{T}_m-$condition and the fact that $(H,\circ)$ is an m-inverse quasigroup.

2. If $(H,\circ)$ is an m-inverse quasigroup, then
$$L'_xJ_\rho'^m R'_{xJ_\rho'^{m+1}} = J_\rho'^m \Leftrightarrow R'_xJ_\lambda'^m L'_{xJ_\lambda'^{m+1}} = J_\lambda'^m\ \forall\ x \in H, \tag{7}$$

while since G is an m-inverse quasigroup,

$$L_xJ_\rho^mR_{xJ_\rho^{m+1}}=J_\rho^m\Leftrightarrow R_xJ_\lambda^mL_{xJ_\lambda^{m+1}}=J_\lambda^m\ \forall\ x\in G. \tag{8}$$

From (7), we get

$$R'_x=J'^m_\lambda L'^{-1}_{xJ'^{m+1}_\lambda}J'^m_\rho\Leftrightarrow L'_x=J'^m_\rho R'^{-1}_{xJ'^{m+1}_\rho}J'^m_\lambda\ \forall\ x\in H, \tag{9}$$

while from (8), we obtain

$$R_x=J_\lambda^mL^{-1}_{xJ_\lambda^{m+1}}J_\rho^m\Leftrightarrow L_x=J_\rho^mR^{-1}_{xJ_\rho^{m+1}}J_\lambda^m\ \forall\ x\in G. \tag{10}$$

The fact that G and H are isotopic implies that

$$L_x=BL'_{xA}C^{-1}\ \forall\ x\in G \tag{11}$$

and

$$R_x=AR'_{xB}C^{-1}\ \forall\ x\in G. \tag{12}$$

So, using (9) and (10) in (11), we get

$$J_\rho^mR^{-1}_{xJ_\rho^{m+1}}J_\lambda^m=BJ'^m_\rho R'^{-1}_{xAJ'^{m+1}_\rho}J'^m_\lambda C^{-1}\ \forall\ x\in G, \tag{13}$$

while, using (9) and (10) in (12), we get

$$J_\lambda^mL^{-1}_{xJ_\lambda^{m+1}}J_\rho^m=AJ'^m_\lambda L'^{-1}_{xBJ'^{m+1}_\lambda}J'^m_\rho C^{-1}\ \forall\ x\in G. \tag{14}$$

Thus, (13) becomes

$$J_\rho^mR_{xJ_\rho^{m+1}}J_\lambda^m=CJ'^m_\rho R'_{xAJ'^{m+1}_\rho}J'^m_\lambda B^{-1}\Leftrightarrow$$

$$\Leftrightarrow J_\rho^mR_{xJ_\rho^{m+1}}J_\lambda^mB=CJ'^m_\rho R'_{xAJ'^{m+1}_\rho}J'^m_\lambda\ \forall\ x\in G$$

while (14) becomes

$$J_\lambda^mL_{xJ_\lambda^{m+1}}J_\rho^m=CJ'^m_\lambda L'_{xBJ'^{m+1}_\lambda}J'^m_\rho A^{-1}\Leftrightarrow$$

$$\Leftrightarrow J_\lambda^mL_{xJ_\lambda^{m+1}}J_\rho^mA=CJ'^m_\lambda L'_{xBJ'^{m+1}_\lambda}J'^m_\rho\ \forall\ x\in G.$$

These completes the proof.

Theorem 3.2 *Let $(G,\cdot)$ be an m-inverse loop with identity element e and $(H,\circ)$ be an arbitrary loop isotope of $(G,\cdot)$ with identity element e' under the triple $\alpha=(A,B,C)$. If $(H,\circ)$ is an m-inverse loop then*

1. $(G,\cdot)\overset{C}{\cong}(H,\circ)\Leftrightarrow\left(J_\lambda^m L_{bJ_\lambda^{m+1}}J_\rho^m, J_\rho^m R_{aJ_\rho^{m+1}}J_\lambda^m, I\right)\in AUT(G,\cdot)$ *where* $a=e'A^{-1}, b=e'B^{-1}$.

2. $(G,\cdot)\overset{C}{\cong}(H,\circ)\Leftrightarrow\left(J_\lambda'^m L'_{b'J_\lambda'^{m+1}}J_\rho'^m, J_\rho'^m R'_{a'J_\rho'^{m+1}}J_\lambda'^m, I\right)\in AUT(H,\circ)$ *where* $a'=eA, b'=eB$.

3. $(G,\cdot)\overset{C}{\cong}(H,\circ)\Leftrightarrow(L_{b^{\lambda^{m+1}}}, R_{a^{\rho^{m+1}}}, I)\in AUT(G,\cdot)$, $a=e'A^{-1}, b=e'B^{-1}$ *provided* $(x\cdot y)^{\rho^m}=x^{\rho^m}\cdot y^{\lambda^m}$ *or* $(x\cdot y)^{\lambda^m}=x^{\lambda^m}\cdot y^{\rho^m}\ \forall\ x,y\in G$. *Hence,* $(G,\cdot)$ *and* $(H,\circ)$ *are isomorphic* m*-inverse loops while* $R_{a^{\rho^{m+1}}}L_{b^{\lambda^{m+1}}}=I$, $b^{\lambda^{m+1}}a^{\rho^{m+1}}=e$.

4. $(G,\cdot)\overset{C}{\cong}(H,\circ)\Leftrightarrow(L'_{b'^{\lambda'^{m+1}}}, R'_{a'^{\rho^{m+1}}}, I)\in AUT(H,\circ)$, $a'=eA, b'=eB$ *provided* $(x\circ y)^{\rho'^m}=x^{\rho'^m}\circ y^{\lambda'^m}$ *or* $(x\circ y)^{\lambda'^m}=x^{\lambda'^m}\circ y^{\rho'^m}\ \forall\ x,y\in H$. *Hence,* $(G,\cdot)$ *and* $(H,\circ)$ *are isomorphic* m*-inverse loops while* $R'_{a'^{\rho^{m+1}}}L'_{b'^{\lambda'^{m+1}}}=I$, $b'^{\lambda'^{m+1}}a'^{\rho^{m+1}}=e'$.

Proof
Consider the second part of Theorem 3.1.

1. Let $y=xA$ in (13) and replace y by e'. Then $J_\rho^m R_{e'A^{-1}J_\rho^{m+1}}J_\lambda^m B=CJ_\rho'^m R'_{e'J_\rho'^{m+1}}J_\lambda'^m=C\Rightarrow C=J_\rho^m R_{aJ_\rho^{m+1}}J_\lambda^m B\Rightarrow B=J_\rho^m R^{-1}_{aJ_\rho^{m+1}}J_\lambda^m C$. Let $y=xB$ in (14) and replace y by e'. Then $J_\lambda^m L_{e'B^{-1}J_\lambda^{m+1}}J_\rho^m A=CJ_\lambda'^m L'_{e'J_\lambda'^{m+1}}J_\rho'^m=C\Rightarrow C=J_\lambda^m L_{bJ_\lambda^{m+1}}J_\rho^m A\Rightarrow A=J_\lambda^m L^{-1}_{bJ_\lambda^{m+1}}J_\rho^m C$.
   So, $\alpha=(A,B,C)=(J_\lambda^m L^{-1}_{bJ_\lambda^{m+1}}J_\rho^m C, J_\rho^m R^{-1}_{aJ_\rho^{m+1}}J_\lambda^m C, C)=$
   $=(J_\lambda^m L^{-1}_{bJ_\lambda^{m+1}}J_\rho^m, J_\rho^m R^{-1}_{aJ_\rho^{m+1}}J_\lambda^m, I)(C,C,C)$. Thus,
   $(J_\lambda^m L_{bJ_\lambda^{m+1}}J_\rho^m, J_\rho^m R_{aJ_\rho^{m+1}}J_\lambda^m, I)\in AUT(G,\cdot)\Leftrightarrow(G,\cdot)\overset{C}{\cong}(H,\circ)$.

2. This is similar to the above proof for (1) but we only need to replace x by e in (13) and (14).

3. This is achieved by simply breaking the autotopism in (1.).

4. Do what was done in (3.) to (2.).

Corollary 3.1 *Let* $(G,\cdot)$ *and* $(H,\circ)$ *be two distinct quasigroups that are isotopic under the triple* (A,B,C). *If* G *is an* m*-inverse quasigroup with the* $\mathcal{T}_m-$*condition, then* H *is an* m*-inverse quasigroup and so:*

1. *there exist* $\alpha,\beta\in S_m'(G)$ *i.e* α *and* β *are* m*-weak inverse permutations and*

2. $J'_\rho = J'_\lambda \Leftrightarrow J_\rho = J_\lambda$.

Proof
By Theorem 3.1, $A = B$ and $J'^m_\rho = C^{-1}J^m_\rho A = B^{-1}J^m_\rho C$ or $J'^m_\lambda = C^{-1}J^m_\lambda B = A^{-1}J^m_\lambda C$.

1. $C^{-1}J^m_\rho A = B^{-1}J^m_\rho C \Leftrightarrow J^m_\rho A = CB^{-1}J^m_\rho C \Leftrightarrow J^m_\rho = CB^{-1}J^m_\rho CA^{-1} = CA^{-1}J^m_\rho CA^{-1} = \alpha J^m_\rho \alpha$ where $\alpha = CA^{-1}$. This implies that $\alpha = CA^{-1} \in S'_m(G, \cdot)$.

2. $C^{-1}J^m_\lambda B = A^{-1}J^m_\lambda C \Leftrightarrow J^m_\lambda B = CA^{-1}J^m_\lambda C \Leftrightarrow J^m_\lambda = CA^{-1}J^m_\lambda CB^{-1} = CB^{-1}J^m_\lambda CB^{-1} = \beta J^m_\lambda \beta$ where $\beta = CB^{-1}$. This implies that $\alpha = \beta = CB^{-1} \in S'_m(G, \cdot)$.

3. $J'^m_\rho = C^{-1}J^m_\rho A = B^{-1}J^m_\rho$, $J'^m_\lambda = C^{-1}J^m_\lambda B$. $J'_\rho = J'_\lambda \Leftrightarrow J'^m_\rho = J'^m_\lambda \Leftrightarrow C^{-1}J^m_\rho A = C^{-1}J^m_\lambda B = C^{-1}J^m_\lambda A \Leftrightarrow J^m_\lambda = J^m_\rho \Leftrightarrow J_\lambda = J_\rho$.

Lemma 3.1 *Let $(G, \cdot)$ be an m-inverse quasigroup with the $\mathcal{T}_m-$condition and isotopic to another quasigroup $(H, \circ)$. $(H, \circ)$ is an m-inverse quasigroup and G has a weak inverse permutation.*

Proof
From the proof of Corollary 3.1, $\alpha = \beta$, hence the conclusion.

Theorem 3.3 *If two distinct quasigroups are isotopic under the $\mathcal{T}-$condition and any one of them is an m-inverse quasigroup and has a trivial set of m-weak inverse permutations, then the two quasigroups are both m-inverse quasigroups that are isomorphic.*

Proof
From Lemma 3.1, $\alpha = I$ is a weak inverse permutation. In the proof of Corollary 3.1, $\alpha = CA^{-1} = I \Rightarrow A = C$. Already, $A = B$, hence $(G, \cdot) \cong (H, \circ)$.

Remark 3.1 *Theorem 3.2 and Theorem 3.3 describes isotopic m-inverse quasigroups and m-inverse loops that are isomorphic by*

1. *an autotopism in either the domain loop or the co-domain loop and*

2. *the $\mathcal{T}_m-$condition(for a special case).*

4 Conclusion and Future Study

Keedwell and Shcherbacov [13, 14] have also generalized m-inverse quasigroups to quasigroups called (r, s, t)-inverse quasigroups. It will be interesting to study the universality of m-inverse loops and (r, s, t)-inverse quasigroups in general sense. These will generalize the works of J. M. Osborn and R. Artzy on universal WIPLs and CIPLs respectively.

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