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A note on the total projectivity of normed units in modular group rings with nilpotents

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Abstract

We find a new concrete example of an imperfect and uncountable commutative unital ring R of prime characteristic p which contains nilpotent elements so that the group $V(RG)$ of all normalized invertible elements in the group ring RG cannot be inseparable totally projective. This continues our recent investigations by us in (Vietnam J. Math., 2006) and (Kyungpook Math. J., 2006) as well as by May in (Forum Math., 2006).

In the present short paper, RG will always denote the group ring of a p-primary multiplicatively written abelian group over a commutative ring with identity (often called as *unital* or *unitary*) of prime characteristic p. As in the existing literature on the subject, we shall let $V(RG)$ to denote the group of normed units (often termed by *normalized units*) in RG.

The problem for *total projectivity* of $V(RG)$ over perfect R has been started by W. May (see, for more details, the bibliography in [3]). However, we first had seen that the ring perfection is not necessarily for $V(RG)$ to be totally projective (see, e.g., [1] and [2]); it is worthwhile noticing that later on May has also observed in [3] that the property of R being perfect is not necessary. In fact, we have constructed there a ring R which is not weakly perfect and countable and which contains nilpotents such that $V(RG)$ is a direct sum of countable groups if and only if G is. Because of its importance, we recall once again the definition of such a ring, named by us as *highly-generated* (see cf. [2]).

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Definition. The ring R is said to be highly-generated if

$$R = \cup_{n<\omega} R_n, R_n \subseteq R_{n+1} \leq R, \forall n < \omega : R_n \cap R^{p^n} \subseteq R^{p^\omega}.$$

Note that in this definition all members of the union are subrings of R. In our main theorem from [2], the ring R is highly-generated so that $R^{p^\omega} = R^{p^{\omega+1}}$. It is a straightforward argument that all weakly perfect rings R (i.e. for which $R^{p^n} = R^{p^{n+1}}$, that is R^{p^n} is perfect, for some natural number n) are highly-generated. If we take each R_n to be only with an additive structure or, in other words, to be an additive subgroup only, it is routine to see that every countable ring is highly-generated (for some more details, compare with [2]).

On the other hand, in his work [3], May independently defined the symbol $\lambda(R)$ which characterizes the nice composition series between the p-powers of the ring R regarded as a valuated additive abelian group (for a more complete information, we refer to the original source [3]). He also proved the following technicality.

Lemma ([2]). *For a ring R as above, $\lambda(R) = \infty \iff \exists$ a chain of additive subgroups $R^{p^\omega} = B_0 \subseteq B_1 \subseteq \cdots \subseteq B_i \subseteq \cdots \subseteq R$ such that $\cup_{i<\omega} B_i = R$, and $\forall i \geq 1$ the heights of elements of $B_i \setminus R^{p^\omega}$ are bounded.*

Thereby, we observe that $\lambda(R) = \infty$ for any highly-generated ring and thus, according to [3], one may generalize the already discussed theorem from [2] to totally projective groups without the limitation on perfection of R^{p^ω}.

As remarked in [2], each highly-generated ring R with $R^{p^\omega} = R^{p^{\omega+1}}$ (i.e. the ring in our chief theorem of [2]) can be represented as $R = R^{p^\omega} + N(R)$, where $N(R)$ is the nil-radical of R. The purpose of this brief article is to show that the proper total projectivity of $V(RG)$ is no longer valid for such rings R of latter type, thus showing that the main result from [2] cannot be extended for these more general rings.

In order to illustrate this, we claim that there is a ring R such that R^{p^ω} is perfect, $R = R^{p^\omega} + N(R)$, and $\lambda(R) = \omega$. Indeed, we give the following

EXAMPLE: Let F be a perfect field of characteristic p, let $F[[X]]$ be the formal power series ring, and let G be a countably infinite divisible p-primary abelian group. Let $I = I(F[[X]]G; G)$ be the augmentation ideal in the group algebra $F[[X]]G$. Put $R = FG + I$. Then R is a subring of the group ring $F[[X]]G$ and it is clear that $R^{p^n} = FG + I(F[[X^{p^n}]]G; G)$ since G is divisible and F is perfect. In particular, we see that $R^{p^\omega} = FG$, which is obviously perfect. Since I consists of nilpotent elements, we also have that $R = R^{p^\omega} + N(R)$.

Now, let F_p be the prime subfield of F and define the additive subgroup $L = \{\sum_{i\geq 0} a_i X^{p^i} | a_i \in F_p\}$, and let $L_k = \{\sum_{i\geq k} a_i X^{p^i} | a_i \in F_p\}$. Put M to be the set of all elements from I which have all their coefficients in L. Consequently, M is an uncountable additive subgroup of R. To see that $\lambda(R) = \omega$, if not, then there is a countable chain as indicated above in the Lemma from R^{p^ω} to R. Intersect this chain with M to get a chain $\{C_i\}_{i\geq 0}$ in M. Note that $M \subseteq \oplus_{g\in G} Lg$. By the height conditions on the chain, for a given i, there is an index k such that $C_i \cap \oplus_{g\in G} L_k g = 0$. Hence C_i is isomorphic to a subgroup of $\oplus_{g\in G} Lg / \oplus_{g\in G} L_k g$, which is countable. This gives the desired contradiction that M is countable.

Therefore, appealing to [3, Theorem 2], $V(RG)$ is totally projective over that ring R precisely when it is separable, i.e. a direct sum of cyclic groups, which is equivalent to G is a direct sum of cyclic groups.

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