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A note on decompositions in abelian group rings

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Abstract

We find a necessary and sufficient condition for a normal decomposition of the group of normed units in a commutative group ring (of prime characteristic) into certain its subgroups. This extends a recent assertion of ours in (Vladikavkaz Math. J., 2007). We also give some new proofs of own recent results published in (Miskolc Math. Notes, 2005).

I. Introduction

Throughout the rest of the present paper, let G be an abelian group with subgroups A and B, possibly proper, and let R be a commutative unitary ring. As usual, the letter $V(RG)$ denotes the normalized unit group in the group ring RG and $S(RG)$ is its Sylow p-subgroup, for some arbitrary but a fixed prime p. For $B \leq G$, the symbol $I(RG;B)$ designates the relative augmentation ideal of RG with respect to B, and $I_p(RG;B)$ is its nil-radical. It is apparent that $I(RG;B) = I_p(RG;B)$ whenever B is a p-group and $char(R) = p$.

A theme that arises naturally is for the decomposition of $V(RG)$ and, in particular, of $S(RG)$ (e.g. [1]-[5]). It was intensively studied in a subsequent series of articles [6] and [8] as well as in the current one. The aim of such studies is of finding a connection between appropriate decompositions of $V(RG)$, respectively $S(RG)$, and G. When these decompositions are direct, they are rather useful for the investigation of direct sums of subgroups with a special structure (for instance, subgroups with cardinality not exceeding $\aleph_1$ - see [2] and [3]).

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The purpose of this exploration is to systematize the attainments from [6] and [8] such that, as a new moment, an explicit criterion for a certain decomposition of $V(RG)$ is given.

All unexplained notions and notations will be in agreement with those from [9].

II. Main affirmations

In this section, we state our main results and their corollaries. We start with the following decomposition property of $V(RG)$.

Theorem.

(1) $V(RG) = V(RA)[(1 + I(RG; B)) \cap V(RG)] \iff G = AB$

(2) $V(RG) = V(RA) \times [(1 + I(RG; B)) \cap V(RG)] \iff G = A \times B$.

Proof. We foremost concentrate on the first relationship. First of all, we concern the necessity. For this goal, we know that the natural map $\phi : G \to G/B$ can be linearly extended to a group homomorphism $\Phi : V(RG) \to V(R(G/B))$ with kernel $(1 + I(RG; B)) \cap V(RG)$ and its restriction $\Phi_{V(RA)} : V(RA) \to V(R(AB/B))$. Next, given $g \in G \subseteq V(RG)$ whence $g \in V(RA)(1+I(RG; B))$. Thus, by acting both sides with Φ, we deduce that $gB \in V(R(AB/B))$. Hence $gB \in (G/B) \cap V(R(AB/B)) = AB/B$, which trivially forces that $g \in AB$, as required.

After this, we deal with the sufficiency. Because $G = AB$, we infer that $\phi(G) = \phi(A)$. Let now $v \in V(RG)$ with $v = \sum_{g \in G} r_g g$, where $r_g \in R$. Consequently, via the action of Φ, we have that $\Phi(v) = \Phi(\sum_{g \in G} r_g g) = \sum_{g \in G} r_g \Phi(g) = \sum_{g \in G} r_g \phi(g) = \sum_{a \in A} \alpha_a \phi(a) = \sum_{a \in A} \alpha_a \Phi(a) =$
$= \Phi(\sum_{a \in A} \alpha_a a) = \Phi(u)$, where we put $u = \sum_{a \in A} \alpha_a a \in RA$. Since v is invertible in RG, it is therefore a straightforward argument to see that u is also invertible in RA. Thus $\Phi(v) = \Phi(u)$, where $u \in V(RA)$. Finally, $vu^{-1} \in ker\Phi = (1 + I(RG; B)) \cap V(RG)$, and thereby we are done.

As for the second dependence, we routine observe with the aid of Intersection Lemma proved in [3] that $(1+I(RG; B)) \cap V(RA) \subseteq 1+I(RA; A \cap B) = 1$ provided that $A \cap B = 1$. So, the previous point works and this concludes the proof.

Remark. The relation (2) actually generalizes the Claim in [8] by adding the reverse implication "$\Leftarrow$".

As immediate consequences, we yield

Corollary ([1],[2],[3],[4],[5]). *Let* $B \leq G_p$ *and* $char(R) = p$*, a prime integer. Then*

(3) $V(RG) = V(RA)(1 + I(RG; B)) \iff G = AB$

(4) $V(RG) = V(RA) \times (1 + I(RG; B)) \iff G = A \times B$.

Proof. Since B is p-primary, it is elementary to see that $1+I(RG;B)$ is a nil-ideal, whence $1+I(RG;B) \subseteq S(RG)$. Furthermore, the preceding theorem is applicable, and we are finished.

We pose two questions of interest.

Problem 1. Suppose that $char(R) = p$. Then find a suitable criterion (in terms of a decomposition for G if possible) when $V(RG) = V(RA)(1 + I_p(RG;B))$ and, in particular, when $V(RG) = V(RA) \times (1 + I_p(RG;B))$.

Owing to the Corollary, alluded to above, we should consider only the situation $B \neq B_p$.

Problem 2. Suppose that $char(R) = p$. Then find a suitable criterion (in terms of a decomposition for G if possible) when $V(RG) = V(RA)[B(1 + I_p(RG;B))]$ and, in particular, when $V(RG) = V(RA) \times [B(1 + I_p(RG;B))]$.

It is worthwhile noting for the latter formula of the last problem that, when G is p-mixed that is the only torsion is p-torsion, and R is with no idempotents (in particular with no zero divisors), such a necessary and sufficient condition, namely $G = A \times B$, was demonstrated in [8]. As aforementioned, only the case $B \neq B_p$ must be examined.

Finally, we ensure a new confirmation of own statements from [6]. Specifically, we proceed by proving the following

Proposition ([6]). *Let $G = AB$ where $A \leq G$ and $B \leq G$ and let R be of prime $char(R) = p$. Then*

$$S(RG) = S(RA)(1 + I_p(RG;B)) \iff G_p = A_pB_p.$$

Proof. For the first implication, the natural map $G \to G/B$ induces a homomorphism $\pi : S(RG) \to S(R(G/B))$. The given formula for $S(RG)$ simply says the image of $S(RG)$ is the same as the image of $S(RA)$. If we assume the formula and $g_p \in G_p$, then $\pi(g_p) = \pi(s), s \in S(RA)$. Thus there exists $a_p \in A_p$ such that $\pi(g_p) = \pi(a_p)$, hence $g_pa_p^{-1} \in B_p = G_p \cap (1 + I(RG;B))$. That is why, $G_p = A_pB_p$, as desired.

Conversely, for the other implication, if $G_p = A_pB_p$ and $x \in S(RG)$, then writing x out and replacing under the action of π every p-torsion element of form $t = t_at_b \in A_pB_p$ by t_a, and every $g = g_ag_b \in AB$ by g_b, we obtain the existence of $y \in S(RA)$ such that $\pi(y) = \pi(x)$, whence $xy^{-1} \in 1 + I_p(RG;B)$. Thus, $S(RG) = S(RA)(1 + I_p(RG;B))$, as wanted.

Remark. Note that a more general version of the last equivalence was established in [8].

We also provide a new argumentation of the niceness proposition in [6] for a ring without nilpotent elements.

Proposition ([6]). *Suppose N is a p-balanced, that is a p-nice and p-isotype, subgroup of G. Then $1 + I_p(RG;N)$ is nice in $S(RG)$ provided R is perfect with no nilpotents of prime $char(R) = p$.*

Proof. Put $H = G/N$. Then the canonical map $G \to H$ induces a homomorphism $\pi : S(RG) \to S(RH)$ and, to finish the proof, it suffices to show that every element of $S(RH)$ has a pre-image in $S(RG)$ of the same p-height. But elements in $S(RH)$ are linear combinations of elements of form h_p and $h - hh_p$, where $h_p \in H_p$ and $h \in H$. But the assumptions on N of p-niceness and p-isotypeness guarantee that elements of H have pre-images in G of the same p-height, and that the pre-images of p-torsion elements may be taken to be p-torsion as well. Thus, in conclusion, every element of $S(RH)$ has a pre-image in $S(RG)$ of the same p-height, as needed.

Remark. Notice that the same technique can be employed to prove niceness of some other groups of type $S(RA)$, which were considered and attacked via a different approach in [7].

References

[1] P. V. Danchev, *Unit groups of abelian group rings with prime characteristic*, Compt. Rend. Acad. Bulg. Sci., **48** (8) (1995), 5-8.

[2] P. V. Danchev, *Isomorphism of modular group algebras of direct sums of torsion-complete abelian p-groups*, Rend. Sem. Mat. Univ. Padova, **101** (1999), 51-58.

[3] P. V. Danchev, *Modular group algebras of coproducts of countable abelian groups*, Hokkaido Math. J., **29** (2) (2000), 255-262.

[4] P. V. Danchev, *The splitting problem and the direct factor problem in modular abelian group algebras*, Math. Balkanica, **14** (3-4) (2000), 217-226.

[5] P. V. Danchev, *Normed units in abelian group rings,* Glasgow Math. J., **43** (3) (2001), 365-373.

[6] P. V. Danchev, *On a decomposition formula in commutative group rings,* Miskolc Math. Notes, **6** (2) (2005), 153-159.

[7] P. V. Danchev, *On the balanced subgroups of modular group rings,* Vladikavkaz Math. J., **8** (2) (2006), 29-32.

[8] P. V. Danchev, *On a decomposition equality in modular group rings,* Vladikavkaz Math. J., **9** (2) (2007), 3-8.

[9] L. Fuchs, *Infinite Abelian Groups*, volumes I and II, Mir, Moskva, 1974 and 1977 (in Russian).

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