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On a remark of Loday about the Associahedron and Algebraic K-Theory

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Abstract

In his 2006 Cyclic Homology Course from Poland, J.L. Loday stated that the edges of the associahedron of any dimension can be labelled by elements of the Steinberg Group such that any 2-dimensional face represents a relation in the Steinberg Group. We prove his statement. We define a new group $R(n)$ relevant in the study of the rotation distance between rooted planar binary trees .

1 Introduction

1.1 Combinatorics: binary rooted trees and the associahedron

Our primary objects of study are planar rooted binary trees. By definition, they are connected graphs without cycles, with n trivalent vertices and n+2 univalent vertices, one of them being marked as the root.
There are $\frac{1}{n+1}\binom{2n}{n}$ binary trees with n internal vertices.

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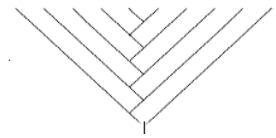

Figure 1.

The associahedron K_n is an $(n-1)$-dimensional convex polytope whose vertices are labelled by rooted planar binary trees with n internal vertices. Two labelled vertices are connected by an edge if and only if the corresponding trees are connected by an *elementary move* called rotation.

Two trees are connected by a rotation if they are identical, except the zones from the figure below, where one edge is moved into another position; we erase an internal edge and one internal vertex and we glue it again in a different location, to get a rooted binary tree.

Figure 2.

The 1-skeleton of the associahedron is called the Rotation Graph G_n [7]. The vertices of this graph are all rooted binary trees with n internal vertices, two trees being adjacent if they are united by a rotation. G_n has the structure of a lattice, called the Tamari lattice: there is an oriented edge from the tree a to the tree b if b is obtained from a using a right to the left rotation.

Devadoss [2], Stanley [9] and Loday [6] gave a geometric realization of this polytope. Its faces are products of lower dimensional associahedra.

There is a bijection between rooted binary trees and paranthetizations of n+1 variables. Various ways to paranthetize tensor products of $(n+1)$ objects in a monoidal category are controlled by so called associativity constrains, whose the source and the target are encoded by two trees connected by a rotation: $....((AB)C)... \to ....(A(BC))....$

1.2 Algebraic K-Theory

Let A be a unital ring which is not necessarly commutative. The Steinberg group $St_n(A)$ is defined as the group with the following generators and relations:
generators: $e_{ij}(a)$ for any a $\in$ A and 1 $\leq$ i $\neq$ j $\leq$ n.
relations:
(I) $e_{ij}(a)e_{ij}(b)$=$e_{ij}(a+b)$
(II) $e_{ij}(a)e_{kl}(b)$=$e_{kl}(b)e_{ij}(a)$ for any set of indices {i,j,k,l} of cardinality 4
(III) $e_{ij}(a)e_{jk}(b)$=$e_{jk}(b)e_{ik}(ab)e_{ij}(a)$ for any set of indices {i,j,k} of cardinality 3.

These are exactly the relations satisfied by the elementary matrices $m_{ij}(a)$ in $Gl_n(A)$, where $m_{ij}(a)$ has 1's on the diagonal and the element from the ring a on the ij entry.

The second K-Theory group of A, $K_2(A)$ is defined to be the kernel of the epimorphism $St_n(A) \to Gl_n(A)$, which send $e_{ij}(a)$ to $m_{ij}(a)$. The abelian group $K_2(A)$ is also isomorphic with H_2(St(A)) (Theorem Whitehead-Kervaire).

This is the appeareance of the Steinberg group in the realm of Algebraic K-Theory [8]. We will prove the statement of Loday: (page 42- Cyclic Homology Theory-Poland 2006) [4]:

Theorem 1 *The edges of the associahedron K_n can be labelled by elements of the Steinberg Group, such that any cycle of length 4 or 5 in its 1-skeleton (The Rotation Graph) represents a relation in $St_n(A)$.*

2 Algebraic realization of the associahedron

To every rooted binary tree X, we associate a permutation. During our paper, we denote this permutation as $p(X)$. Based on this function, we build elements indexed by trees, and by pair of trees in different groups. In the same way there are different geometric realizations of the associahedron, the convex polytope whose 1-skeleton is the rotation graph [6] [9] [2] [3], we can say we realize the associahedron on different algebraic structures.

Let T be a rooted binary tree, with n internal vertices. The univalent vertices are labelled $1, 2, ...n+1$.
The internal zone between the univalent vertices k and k+1 is called zone k. Every internal zone has a unique internal vertex which is the head of that zone.

To every tree T we associate a permutation, denoted $p(T)$ using the following recursive procedure:
If t_1 and t_2 are the left and the right trees of T, then the permutation associated with T is p(T)=mp(t_1)p(t_2), where the zone between the left tree t_1 and the right tree t_2 is located between the univalent vertices m and m+1. This rule is sufficent to associate a permutation to every tree, by induction over n using its subtrees, but we would like to be more specific:

There is a unique directed path $P(k)$ which joins the root of the tree to every univalent vertex k. We label the internal vertices with numbers $1, 2...n$ in the order they appear on the paths $P(1)$, $P(2)...P(n+1)$, from the root to the univalent vertices. So we define a total order on the set of internal vertices: x $\leq$ y if x $\in P(k)$ and y $\in P(m)$ and k$\leq$ m or, k=m and x is on the path from y to the root of the tree.
In the permutation associated with the tree, p(T)(x)=y, if the internal vertex labelled with x is the vertex associated to the zone y, between the univalent vertices y and y+1. We call x the head, or the lowest point of the zone y.

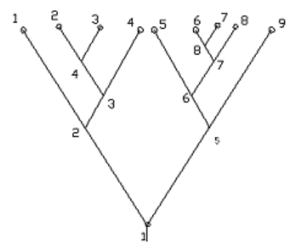

Figure 3.

An insertion in a permutation p:= p(1)p(2)....p(n) is the following transformation applied to p: insert the element p(y) between two consecutive elements in p: p(x-1) and p(x). Under this transformation, we get the permutation pr(x,y), where r(x,y) is the insertion:

$$r(x, y) = \begin{cases} 1, 2, ...x, x+1, ...y....n \\ 1, 2, ...y, x, ...y-1...n \end{cases}$$

Note: In the Symmetric Group $S(n)$, we use the group-like notation fg to denote the permutation g$\circ$f.

Lemma 1 *At a rotation , the permutation p(T) evolves by an insertion.*

Proof. After a rotation ,locally, the permutation of a tree p(T):... yxp(T_1)p(T_2)p(T_3)... is changed into ...xp(T_1)yp(T_2)p(T_3).

The insertion is from right to the left if the rotation is from right to the left.

Some of the partial trees T_i from the figure below can be empty.

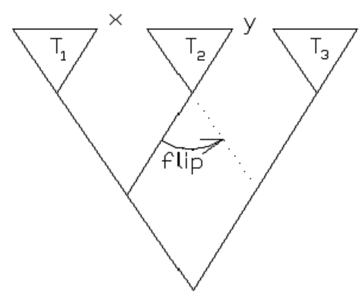

Figure 4 .

The rotation graph is the skeleton of a lattice, called Tamari lattice [6]. The minimal element is the right comb tree S whose permutation is the identity permutation. $T_1 < T_2$ if we use a "right to the left "rotation to obtain T_2 from T_1 .

The fact the Rotation Graph is a lattice allow us to record the direction of the rotation. If $a < b$ (b is obtained from a using a right to the left elementary move) and ab is an edge in the Rotation Graph (the 1-skeleton of the associahedron), then $p(b) = p(a)r(x, y)$.

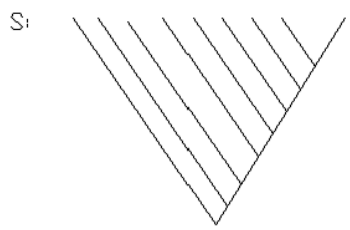

Figure 5.

We label the vertices T of the Rotation Graph by these permutations $p(T)$ and the edges by the left insertions r(x,y).

Remark 1 *Given two trees T_1 and T_2, and a sequence of edges between them - a directed path in the Rotation Graph, then $p(T_2) = p(T_1)\prod r(x,y)^{\alpha}$, the product of the insertions which label the edges. We used the convention that if the rotation between two consecutive trees of the path is from right to left , then we use $r(x,y)$. Otherwise, we use the inverse of $r(x,y)$in the product above, so the exponent α will be -1.*

2.1 The proof of the Theorem 1

Let a_{12} a_{23}... a_{n-1n} be arbitrary elements from $St_n(A)$. Let c_{ij}= $\prod_{k=i}^{j-1} a_{kk+1}$

Remark 2 *[5] The permutation group S_n acts on $St_n(A)$ in the following way:*
Let p be a permutation. $p.e_{uv}(a)$=$e_{ij}(a)$ if p(i)=u and p(j)=v.

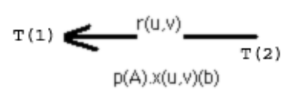

Figure 6.

Let T_1 and T_2 be two trees with n internal vertices, united by a rotation, such that p(T_2)=p(T_1)r(x,y).
They are the vertices of an edge in the Rotation Graph (or equivalently in the associahedron). We label this edge with b(i,j):= p(T_1).$e_{xy}(c_{ij})$, where p(T_1)(i)=x and p(T_1)(j)=y. Let us call this assignement **rule (IV)**.
So, the indices i and j are determined by the permutation of the tree T_1 and by the indices of the insertion $r(x,y)$.
We will prove that this labelling of the 1-skeleton of the associahedron with elements from $St_n(A)$ is coherent: every directed cycle of length 4 or 5 is labelled by a relation from $St_n(A)$. The product of the elements which label the edges of any cycle is identity, where we will use exponent -1 if the direction of the edge is opposite to the direction of the cycle.

A cycle of length 5 from the 1-skeleton of the associahedron is described by the following 5 trees united by rotations from the figure below.

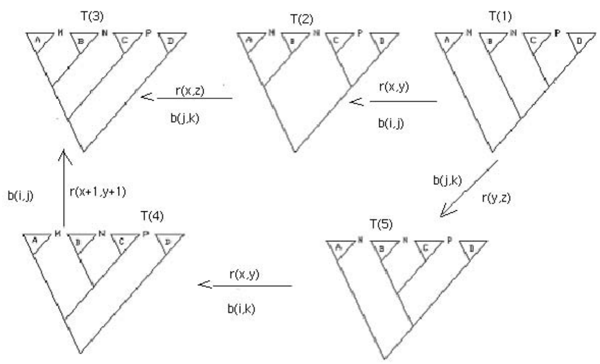

Figure 7.

Locally, the permutations of these trees are:
p=p(T(1))=...M p(A) N p(B) P p(C) p(D)... where the zones M, N and P are on the positions x,y and z from the permutation, which means: p(x)=M, p(y)=N p(z)=P; zone k is between the univalent vertices k and k+1.

p(T(2))=...NM p(A) p(B) P p(C) p(D)...
p(T(3))=...PNM p(A) p(B) p(C) p(D)...
p(T(4))=...PM p(A) N p(B) p(C) p(D)...
p(T(5))=...M p(A) PN p(B) p(C) p(D)...

The insertions $r(u, v)$ which connect these permutations are written in the figure above.
In the symmetric group, this 5-cycle represents the relation : $r(x, y)r(x, z) = r(y, z)r(x, y)r(x + 1, y + 1)$.

Let us find the elements of the Steinberg Group which label the 5 edges, according to the rule (IV) from page 5.
Assume p(i)=x; p(j)=y p(k)=z where p=p(T(1)). The element b(i,j) of the edge T(1)T(2) is given by the action of p on $e_{xy}(c_{ij})$ (Remark 2, page 6).
p(i)=x and p(j)=y

The element of the edge T(2)T(3) is given by the action of p(T(2))=pr(x,y) on $e_{xz}(c_{vw})$.
p(j)=y and r(x,y)(y)=x , so v=j
p(k)=z and r(x,y)(z)=z , so w=k

The element of the edge T(1)T(5) is given by the action of p(T(1))=p on $e_{yz}(c_{vw})$.
p(j)=y, so v=j; p(k)=z so w=k

The element of the edge T(5)T(4)is given by the action of p(T(5))=pr(y,z) on $e_{xy}(c_{vw})$.
p(i)=x r(y,z)(x)=x, so v=i
p(k)=z r(y,z)(z)=y so w=k

The element of the edge T(4)T(3)is given by the action of pr(y,z)r(x,y) on $e_{x+1y+1}(c_{vw})$
p(i)=x r(y,z)(x)=x r(x,z)(x)=x+1, so v=i
p(j)=y r(y,z)(y)=y+1 r(x,y)(y+1)=y+1 so w=j

So, the edges of the 5-cycle are decorated in a coherent way with the relation III from the definition of the Steinberg Group, where the ring coefficients are a:=$c(i,j)$ b:=$c(j,k)$ ab=$c(i,j)c(j,k) = c(i,k)$

There are two types of 4-cycles ,described in the on the next page:

Locally, the permutations associated with the trees from the first figure below are:
p(T(1))=...mnq(A)p(B)(C)...
p(T(2))=...mnpq(A)(B)(C)...
p(T(3))=...npq(A)(B)(C)m...
p(T(4))=...nq(A)p(B)(C)m...
(X) denote the permutation associated with the tree X.

The following relation among insertions in the Symmetric Group label this cycle :
r(x,y)r(x-1,z)=r(x-1,z)r(x+1,y+1)

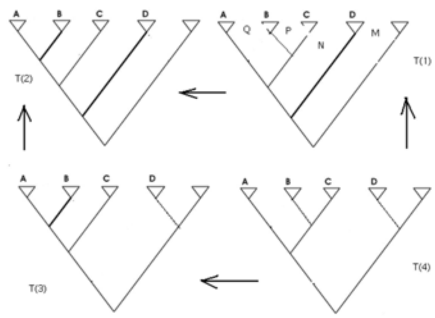

Figure 8.

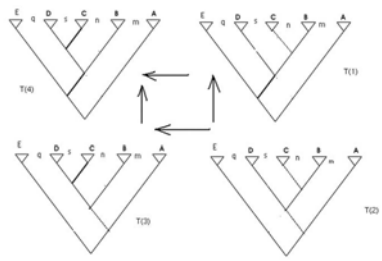

Figure 9.

Locally, the permutations associated with the trees above, from the second 4-cycle are:
p(T(1))=...mq(E)s(D)n(C)(B)...
p(T(2))=...q(E)ms(D)n(C)(B)...
p(T(3))=...q(E)mns(D)(C)(B)...
p(T(4))=...mq(E)ns(D)(C)(B)...

The following relation among insertions in the symmetric group label the second 4-cycle from the last page:
r(x,y)r(y+1,z)=r(y+1,z)r(x,y)

We will compute the Steinberg Group elements which label the edges from the last 4-cycle, for the first one being the same type of commutativity relation (II).

Let p be $p(T(2))$. The indices i, j, k and l satisfy: p(i)=x; p(j)=y; p(k)=y+1; p(l)=z.
So, the edge $T(2)T(1)$ is labelled by $p.e_{xy}(c_{vw})$ (the action of the permutation p on the element e_{xy} from the Steinberg Group. p(v)=x and p(w)=y, so v=i and w=j.

The edge T(1)T(4) is labelled by $(pr(x,y)).e_{y+1z}(c_{vw})$. v=k and w=l, because they are the inverse images of the indices of e_{y+1z} under the permutation which acts on the Steinberg generator.

In the same way, the other two edges are labelled by the same generators. This 4-cycle encodes the defining relation II from $St_n(A)$.

2.2 Generalizations. Further directions

We look at the insertions which label the cycles of length 4 and 5 in the associahedron; we generalize the construction above, replacing the Symmetric Group by other groups.

Let $B(n)$ be the Artin's Braid group.
Let $R(n)$ be the following group, given by generators and relations:
generators: R(x,y), where 1≤x<y≤n
relations: R(x,y)R(x,z)=R(y,z)R(x,y)R(x+1,y+1) if x<y<z
R(x,y)R(z,t)=R(z,t)R(x,y) if x<y<z<t and
R(a,y)R(x,z)=R(x,z)R(a+1,y+1), where x<a<y<z

Because of the 5-term relation above, the group is in fact generated by R(x,x+1), 0<x<n

Instead of the labelling of the edges of the associahedron by right insertions, we label the edges of the Rotation Graph by the elements R(x,y) from $R(n)$. The lattice structure of the Rotation Graph will tell us if, given an edge ab labelled by R(x,y) we have the relation $p(a) = p(a)$R(x,y) or $p(b) = p(a)$R(x,y).

Let t be a tree. Let $d(t)$ be a particular path which joins t to S- the right comb tree (page 4), using rotations. Let g_t be the product of the labels of the directed edges which form $d(t)$. The result will be an element from the group $R(n)$ (instead of a permutation from the Symmetric group).
We associate to every pair of trees (t_1, t_2) the element $g_1^{-1} \circ g_2$.

Lemma *For any path between t_1 and t_2, the product of the elements $R(x,y)^{\alpha}$ associated to the edges of the path is $g_1^{-1} \circ g_2$. The exponent α is 1 or –1.* This lemma states that the product above does not depend on the path, it depends only on the extremities of the path.

Proof. Any pentagon and 4-gon from the associahedron represents a relation in R(n) i.e. the edges are labelled by elements of $R(n)$ (instead of the insertions from the symmetric group), whose product is a defining relation in R(n).

Two representations of the same element g $\in R(n)$ by generators $R(u,v)$ are connected by a finite sequence of steps where we apply the relations in the group $R(n)$. Also, any two paths from t_1 and t_2 are connected by a sequence of paths, where every consecutive two paths are connected by a pentagon or a 4-gon. (the two paths form the boundary of a disk paved by pentagons and 4-gons). So, we gradually change the first path by glueing pentagons or 4-gons, to finally obtain the second path. This is a consequence of the MacLane Coherence Theorem or the simply-connectedness of the associahedron.

So, for any two paths $x(1)$ and $x(2)$ with the same ends, the products of their labels are equal to the same element from $R(n)$ because gradually we change the representation of the product from $x(1)$, using the group relations which encircle the pentagons or 4-gons from the paved disk. In particular, the product of the elements from any path which joins t_1 and t_2 is equal to the product of the elements from two paths, the first one from t_1 to S and the second one from S to t_1. By definition, this product is equal to $g_1^{-1} \circ g_2$.
S is the right comb tree (page 4).

Lemma 2 *We have a morphism b from R(n) to B(n) which associate to R(x,y) the insertion braid $b(x,y)$.*

The insertion braid $b(x,y)$ is defined as $b(x,y)= \prod_{k=x}^{y-1} s_{x+y-1-k}$, where s_i are the Artin generators of the Braid group.

Proof. The insertion braids satisfy the defining relations of R(n). The figure below shows the relation (III) from the Steinberg Group also satisfied in the Braid Group: the strings from the left hand side of the figure can be deformed in the 3-dimensional space to the strings from the right hand side.

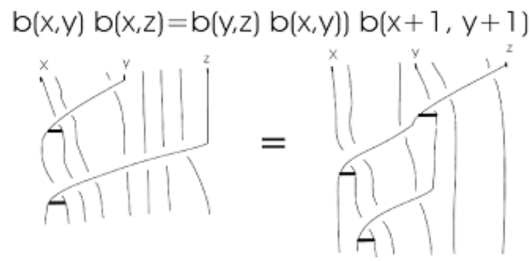

Figure 10.

Corollary 1 *We can label the edges of the associahedron by the elements of the Artin's Braid group B(n), such that any cycle represents a relation from B(n). Any group G to which there is a morphism from R(n) to G can be used to label the edges of the associahedron coherently. In particular we can take G:= the semidirect product between B(n) and $St_n(A)$, a group already studied in [5].*

Corollary 2 *The rotation distance between the trees t_1 and t_2 is greater or equal than the length of the $g_1^{-1} \circ g_2$ in the group $R(n)$, with respect to generators R(x,y).*

In a forthcoming paper, we are using the groups R(n) and $St_n(A)$ to give a combinatorial proof of the result of Thurston , Sleator and Tarjan which states that the diameter of the Rotation Graph is 2n-6, for n large enough, one of the very few combinatorial problems whose solution was based on Hyperbolic Geometry.

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