An. Şt. Univ. Ovidius Constanţa Vol. 15(2), 2007, 83–90

Post algebras in 3-rings

Sergiu Rudeanu

To Professor, Dorin Popescu, at his 60th anniversary

Abstract

A unitary commutative ring of characteristic 3 and 3-potent is called a 3-ring. We prove that every polynomial of a 3-ring is uniquely determined by its restriction on the subring $\{0, 1, 2\}$ (the Verification Theorem). Then we establish an isomorphism between the category of 3-rings and the category of Post algebras of order 3.

1 Introduction

The concept of a *p-ring*, where p is a prime, was defined by McCoy and Montgomery [3] as a commutative ring satisfying the identities $px = 0$ and $x^p = x$. They proved that every finite p-ring is a direct product of fields $\mathbb{Z}_p$, and every p-ring is isomorphic to a subring of a direct product of fields $\mathbb{Z}_p$. So p-rings generalize Boolean rings, for which $p = 2$.

Moisil [4] proved that every unitary 3-ring can be made into a bounded distributive lattice which, in the case of the ring $\mathbb{Z}_3$, is a centred 3-valued Lukasiewicz(-Moisil) algebra. Note that centred Lukasiewicz-Moisil algebras coincide with Post algebras.

In the paper [7] we determined all the rings that can be constructed on a Post algebra of arbitrary order r by Post functions. All these rings satisfy the identities $rx = 0$ and $x^r = x$. In the case $r = 3$ exactly one of them is term

Key Words: Post algebra; 3-ring
Mathematics Subject Classification: 06D25, 16B99.
Received: March, 2007
Accepted: May, 2007

equivalent to the Post algebra, just like a Boolean ring is equivalent to the Boolean algebra having the same support.

In the present paper we lift the above equivalence to an isomorphism between the category **3Post** of Post algebras of order 3 and the category **3Rng** of 3-*rings*. The latter are understood as commutative unitary rings satisfying the identities

$$x + x + x = 0 \ , \tag{1}$$

$$x^3 = x \ . \tag{2}$$

Since it is proved in [3] that every p-ring can be embedded into a unitary p-ring, our definition is essentially the same as the one given in [3].

To obtain the desired isomorphism we need a preparation: every 3-ring R includes $\mathbb{Z}_3$ as a subring and two polynomials of R coincide if and only if their restrictions to $\mathbb{Z}_3$ coincide. We call this result the *Verification Theorem* for 3-rings (corollary of Theorem 1) and it will be the crucial tool for our main results, just like the Verification Theorem for Post algebras was the crucial tool in [7].

It is easy to see (Proposition 2) that Theorem 3 in [7] yields in fact a functor F : **3Post** $\longrightarrow$ **3Rng**, while in this paper we construct a functor G : **3Rng** $\longrightarrow$ **3Post** (Proposition 3) and prove that F and G establish an isomorphism of categories (Theorem 2).

2 The Verification Theorem

Let $(R, +, \cdot, 0, 1)$ be a 3-ring.

Lemma 1 *The element*

$$2 = 1 + 1 \tag{3}$$

satisfies

$$2 + 1 = 0 \ , \tag{4}$$

$$2 + 2 = 1 \ , \tag{5}$$

$$2^2 = 1 \ . \tag{6}$$

PROOF: Properties (4) and (5) follow from $1 + 1 + 1 = 0$. Then (3) implies $2^2 = 2 + 2 = 1$. □

Proposition 1 *The set* $E = \{0, 2, 1\}$ *is a subring isomorphic to* $\mathbb{Z}_3$.

Proof: Immediate by Lemma 1. □

Lemma 2 *Every polynomial* $p : R \longrightarrow R$ *can be written in the form* $p(x) = ax^2 + bx + c$*, the coefficients being uniquely determined by*

$$a = 2p(1) + 2p(2) + 2p(0)\,,\; b = 2p(1) + p(2)\,,\; c = p(0)\;. \tag{7}$$

Proof: The existence of the representation follows from (2). Taking in turn $x := 0, 2, 1$, we obtain

$$p(0) = c\;,$$
$$p(2) = a + 2b + c\;,$$
$$p(1) = a + b + c\;,$$

hence $2p(1) + p(2) = p(2) - p(1) = b$, therefore $a = p(1) + 2b + 2c$, which is the first equality (7). □

Theorem 1 *Every polynomial* $p : R^n \longrightarrow R$ *is uniquely determined by its restriction to* E^n*.*

Proof: For $n = 1$ this follows from Lemma 2. At the inductive step $n-1 \mapsto n$ we fix momentarily $x_2, \ldots, x_n$, apply again Lemma 2 and obtain

$$\begin{aligned} p(x_1, \ldots, x_n) = (&2p(1, x_2, \ldots, x_n) + 2p(2, x_2, \ldots, x_n) + 2p(0, x_2, \ldots, x_n))x_1^2 + \\ &+ (2p(1, x_2, \ldots, x_n) + p(2, x_2, \ldots, x_n))x_1 + p(0, x_2, \ldots, x_n)\;, \end{aligned} \tag{8}$$

then let $x_2, \ldots, x_n$ vary arbitrarily, so that (8) holds for all $x_1, x_2, \ldots, x_n$. Now if p' is a polynomial which coincides with p on E^n, then for each $a \in E$,

$$p(a, x_2, \ldots, x_n) = p'(a, x_2, \ldots, x_n) \tag{9}$$

by the inductive hypothesis. Finally we apply (8) to p and p', taking into account (9); this yields $p = p'$. □

Corollary 1 (Verification Theorem) *A polynomial identity* $p(x_1, \ldots, x_n) = q(x_1,$
$\ldots, x_n)$ *holds in* R *if and only if it is verified on* E*.*

3 The categories 3Post and 3Rng are isomorphic

We recall that a *Post algebra of order* 3 is an algebra $(P, \vee, \wedge, ^{(0)}, ^{(1)}, ^{(2)}, 0, e, 1)$ of type (2,2,1,1,1,0,0,0) such that $(P, \vee, \wedge, 0, 1)$ is a bounded distributive lattice and the following identities hold:

$$x^{(0)} \wedge x^{(1)} = x^{(0)} \wedge x^{(2)} = x^{(1)} \wedge x^{(2)} = 0 \,, \tag{10}$$

$$x^{(0)} \vee x^{(1)} \vee x^{(2)} = 1 \,, \tag{11}$$

$$x = (e \wedge x^{(1)}) \vee x^{(2)} \,. \tag{12}$$

For the general concept of Post algebra of order r the reader is referred to [2], [8], [1], [6] or [7]. Our presentation of Post algebras in the monograph [6] and the paper [7]* is strongly influenced by the book [8].

Let **3Post** and **3Rng** denote the category of Post algebras of order 3 and the category of 3-rings, respectively. The morphisms are defined as prescribed by universal algebra.

Proposition 2 *A functor* $F : \mathbf{3Post} \longrightarrow \mathbf{3Rng}$ *is defined by*

$$F(P, \vee, \wedge, ^{(0)}, ^{(1)}, ^{(2)}, 0, e, 1) = (P, \oplus, \odot, 0, 1)\,, \qquad Fu = u \,, \tag{13}$$

where

$$\begin{aligned} x \oplus y = (e \wedge ((x^{(2)} \wedge y^{(2)}) \vee (x^{(0)} \wedge y^{(1)}) \vee (x^{(1)} \wedge y^{(0)})))\vee \\ \vee (x^{(1)} \wedge y^{(1)}) \vee (x^{(0)} \wedge y^{(2)}) \vee (x^{(2)} \wedge y^{(0)}) \,, \end{aligned} \tag{14}$$

$$x \odot y = (e \wedge ((x^{(1)} \wedge y^{(2)}) \vee (x^{(2)} \wedge y^{(1)}))) \vee (x^{(1)} \wedge y^{(1)}) \vee (x^{(2)} \wedge y^{(2)}). \tag{15}$$

PROOF: The fact that F is correctly defined on objects is part of Theorem 3 in [7]. It follows from (14) and (15) that if $u : P \longrightarrow P'$ is a morphism in **3Post**, then $u : FP \longrightarrow FP'$ is a morphism in **3Rng**. □

Proposition 3 *A functor* $G : \mathbf{3Rng} \longrightarrow \mathbf{3Post}$ *is defined by*

$$G(R, +, \cdot, 0, 1) = (R, \vee, \wedge, ^{(0)}, ^{(1)}, ^{(2)}, 0, 2, 1)\,, \qquad Gv = v \,, \tag{16}$$

where

$$x \vee y = 2x^2y^2 + x^2y + xy^2 + xy + x + y \,, \tag{17}$$

$$x \wedge y = x^2y^2 + 2x^2y + 2xy^2 + 2xy \,, \tag{18}$$

$$x^{(0)} = 2x^2 + 1 \,, \tag{19}$$

$$x^{(1)} = 2x^2 + x \,, \tag{20}$$

$$x^{(2)} = 2x^2 + 2x \,. \tag{21}$$

*In [7] we have denoted meet by $\cdot$ or concatenation and the disjunctive components by x^0, x^1, x^2.

Comment Moisil [4] proved that formulas (17), (18) make a 3-ring into a distributive lattice which, for the ring $\mathbb{Z}_3$, is a centred 3-valued Lukasiewicz(-Moisil) algebra. Centred Lukasiewicz-Moisil algebras coincide with Post algebras; cf. [1], Corollary 4.1.9.

PROOF: We have to prove that the algebra in (16) is a bounded distributive lattice which satisfies (17)-(21). A well-known theorem due to Sholander [9] shows that the former property is equivalent to the axioms

$$x \wedge (x \vee y) = x \ , \tag{22}$$

$$x \wedge (y \vee z) = (z \wedge x) \vee (y \wedge x) \ . \tag{23}$$

In view of the Verification Theorem it suffices to check properties (10),(11),(12), (22),(23) on the subset $E = \{0, 2, 1\}$.

Note first that (17) and (18) imply the identities

$$x \vee 0 = x \, , \qquad x \wedge 0 = 0 \ ,$$

$$x \vee 1 = 2x^2 + x^2 + x + x + x + 1 = 1 \ ,$$

$$x \wedge 1 = x^2 + 2x^2 + 2x + 2x = x \ ,$$

$$2 \vee 2 = 2 + 2 + 2 + 1 + 2 + 2 = 2 \ ,$$

$$2 \wedge 2 = 1 + 1 + 1 + 2 = 2 \ ,$$

therefore

$$a \vee b = \max(a, b) \ , \ a \wedge b = \min(a, b) \qquad (\forall\, a, b \in E) \ . \tag{24}$$

In other words, $(E, \vee, \wedge, 0, 1)$ is the chain $0 < 2 < 1$. Since every chain is a distributive lattice, properties (22), (23) are verified on E.

Furthermore, it follows easily from (19)-(21) that

$$0^{(0)} = 1 \ , \ a^{(0)} = 0 \ \text{ for } \ a \in \{1, 2\} \ , \tag{25.1}$$

$$2^{(1)} = 1 \ , \ a^{(1)} = 0 \ \text{ for } \ a \in \{0, 1\} \ , \tag{25.2}$$

$$1^{(2)} = 1 \ , \ a^{(2)} = 0 \ \text{ for } \ a \in \{0, 2\} \ . \tag{25.3}$$

Clearly (24) and (25) prove (11), while suitable combinations prove (10), for instance

$$0^{(0)} \wedge 0^{(1)} = 2^{(0)} \wedge 2^{(1)} = 1^{(0)} \wedge 1^{(1)} = 0 \ ,$$

etc. From

$$(2 \wedge 0^{(1)}) \vee 0^{(2)} = 0 \ ,$$

$$(2 \wedge 2^{(1)}) \vee 2^{(2)} = 2 \vee 0 = 2 \ ,$$

$$(2 \wedge 1^{(1)}) \vee 1^{(2)} = 0 \vee 1 = 1 \ ,$$

we see that property (12) with $e = 2$ holds on E.

We have thus proved that the algebra in (16) is a Post algebra of order 3. Finally it follows from (17), (18) that if $v : R \longrightarrow R'$ is a morphism in **3Rng**, then $v : GR \longrightarrow GR'$ is a morphism in **3Post**. □

Theorem 2 *The functors F and G establish an isomorphism.*

Proof: The relation $GF = 1_{\mathbf{3Post}}$ is a paraphrase of Theorem 4 in [7]. It remains to prove $FG = 1_{\mathbf{3Rng}}$. This is clear on morphisms.

Let $(R, +, \cdot, 0, 1)$ be a 3-ring. The algebra GR is given by formulas (16)-(21), hence the ring FGR is constructed by formulas (13)-(15) with $P = R$ and $e = 2$. We must prove that $FGR = R$, which amounts to $x \oplus y = x + y$ and $x \odot y = x \cdot y$.

Since 0 and 1 are the zero and unit of the ring FGR, we have

$$x \oplus 0 = x \ , \ x \odot 1 = x \ , \ x \odot 0 = 0 \ . \tag{26}$$

Then for every $a, b \in E$ we use (14), (15), (24), (25) and obtain

$$a \oplus 1 = (2 \wedge a^{(2)}) \vee a^{(0)} \ ,$$

$$a \oplus 2 = (2 \wedge a^{(0)}) \vee a^{(1)} \ ,$$

$$a \odot 2 = (2 \wedge a^{(2)}) \vee a^{(1)} \ ,$$

which implies further

$$2 \oplus 1 = 0 \ , \ 1 \oplus 1 = 2 \vee 0 = 2 \ , \ 2 \oplus 2 = 1 \ , \ 2 \odot 2 = 1 \ . \tag{27}$$

Relations (26), (27) show that $a \oplus b = a + b$ and $a \odot b = a \cdot b$ for every $a, b \in E$. In view of the Verification Theorem, this implies $x \oplus y = x + y$ and $x \odot y = x \cdot y$ for all $x, y \in R$. □

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Faculty of Mathematics-Informatics, University of Bucharest
Str. Academiei 14, 010014 Bucharest,
Romania
email: srudeanu@yahoo.com