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THE CONES ASSOCIATED TO SOME TRANSVERSAL POLYMATROIDS

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Abstract

In this paper we describe the facets cone associated to transversal polymatroid presented by $\mathcal{A} = \{\{1,2\},\{2,3\},\ldots,\{n-1,n\},\{n,1\}\}$. Using the Danilov-Stanley theorem to characterize the canonicale module, we deduce that the base ring associated to this polymatroid is Gorenstein ring. Also, starting from this polymatroid we describe the transversal polymatroids with Gorenstein base ring in dimension 3 and with the help *Normaliz* in dimension 4.

1 Preliminaries on polyhedral geometry

An affine space generated by $A \subset \mathbb{R}^n$ is a translation of a linear subspace of $\mathbb{R}^n$. If $0 \neq a \in \mathbb{R}^n$, then H_a will denote the hyperplane of $\mathbb{R}^n$ through the origin with normal vector a, that is,

$$H_a = \{x \in \mathbb{R}^n \mid \ < x, a >= 0\},$$

where $<,>$ is the usual inner product in $\mathbb{R}^n$. The two closed half spaces bounded by H_a are:

$$H_a^+ = \{x \in \mathbb{R}^n \mid \ < x, a > \geq 0\} \ and \ H_a^- = \{x \in \mathbb{R}^n \mid \ < x, a > \leq 0\}.$$

Recall that a *polyhedral cone* $Q \subset \mathbb{R}^n$ is the intersection of a finite number of closed subspaces of the form H_a^+. If $A = \{\gamma_1, \ldots, \gamma_r\}$ is a finite set of points in $\mathbb{R}^n$ the *cone* generated by A, denoted by $\mathbf{R}_+A$, is defined as

$$\mathbf{R}_+A = \{\sum_{i=1}^{r} a_i\gamma_i \mid a_i \in \mathbb{R}_+, \ with \ 1 \leq i \leq n\}.$$

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An important fact is that Q is a polyhedral cone in $\mathbb{R}^n$ if and only if there exists a finite set $A \subset \mathbb{R}^n$ such that $Q = \mathbf{R}_+ A$, see ([15],theorem 4.1.1.).

Definition 1.1. A proper face of a polyhedral cone is a subset $F \subset Q$ such that there is a supporting hyperplane H_a satisfying:
1) $F = Q \cap H_a \neq \emptyset$;
2) $Q \nsubseteq H_a$ and $Q \subset H_a^+$.

Definition 1.2. A proper face F of a polyhedral cone $Q \subset \mathbb{R}^n$ is called a $facet$ of Q if $dim(F) = dim(Q) - 1$.

2 Polymatroids

Let K be an infinite field, n and m be positive integers, $[n] = \{1, 2, \ldots, n\}$. A nonempty finite set B of $\mathbf{N}^n$ is the base set of a discrete polymatroid $\mathcal{P}$ if, for all $u = (u_1, u_2, \ldots, u_n)$, $v = (v_1, v_2, \ldots, v_n) \in B$, one has $u_1 + u_2 + \ldots + u_n = v_1 + v_2 + \ldots + v_n$ and, for all i such that $u_i > v_i$, there exists j such that $u_j < v_j$ and $u + e_j - e_i \in B$, where e_k denotes the k^{th} vector of the standard basis of $\mathbf{N}^n$. The notion of discrete polymatroid is a generalization of the classical notion of matroid, see [6] [9] [8] [16]. Associated with the base B of a discret polymatroid $\mathcal{P}$ one has a $K-$algebra $K[B]$ - called the base ring of $\mathcal{P}$ - defined to be the $K-$subalgebra of the polynomial ring in n indeterminates $K[x_1, x_2, \ldots, x_n]$ generated by the monomials x^u with $u \in B$. From [9], the algebra $K[B]$ is known to be normal and hence Cohen-Macaulay.

If A_i are some non-empty subsets of $[n]$, for $1 \leq i \leq m$, $\mathcal{A} = \{A_1, A_2, \ldots, A_m\}$, then the set of the vectors $\sum_{k=1}^m e_{i_k}$ with $i_k \in A_k$, is the base of a polymatroid, called transversal polymatroid presented by $\mathcal{A}$. The base ring of a transversal polymatroid presented by $\mathcal{A}$ denoted by $K[\mathcal{A}]$ is the ring :

$$K[\mathcal{A}] := K[x_{i_1} x_{i_2} \ldots x_{i_m} : i_j \in A_j, 1 \leq j \leq m].$$

3 Some Linear Algebra

Let $n \in \mathbf{N}$ be an integer number, $n \geq 3$ and let be given the following set with $2n - 3$ points with positive integer coordinates :

$$\{\ R_{0,1}(2,1,1,\ldots,1,1,0), R_{0,2}(2,1,1,\ldots,1,0,1), \ldots, R_{0,n-2}(2,1,0,\ldots,1,1,1),$$

$$R_{0,n-1}(2,0,1,\ldots,1,1,1), Q_{0,1}(1,2,1,1,\ldots,1,1,0), Q_{0,2}(1,1,2,1,\ldots,1,1,0),$$

$$\ldots\ldots,\ Q_{0,n-3}(1,1,1,1,\ldots,2,1,0), Q_{0,n-2}(1,1,1,1,\ldots,1,2,0)\} \subset \mathbf{N}^n.$$

We shall denote by $A_1 \in M_{n-1,n}(\mathbb{R})$ the matrix with rows the coordinates of points $\{ R_{0,1}, R_{0,2}, \ldots, R_{0,n-1}\}$ and for $2 \leq i \leq n-1$, $A_i \in M_{n-1,n}(\mathbb{R})$ the matrix with rows the coordinates of the points

$$\{ R_{0,1}, \ldots, R_{0,n-i}, Q_{0,1}, Q_{0,2}, \ldots, Q_{0,i-1}\},$$

that is:

$$A_1 = \begin{pmatrix} 2 & 1 & 1 & 1 & \ldots & \ldots & 1 & 1 & 0 \\ 2 & 1 & 1 & 1 & \ldots & \ldots & 1 & 0 & 1 \\ 2 & 1 & 1 & 1 & \ldots & \ldots & 0 & 1 & 1 \\ . & . & . & . & . & . & . & . & . \\ 2 & 1 & 1 & 0 & \ldots & \ldots & 1 & 1 & 1 \\ 2 & 1 & 0 & 1 & \ldots & \ldots & 1 & 1 & 1 \\ 2 & 0 & 1 & 1 & \ldots & \ldots & 1 & 1 & 1 \end{pmatrix}$$

and, for $2 \leq i \leq n-1$,

$$\downarrow (n-i)^{th} column$$

$$A_i = \begin{pmatrix} 2 & 1 & 1 & \ldots & 1 & 1 & 1 \ldots & 1 & 1 & 0 \\ 2 & 1 & 1 & \ldots & 1 & 1 & 1 \ldots & 1 & 0 & 1 \\ 2 & 1 & 1 & \ldots & 1 & 1 & 1 \ldots & 0 & 1 & 1 \\ . & . & . & \ldots & . & . & . \ldots & . & . & . \\ 2 & 1 & 1 & \ldots & 1 & 1 & 0 \ldots & 1 & 1 & 1 \\ 1 & 2 & 1 & \ldots & 1 & 1 & 1 \ldots & 1 & 1 & 0 \\ 1 & 1 & 2 & \ldots & 1 & 1 & 1 \ldots & 1 & 1 & 0 \\ . & . & . & \ldots & . & . & . \ldots & . & . & . \\ 1 & 1 & 1 & \ldots & 2 & 1 & 1 \ldots & 1 & 1 & 0 \\ 1 & 1 & 1 & \ldots & 1 & 2 & 1 \ldots & 1 & 1 & 0 \end{pmatrix} \begin{matrix} \\ \\ \\ \\ \\ \leftarrow (n-i+1)^{th} \ row \\ \\ \\ \\ \\ \end{matrix}.$$

Let T_i be the linear transformation from $\mathbb{R}^n$ into $\mathbb{R}^{n-1}$ defined by $T_i(x) = A_i x$ for all $1 \leq i \leq n-1$.

Let $i, j \in \mathbf{N}$, $1 \leq i, j \leq n$. We denote by $e_{i,j}$ the matrix in $M_{n-1}(\mathbb{R})$ with the entries: 1, for the (i,j)-entry, and 0 for the other entries. We denote by $T_{i,j}(a)$ the matrix

$$T_{i,j}(a) = I_{n-1} + a e_{i,j} \in M_{n-1}(\mathbb{R}).$$

By $P_{i,j}$ we denote the matrix in $M_{n-1}(\mathbb{R})$ defined by

$$P_{i,j} = I_{n-1} - e_{i,i} - e_{j,j} + e_{i,j} + e_{j,i}.$$

Lemma 3.1. *a) The set of points* $\{ R_{0,1},\dots,R_{0,n-i},Q_{0,1},Q_{0,2},\dots,Q_{0,i-1}\}$, *for* $2\leq\ i\leq\ n-1$ *and* $\{ R_{0,1},R_{0,2},\dots,R_{0,n-1}\}$ *are linearly independent.*
b) For $1\leq\ i\leq\ n-1$, *the equations of the hyperplanes generated by the points* $\{ O,R_{0,1},R_{0,2}\dots,R_{0,n-i},Q_{0,1},Q_{0,2},\dots,Q_{0,i-1}\}$ *are :*

$$H_{[i]}:=-(n-i-1)\sum_{j=1}^{i}x_j+(i+1)\sum_{j=i+1}^{n}x_j=0,$$

where $[i]$ *is the set* $[i]:=\{1,\dots,i\}$.

Proof. *a*) The set of points are linearly independent if the matrices with rows the coordinates of the points have the rank $n-1$.
Using elementary row transformations on the matrix A_1, we have:
$B_1=U_1A_1$, where $U_1\in\ M_{n-1}(\mathbb{R})$is given by:

$$U_1=\prod_{2\leq i\leq\lfloor\frac{n}{2}\rfloor}P_{i,n-i+1}\prod_{i=2}^{n-1}T_{n-i+1,1}(-1),$$

and $\lfloor c\rfloor$ is the greatest integer $\leq c$. So B_1 is :

$$B_1=\begin{pmatrix} 2 & 1 & 1 & 1 & \dots & \dots & 1 & 1 & 0\\ 0 & -1 & 0 & 0 & \dots & \dots & 0 & 0 & 1\\ 0 & 0 & -1 & 0 & \dots & \dots & 0 & 0 & 1\\ . & . & . & . & . & . & . & . & .\\ 0 & 0 & 0 & 0 & \dots & \dots & 0 & 0 & 1\\ 0 & 0 & 0 & 0 & \dots & \dots & -1 & 0 & 1\\ 0 & 0 & 0 & 0 & \dots & \dots & 0 & -1 & 1 \end{pmatrix}.$$

For $2\leq\ i\leq\ n-1$, using elementary row transformations on the matrix A_i, we have: $B_i=U_iA_i$, where $U_i\in\ M_{n-1}(\mathbb{R})$,

$$U_i=[\prod_{j=i}^{n-2}(\prod_{k=1}^{i-1})P_{n-j+k-1,n-j+k}][\prod_{k=2}^{i-1}(\prod_{j=n-i+k}^{n-1}T_{j,n-i+k-1}(-\frac{1}{k+1}))]\cdot$$

$$\cdot(\prod_{j=n-i+1}^{n-1}T_{j,1}(-\frac{1}{2}))(\prod_{j=1}^{n-i}T_{j,1}(-1)),$$

and so B_i is :

$$B_i = \begin{array}{cc} \begin{array}{cccccccccccc} & & & & & & \downarrow (i+1)^{th} column & & & & & \end{array} & \\ \begin{pmatrix} 2 & 1 & 1 & 1 & \dots & 1 & 1 & 1 & \dots & 1 & 1 & 0 \\ 0 & \frac{3}{2} & \frac{1}{2} & \frac{1}{2} & \dots & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \dots & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & \frac{4}{3} & \frac{1}{3} & \dots & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \dots & \frac{1}{3} & \frac{1}{3} & 0 \\ . & . & . & . & \dots & . & . & & \dots & & . & . \\ 0 & 0 & 0 & 0 & \dots & \frac{i}{i-1} & \frac{1}{i-1} & \frac{1}{i-1} & \dots & \frac{1}{i-1} & \frac{1}{i-1} & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & \frac{i+1}{i} & \frac{1}{i} & \dots & \frac{1}{i} & \frac{1}{i} & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & -1 & \dots & 0 & 0 & 1 \\ . & . & . & . & \dots & . & . & . & \dots & . & . & . \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & -1 & 1 \end{pmatrix} & \begin{array}{c} \\ \\ \\ \\ \\ \leftarrow\ i^{th}\ row \\ \\ \\ \\ \\ \end{array} \end{array} .$$

Since the rank of B_i is $n-1$, the rank of A_i is $n-1$, for all $1 \leq\ i \leq\ n-1$.

$b)$ The hyperplane generated by the points

$$\{\ R_{0,1}, \dots, R_{0,n-i}, Q_{0,1}, Q_{0,2}, \dots, Q_{0,i-1}\}$$

has the normal vector the generator of the subspace $Ker(T_i)$.

For $1 \leq\ i \leq\ n-1$, using $a)$, we obtain that

$$Ker(T_i) = \{x \in \mathbb{R}^n | T_i(x) = 0\} = \{x \in \mathbb{R}^n | A_i x = 0\} = \{x \in \mathbb{R}^n | B_i x = 0\},$$

that is

$$x_n = x_{n-1} = \dots = x_{i+1} = (i+1)\alpha$$

and

$$x_i = x_{i-1} = \dots = x_1 = -(n-i-1)\alpha,$$

where $\alpha \in \mathbb{R}$.

Thus, for $1 \leq\ i \leq\ n-1$, the equations of the hyperplanes generated by the points $\{\ R_{0,1}, \dots, R_{0,n-i}, Q_{0,1}, Q_{0,2}, \dots, Q_{0,i-1}\}$ are :

$$H_{[i]} := -(n-i-1)\sum_{j=1}^{i} x_j + (i+1)\sum_{j=i+1}^{n} x_j = 0.$$

$\square$

For $1 \leq k \leq n-1$, we define two types of sets of points:

1)

$$\{\ R_{k,1}, R_{k,2}, \ldots, R_{k,n-1}\}$$

is the set of points whose coordinates are the rows of the matrix $A_1 P_{1k+1}$;

2)

$$\{\ Q_{k,1}, Q_{k,2}, \ldots, Q_{k,n-2}\}$$

is the set of points whose coordinates are the rows of the matrix QM^k, where M is the matrix

$$M \in\ M_n(\mathbb{R}), M = \prod_{i=1}^{n-1} P_{n-i,n-i+1}$$

and $Q \in\ M_{n-2,n}(\mathbb{R})$ is the matrix with rows the coordinates of points $\{\ Q_1, Q_2, \ldots, Q_{n-2}\}$.

For every $1 \leq i \leq n-1$, we shall denote by $\nu_{[i]}$ the normal of the hyperplane $H_{[i]}$:

$$\begin{array}{c} \qquad\qquad\qquad\qquad\qquad \downarrow i^{th} column \\ \nu_{[i]} = \big(\ -(n-i-1), \quad \ldots \quad , -(n-i-1), \quad (i+1), \quad \ldots \quad , (i+1)\ \big) \in \mathbb{R}^n. \end{array}$$

For $i = 1$, we denote by $H_{\{k+1\}}$ the hyperplane having the normal :

$$\nu_{\{k+1\}} := \nu_{[i]} P_{1,k+1} = \nu_{[1]} P_{1,k+1},$$

for all $1 \leq k \leq n-1$.

For $2 \leq i \leq n-1$ and $1 \leq k \leq n-1$, we denote by $H_{\{\sigma^k(1),\sigma^k(2),\ldots,\sigma^k(i)\}}$ the hyperplane which has the normal :

$$\nu_{\{\sigma^k(1),\sigma^k(2),\ldots,\sigma^k(i)\}} := \nu_{[i]} M^k,$$

where $\sigma \in S_n$ is the product of transpositions :

$$\sigma := \prod_{i=1}^{n-1} (i, i+1).$$

Lemma 3.2. *a) For $1 \leq k \leq n-1$ and $2 \leq\ i \leq\ n-1$, the set of points $\{\ R_{k,1}, \ldots, R_{k,n-i}, Q_{k,1}, Q_{k,2}, \ldots, Q_{k,i-1}\}$ and $\{\ R_{k,1}, R_{k,2}, \ldots, R_{k,n-1}\}$ are linearly independent.*

b) For $1 \leq k \leq n-1$ and $2 \leq\ i \leq\ n-1$, the equation of the hyperplane generated by the points $\{\ O, R_{k,1}, R_{k,2} \ldots, R_{k,n-i}, Q_{k,1}, Q_{k,2}, \ldots, Q_{k,i-1}\}$ is :

$$H_{\{\sigma^k(1),\sigma^k(2),\ldots,\sigma^k(i)\}} :=< \nu_{\{\sigma^k(1),\sigma^k(2),\ldots,\sigma^k(i)\}}, x >= 0,$$

where O is zero point, $O(0,0,\ldots,0)$ and $\sigma \in S_n$ is the product of transpositions:

$$\sigma := \prod_{i=1}^{n-1} (i, i+1).$$

For $1 \le k \le n-1$, the equation of the hyperplane generated by the points $\{ O, R_{k,1}, R_{k,2} \ldots, R_{k,n-1}\}$ is

$$H_{\{k+1\}} :=< \nu_{\{k+1\}}, x >= 0.$$

Proof. $a)$ Since, for any $1 \le k \le n-1$, the matrix $P_{1,k+1}$, M^k are invertible and the sets of points

$$\{ R_{0,1}, \ldots, R_{0,n-i}, Q_{0,1}, Q_{0,2}, \ldots, Q_{0,i-1}\}, \{ R_{0,1}, R_{0,2}, \ldots, R_{0,n-1}\}$$

are linearly independent then the set of points

$$\{ R_{k,1}, \ldots, R_{k,n-i}, Q_{k,1}, Q_{k,2}, \ldots, Q_{k,i-1}\}, \{ R_{k,1}, R_{k,2}, \ldots, R_{k,n-1}\}$$

are linearly independent.

$b)$ Since, for any $1 \le k \le n-1$ and $2 \le \ i \le \ n-1$, the matrix M^k are invertible, then the hyperplane generated by the points

$$\{ O, R_{k,1}, \ldots, R_{k,n-i}, Q_{k,1}, \ldots, Q_{k,i-1}\}$$

has the normal vector obtained by multiplying the normal vector $\nu_{[k]}$ on the right with M^k. For any $1 \le k \le n-1$, the matrix $P_{1,k+1}$ is invertible, then the hyperplane generated by the points $\{ O, R_{k,1}, R_{k,2} \ldots, R_{k,n-1}\}$ has the normal vector obtained by multiplying on the right the normal vector $\nu_{[1]}$ with $P_{1,k+1}$. □

Lemma 3.3. *Any point $P \in \mathbf{N^n}$, $n \ge 3$ which lies in the hyperplane H : $x_1 + x_2 + \ldots + x_n - n = 0$ such that its coordinates are in the set $\{0,1,2\}$ and has at least one coordinate equal to 2 lies in the hyperplane $H_{\{k\}} = 0$, for an integer $k \in \{1, 2, ..., n\}$.*

Proof. Let $k \in \{1, 2, ..., n\}$ be the first position of "2" that appears in the coordinates of a point $P \in \mathbf{N^n}$. Since the equation of the hyperplane $H_{\{k\}}$ is:

$$H_{\{k\}} = \sum_{i=1}^{k-1} 2x_i - (n-2)x_k + \sum_{i=k+1}^{n} 2x_i = 0,$$

it results that

$$-2(n-2) + 2 \sum_{i=1, i\ne k} na_i = -2(n-2) + 2(n-2) = 0,$$

where $P = (a_1, a_2, \ldots, a_n) \in H$ with $a_i \in \{0,1,2\}$ and which has at least one coordinate equal to 2. □

4 The main result

First let us fix some notations that will be used throughout the remaining of this paper. Let K be a field and $K[x_1, x_2, \ldots, x_n]$ be a polynomial ring with coefficients in K. Let $n \geq 2$ be a positive integer and $\mathcal{A}$ be the collection of sets:

$$\mathcal{A} = \{\{1,2\}, \{2,3\}, \ldots, \{n-1,n\}, \{n,1\}\}.$$

We denote by $K[\mathcal{A}]$ the $K-$algebra generated by $x_{i_1}x_{i_2}...x_{i_n}$, with

$$i_1 \in \{1,2\}, i_2 \in \{2,3\}, \ldots, i_{n-1} \in \{n-1,n\}, i_n \in \{1,n\}.$$

This $K-$algebra represents the base ring associated to transversal polymatroid presented by $\mathcal{A}$.

Given $A \in \mathbf{N^n}$ finite, we define C_A as being the subsemigroup of $\mathbf{N}^n$ generated by A :

$$C_A = \sum_{\alpha \in A} \mathbf{N}\alpha,$$

thus the *cone* generated by C_A is:

$$\mathbf{R}_+ C_A = \mathbf{R}_+ A = \{\sum a_i \gamma_i \mid a_i \in \mathbb{R}_+, \gamma_i \in A\}.$$

With this notation, we state our main result:

Theorem 4.1. *Let* $A = \{log(x_{i_1}x_{i_2}...x_{i_n}) \mid i_1 \in \{1,2\}, i_2 \in \{2,3\}, \ldots, i_{n-1} \in \{n-1,n\}, i_n \in \{1,n\}\} \subset \mathbf{N}^n$ *the exponent set of the generators of* $K-$*algebra* $K[\mathcal{A}]$ *and* $N = \{\nu_{\{k+1\}}, \nu_{\{\sigma^k(1), \sigma^k(2), \ldots, \sigma^k(i)\}} \mid 0 \leq k \leq n-1, 2 \leq i \leq n-1\}$, *then*

$$\mathbf{R}_+ C_A = \bigcap_{a \in N} H_a^+,$$

such that H_a^+ *with* $a \in N$ *are just the facets of the cone* $\mathbf{R}_+ C_A$.

Proof. Since $A = \{\log(x_{i_1}x_{i_2}...x_{i_n}) \mid i_1 \in \{1,2\},\ i_2 \in \{2,3\}\ ,\ldots,\ i_{n-1} \in \{n-1,n\},\ i_n \in \{1,n\}\} \subset \mathbf{N^n}$ is the exponent set of the generators of $K-$algebra $K[\mathcal{A}]$, then the set $\{\ R_{0,1}, R_{0,2}, \ldots, R_{0,n-2}, R_{0,n-1}, I\} \subset A$, where $I(1,1,\ldots,1) \in \mathbf{N}^n$.

First step.

We must show that the dimension of the cone $\mathbf{R}_+ C_A$ is $dim(\mathbf{R}_+ C_A) = n$. We denote by $\widetilde{A} \in M_n(\mathbb{R})$ the matrix with rows the coordinates of the points

$\{ R_{0,1}, R_{0,2}, \ldots, R_{0,n-2}, R_{0,n-1}, I\}$. Using elementary row transformations to the matrix $\widetilde{A}$, we have: $\widetilde{B} = \widetilde{U}\widetilde{A}$, where $\widetilde{U} \in M_n(\mathbb{R})$ is an invertible matrix:

$$\widetilde{U} = (\prod_{i=2}^{n-1} T_{n-i+1,1}(-1)))(T_{n,1}(-\frac{1}{2}))(\prod_{2\leq i\leq \lfloor \frac{n}{2}\rfloor} P_{i,n-i+1})(\prod_{i=2}^{n-1} T_{n,n-i+1}(\frac{1}{2})),$$

where $\lfloor c\rfloor$ is the greatest integer $\leq c$.

So $\widetilde{B}$ is:

$$\widetilde{B} = \begin{pmatrix} 2 & 1 & 1 & 1 & \ldots & \ldots & 1 & 1 & 0 \\ 0 & -1 & 0 & 0 & \ldots & \ldots & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & \ldots & \ldots & 0 & 0 & 1 \\ . & . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & \ldots & \ldots & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & \ldots & \ldots & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & \ldots & \ldots & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & \ldots & \ldots & 0 & 0 & \frac{n}{2} \end{pmatrix}.$$

Then the dimension of the cone $\mathbf{R}_+C_A$ is:

$$dim(\mathbf{R}_+C_A) = rank(\widetilde{A}) = rank(\widetilde{B}) = n,$$

since $det(\widetilde{B}) = (-1)^n n$.

Second step.

We must show that $H_a \cap \mathbf{R}_+C_A$ with $a \in N$ are precisely the facets of the cone $\mathbf{R}_+C_A$. This is equivalent to show that $\mathbf{R}_+C_A \subset H_a^+$ and $dim H_a \cap \mathbf{R}_+C_A = n-1 \ \forall\ a \in N$.

The fact that $dim H_a \cap \mathbf{R}_+C_A = n-1 \ \forall\ a \in N$ it is clear, from Lemma 3.1 and Lemma 3.2.

For $1 \leq k \leq \lfloor \frac{n}{2}\rfloor$ and $1 \leq i_1 < i_2 < \ldots < i_{2k-1} < i_{2k} \leq n$, let

$$I_{i_1 i_2 \ldots i_{2k-1} i_{2k}} = I + (e_{i_1} - e_{i_2}) + (e_{i_3} - e_{i_4}) + \ldots + (e_{i_{2k-1}} - e_{i_{2k}})$$

and

$$I'_{i_1 i_2 \ldots i_{2k-1} i_{2k}} = I + (e_{i_2} - e_{i_1}) + (e_{i_4} - e_{i_3}) + \ldots + (e_{i_{2k}} - e_{i_{2k-1}}),$$

where $I = I(1,1,\ldots,1) \in \mathbf{N}^n$ and e_i is the i^{th} unit vector.

We set

$$A' = \{I, I_{i_1 i_2 \ldots i_{2k-1} i_{2k}}, I'_{i_1 i_2 \ldots i_{2k-1} i_{2k}} | 1 \leq k \leq \lfloor \frac{n}{2}\rfloor$$

$$\text{and } 1 \leq i_1 < i_2 < \ldots < i_{2k-1} < i_{2k} \leq n\}.$$

We claim that $A = A^{'}$.

Let

$$m_{i_1 i_2 \ldots i_{2k-1} i_{2k}} = \prod_{s=1}^{\lfloor \frac{n}{2} \rfloor} m_s,\ m^{'}_{i_1 i_2 \ldots i_{2k-1} i_{2k}} = \prod_{s=1}^{\lfloor \frac{n}{2} \rfloor} m^{'}_s,$$

where

$$m_s = x_{k_{i_{2s-2}+1}} \ldots x_{k_{i_{2s-1}-2}} x^2_{i_{2s-1}} x_{k_{i_{2s-1}+1}} \ldots x_{k_{i_{2s}-2}} x_{i_{2s}-1} x_{i_{2s}+1},$$

$$m^{'}_s = x_{k_{i_{2s-2}+1}} \ldots x_{k_{i_{2s-1}-2}} x_{i_{2s-1}-1} x_{i_{2s-1}+1} x_{k_{i_{2s-1}+1}} \ldots x_{k_{i_{2s}-2}} x^2_{i_{2s}},$$

for all $1 \leq k, s \leq \lfloor \frac{n}{2} \rfloor$, $i_0 = 0$ and $k_j \in \{j, j+1\}$, for $1 \leq j \leq n$. Evidently $\log(m_{i_1 i_2 \ldots i_{2k-1} i_{2k}}), \log(m^{'}_{i_1 i_2 \ldots i_{2k-1} i_{2k}}) \in A$.

Since

$$\log(m_{i_1 i_2 \ldots i_{2k-1} i_{2k}}) = I_{i_1 i_2 \ldots i_{2k-1} i_{2k}}$$

and

$$\log(m^{'}_{i_1 i_2 \ldots i_{2k-1} i_{2k}}) = I^{'}_{i_1 i_2 \ldots i_{2k-1} i_{2k}},$$

for all $1 \leq k \leq \lfloor \frac{n}{2} \rfloor$ and $1 \leq i_1 < i_2 < \ldots < i_{2k-1} < i_{2k} \leq n$, then $A^{'} \subset A$.

But the cardinal of A is $\sharp(A) = 2^n - 1$ and since

$$\sum_{s=1}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2s} = 2^{n-1} - 1,$$

the cardinal of $A^{'}$ is:

$$\sharp(A^{'}) = 1 + 2 \sum_{s=1}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2s} = 2^n - 1.$$

Thus $A = A^{'}$.

Now we start to prove that $\mathbf{R}_+ C_A \subset H_a^+$ for all $a \in N$.

Note that

$$< \nu_{\{p+1\}}, I > = < \nu_{\{\sigma^p(1), \sigma^p(2), \ldots, \sigma^p(i)\}}, I > = n > 0,$$

for any $0 \leq p \leq n-1$, $1 \leq i \leq n-1$.

Let $0 \leq p \leq n-1$. We claim that:

$$< \nu_{\{p+1\}}, I_{i_1 i_2 \ldots i_{2k-1} i_{2k}} > \geq 0 \ \mathit{and}\ < \nu_{\{p+1\}}, I^{'}_{i_1 i_2 \ldots i_{2k-1} i_{2k}} > \geq 0,$$

for any $1 \leq k \leq \lfloor \frac{n}{2} \rfloor$ and $1 \leq i_1 < i_2 < \ldots < i_{2k-1} < i_{2k} \leq n$.
We prove the first inequality. The proof of the second inequality will be similar.
We have three possibilities:
1) If $< I_{i_1 i_2 \ldots i_{2k-1} i_{2k}}, e_{p+1} >= 0$, then $< \nu_{\{p+1\}}, I_{i_1 i_2 \ldots i_{2k-1} i_{2k}} >= 2n > 0$;
2) If $< I_{i_1 i_2 \ldots i_{2k-1} i_{2k}}, e_{p+1} >= 1$ then $< \nu_{\{p+1\}}, I_{i_1 i_2 \ldots i_{2k-1} i_{2k}} >= n > 0$;
3) If $< I_{i_1 i_2 \ldots i_{2k-1} i_{2k}}, e_{p+1} >= 2$ then $< \nu_{\{p+1\}}, I_{i_1 i_2 \ldots i_{2k-1} i_{2k}} >= 0$.

Let $0 \leq p \leq n-1$ and $2 \leq i \leq n-1$ be fixed.

We claim that:

$$< \nu_{\{\sigma^p(1), \sigma^p(2), \ldots, \sigma^p(i)\}}, I_{i_1 i_2 \ldots i_{2k-1} i_{2k}} > \ \geq 0$$

and

$$< \nu_{\{\sigma^p(1), \sigma^p(2), \ldots, \sigma^p(i)\}}, I'_{i_1 i_2 \ldots i_{2k-1} i_{2k}} > \ \geq 0,$$

for any $1 \leq k \leq \lfloor \frac{n}{2} \rfloor$ and $1 \leq i_1 < i_2 < \ldots < i_{2k-1} < i_{2k} \leq n$.
We prove the first inequality. The proof of the second inequalities is analogous.

We have:

$$< \nu_{\{\sigma^p(1), \sigma^p(2), \ldots, \sigma^p(i)\}}, I_{i_1 i_2 \ldots i_{2k-1} i_{2k}} >= H_{\{\sigma^p(1), \sigma^p(2), \ldots, \sigma^p(i)\}}(I_{i_1 i_2 \ldots i_{2k-1} i_{2k}}) =$$

$$= -(n-i-1) \sum_{s=1}^{i} < I_{i_1 i_2 \ldots i_{2k-1} i_{2k}}, e_{\sigma^p(s)} > +$$

$$+(i+1) \sum_{s=i+1}^{n} < I_{i_1 i_2 \ldots i_{2k-1} i_{2k}}, e_{\sigma^p(s)} > .$$

Let

$$\Gamma = \{s | < I_{i_1 i_2 \ldots i_{2k-1} i_{2k}}, e_{\sigma^p(s)} >= 2, 1 \leq s \leq i\}$$

be the set of indices of $I_{i_1 i_2 \ldots i_{2k-1} i_{2k}}$, where the coordinates are equal to 2.
If the cardinal of Γ is zero, then there exists at most an index $i_{2t-1} \in \{\sigma^p(1), \sigma^p(2), \ldots, \sigma^p(i)\}$ with $1 \leq t \leq \lfloor \frac{n}{2} \rfloor$. Otherwise we have two possibilities:
1)There exist at least two indices $i_{2t-1}, i_{2t_1-1} \in \{\sigma^p(1), \sigma^p(2), \ldots, \sigma^p(i)\}$, with $1 \leq t < t_1 \leq \lfloor \frac{n}{2} \rfloor$ and, since $\sigma^p(s) = (p+s) \ mod \ n$, then there exists $1 \leq t_2 \leq \lfloor \frac{n}{2} \rfloor$ such that $i_{2t_2} \in \{\sigma^p(1), \sigma^p(2), \ldots, \sigma^p(i)\}$ and thus $< I_{i_1 i_2 \ldots i_{2k-1} i_{2k}}, e_{\sigma^p(i_{2t_2})} >= 2$, which it is false.
2) There exist at least two indices $i_{2t-1}, i_{2t_1} \in \{\sigma^p(1), \sigma^p(2), \ldots, \sigma^p(i)\}$, with $1 \leq t, \ t_1 \leq \lfloor \frac{n}{2} \rfloor$. Then as in the first case, we have $< I_{i_1 i_2 \ldots i_{2k-1} i_{2k}}, e_{\sigma^p(i_{2t_1})} >= 2$, which it is false.
If for any $1 \leq k \leq \lfloor \frac{n}{2} \rfloor$, $i_{2k-1} \notin \{\sigma^p(1), \sigma^p(2), \ldots, \sigma^p(i)\}$, then

$$< \nu_{\{\sigma^p(1), \sigma^p(2), \ldots, \sigma^p(i)\}}, I_{i_1 i_2 \ldots i_{2k-1} i_{2k}} >= -(n-i-1)i + (i+1)(n-i) = n > 0.$$

When there exists just one index $i_{2t-1} \in \{\sigma^p(1), \sigma^p(2), \ldots, \sigma^p(i)\}$ with $1 \leq t \leq \lfloor \frac{n}{2} \rfloor$, then

$$< \nu_{\{\sigma^p(1),\sigma^p(2),\ldots,\sigma^p(i)\}}, I_{i_1 i_2 \ldots i_{2k-1} i_{2k}} >=$$
$$= -(n-i-1)(i-1) + (i+1)(n-i+1) = 2n > 0.$$

If the cardinal of Γ, is $\sharp(\Gamma) = t \geq 1$, then we have two possibilities:
1) If

$$\{1 \leq i_1 < i_2 < \ldots < i_{2t-3} < i_{2t-2} < i_{2t-1} \leq n\} \subset \{\sigma^p(1), \sigma^p(2), \ldots, \sigma^p(i)\},$$

then we have:

$$< \nu_{\{\sigma^p(1),\sigma^p(2),\ldots,\sigma^p(i)\}}, I_{i_1 i_2 \ldots i_{2k-1} i_{2k}} >= -(n-i-1)(i+1)+(i+1)(n-i-1) = 0.$$

2) If

$$\{1 \leq i_1 < i_2 < \ldots < i_{2t-1} < i_{2t} \leq n\} \subset \{\sigma^p(1), \sigma^p(2), \ldots, \sigma^p(i)\},$$

then we have:

$$< \nu_{\{\sigma^p(1),\sigma^p(2),\ldots,\sigma^p(i)\}}, I_{i_1 i_2 \ldots i_{2k-1} i_{2k}} >= -(n-i-1)(i)+(i+1)(n-i) = n > 0.$$

Thus we have:

$$\mathbf{R}_+ C_A \subset \bigcap_{a \in N} H_a^+$$

Finally let us prove the converse inclusion.
This is equivalent with the fact that the extremal rays of the cone

$$\bigcap_{a \in N} H_a^+$$

are in $\mathbf{R}_+ C_A$.

Let $1 \leq k \leq \lfloor \frac{n}{2} \rfloor$, $1 \leq i_1 < i_2 < \ldots < i_{2k-1} < i_{2k} \leq n$. We consider the following hyperplanes:
a) $H_{\{[i_{2s-1}]\setminus[j]\}}$ if $j \in \{i_{2s-2}, \ldots i_{2s-1}-1\}$ and $1 \leq s \leq k$,
b) $H_{\{[j]\setminus[i_{2s-1}-1]\}}$ if $j \in \{i_{2s-1}+1, \ldots, i_{2s}-1\}$ and $1 \leq s \leq k$,
c) $H_{\{[i_{2k-1}]\cup([n]\setminus[j])\}}$ if $j \in \{i_{2k}, \ldots n-1\}$,
d) $H_{\{i_{2s}\}}$ for $1 \leq s \leq k-1$; where $[i] := \{1, \ldots, i\}$, $i_0 = 0$ and $[0] = \emptyset$.
We claim that the point $I_{i_1 i_2 \ldots i_{2k-1} i_{2k}}$ belongs to these hyperplanes.
a) Let $j \in \{i_{2s-2}, \ldots i_{2s-1}-1\}$ and $1 \leq s \leq k$, then

$$< H_{\{[i_{2s-1}]\setminus[j]\}}, I_{i_1 i_2 \ldots i_{2k-1} i_{2k}} >=< H_{\{j+1,\ldots,i_{2s-1}\}}, I_{i_1 i_2 \ldots i_{2k-1} i_{2k}} >=$$

$$=< -(n-(i_{2s-1}-j)-1)$$
$$\sum_{t\in\{j+1,\dots,i_{2s-1}\}} x_t + (i_{2s-1}-j+1) \sum_{t\in[n]\setminus\{j+1,\dots,i_{2s-1}\}} x_t \ , \ I_{i_1 i_2 \dots i_{2k-1} i_{2k}} >=$$
$$= -(n-i_{2s-1}+j-1)(i_{2s-1}-j+1)+$$
$$+(i_{2s-1}-j+1)(n-(i_{2s-1}-j)+1) = 0,$$

since

$$\begin{array}{ccccccccccc} & & & & \downarrow j+1^{th} & & & & \downarrow i_{2s-1}^{th} & & \\ I_{i_1 i_2 \dots i_{2k-1} i_{2k}} & = & (& \dots & , & 1 & , & \dots & , & 1 & , & 2 & , & \dots &). \end{array}$$

b) Let $j \in \{i_{2s-1}+1, \dots i_{2s}-1\}$ and $1 \le s \le k$. Then

$$< H_{\{[j]\setminus[i_{2s-1}-1]\}}, I_{i_1 i_2 \dots i_{2k-1} i_{2k}} >=$$
$$=< H_{\{i_{2s-1},\dots,j\}}, I_{i_1 i_2 \dots i_{2k-1} i_{2k}} >=< -(n-(j-i_{2s-1}+1)-1)$$
$$\sum_{t\in\{i_{2s-1},\dots,j\}} x_t + (j-i_{2s-1}+1+1) \sum_{t\in[n]\setminus\{i_{2s-1},\dots,j\}} x_t \ , \ I_{i_1 i_2 \dots i_{2k-1} i_{2k}} >=$$
$$= -(n-(j-i_{2s-1}+1)-1)(j-i_{2s-1}+1+1)+$$
$$+(j-i_{2s-1}+1+1)(n-(j-i_{2s-1}+1+1)) = 0,$$

since

$$\begin{array}{ccccccccccc} & & & & \downarrow i_{2s-1}^{th} & & & & \downarrow j^{th} & & \\ I_{i_1 i_2 \dots i_{2k-1} i_{2k}} & = & (& \dots & , & 2 & , & 1 & , & \dots & , & 1 & , & \dots &). \end{array}$$

c) Let $j \in \{i_{2k}, \dots n-1\}$. Then

$$< H_{\{[i_{2k-1}]\cup([n]\setminus[j])\}}, I_{i_1 i_2 \dots i_{2k-1} i_{2k}} >=< H_{\{1,\dots,i_{2k-1},j+1,\dots,n\}}, I_{i_1 i_2 \dots i_{2k-1} i_{2k}} >=$$
$$=< -(n-(i_{2k-1}+n-j)-1)$$
$$\sum_{t\in[i_{2k-1}]\cup([n]\setminus[j])} x_t + (i_{2k-1}+n-j+1) \sum_{t\in\{i_{2k-1}+1,\dots,j\}} x_t, \ I_{i_1 i_2 \dots i_{2k-1} i_{2k}} >=$$
$$= -(j-i_{2k-1}-1)(i_{2k-1}+n-j+1)+(i_{2k-1}+n-j+1)(j-(i_{2k-1}+1)+1-1) = 0,$$

since

$$\downarrow i_{2k-1}^{th} \qquad \downarrow i_{2k}^{th} \qquad \downarrow j+1^{th}$$

$$I_{i_1 i_2 \ldots i_{2k-1} i_{2k}} \quad = \quad (\quad \ldots, \quad 2, \quad 1, \quad \ldots, \quad 1, \quad 0, \quad 1, \quad \ldots, \quad 1, \quad \ldots, \quad 1).$$

d) It is clear from Lemma 3.3.
Since the number of hyperplanes is

$$\sum_{s=1}^{k}(i_{2s-1}-1-i_{2s-2}+1)+\sum_{s=1}^{k}(i_{2s}-1-(i_{2s-1}+1)+1)+(n-1-i_{2k}+1)+k-1=$$

$$=\sum_{s=1}^{k}(i_{2s}-i_{2s-2})-k+n-i_{2k}+k-1=n-1,$$

then

$$\bigcap_{s=1}^{k}\Big(\bigcap_{j=i_{2s-2}}^{i_{2s-1}-1}(H_{\{[i_{2s-1}]\setminus[j]\}})\cap\bigcap_{j=i_{2s-1}+1}^{i_{2s}-1}(H_{\{[j]\setminus[i_{2s-1}-1]\}})\Big)\cap$$

$$\bigcap_{j=i_{2k}}^{n-1}(H_{\{[i_{2k-1}]\cup([n]\setminus[j])\}})\cap\bigcap_{s=1}^{k-1}(H_{\{i_{2s}\}})=OI_{i_1 i_2 \ldots i_{2k-1} i_{2k}}$$

is an extremal ray of the cone $\bigcap_{a\in N} H_a^+$. But

$$OI_{i_1 i_2 \ldots i_{2k-1} i_{2k}} \in \mathbf{R}_+ C_A.$$

Thus

$$\bigcap_{a\in N} H_a^+ = \mathbf{R}_+ C_A.$$

□

For using bellow, we recall that $K-$algebra $K[\mathcal{A}]$ is a normal domain according to [9].

Definition 4.2. Let R be a polynomial ring over a field K and F be a finite set of monomials in R. A decomposition

$$K[F]=\bigoplus_{i=0}^{\infty}K[F]_i$$

of the $K-$ vector space $K[F]$ is an *admissible grading* if $k[F]$ is a positively graded $K-$ algebra with respect to this decomposition and each component $K[F]_i$ has a finite $K-$ basis consisting of monomials.

Theorem 4.3 (Danilov, Stanley)**.** *Let $R = K[x_1, \ldots, x_n]$ be a polynomial ring over a field K and F be a finite set of monomials in R. If $K[F]$ is normal, then the canonical module $\omega_{K[F]}$ of $K[F]$, with respect to an arbitrary admissible grading, can be expressed as an ideal of $K[F]$ generated by monomials*

$$\omega_{K[F]} = (\{x^a \mid a \in \mathbf{N}A \cap ri(\mathbf{R}_+A)\}),$$

where $A = log(F)$ and $ri(\mathbf{R}_+A)$ denotes the relative interior of $\mathbf{R}_+A$.

Corollary 4.4. *The canonical module $\omega_{K[\mathcal{A}]}$ of $K[\mathcal{A}]$ is*

$$\omega_{K[\mathcal{A}]} = (x_1x_2\ldots x_n)K[\mathcal{A}].$$

Thus the $K-$ algebra $K[\mathcal{A}]$ is a Gorenstein ring.

Proof. Since

$$< \nu_{\{p+1\}}, I > = < \nu_{\{\sigma^p(1),\sigma^p(2),\ldots,\sigma^p(i)\}}, I > = n > 0,$$

for any $0 \leq p \leq n-1$, $1 \leq i \leq n-1$ and since for any $I_{i_1i_2\ldots i_{2k-1}i_{2k}}, I^{'}_{i_1i_2\ldots i_{2k-1}i_{2k}}$ there exist two hyperplanes $H_a, H_{a'}$ with $a, a^{'} \in N$ such that

$$< I_{i_1i_2\ldots i_{2k-1}i_{2k}}, H_a > = < I^{'}_{i_1i_2\ldots i_{2k-1}i_{2k}}, H_{a'} > = 0,$$

then $I \in ri(\mathbf{R}_+C_A)$ *is the only point in relative interior of the cone* $\mathbf{R}_+C_A$. Thus the canonical module is generated by one generator,

$$\omega_{K[\mathcal{A}]} = (x_1x_2\ldots x_n)K[\mathcal{A}].$$

Therefore the $K-$ algebra $K[\mathcal{A}]$ is a Gorenstein ring. $\square$

Conjecture 4.5. *Let $n, m \in \mathbf{N}, m \leq n, A_i \subsetneqq [n], 1 \leq i \leq m$, and $\widetilde{\mathcal{A}} = \{A_1, A_2, \ldots, A_m\}$. We denote*

$$A = \{\log(x_{i_1}x_{i_2}...x_{i_n}) \mid i_1 \in \{1,2\}, i_2 \in \{2,3\}, \ldots, i_{n-1} \in \{n-1,n\}, i_n \in \{1,n\}\},$$

$$N = \{\nu_{\{k+1\}}, \nu_{\{\sigma^k(1),\sigma^k(2),\ldots,\sigma^k(i)\}} \mid 0 \leq k \leq n-1, 2 \leq i \leq n-1\},$$

and

$$\widetilde{A} = \{\log(x_{i_1}x_{i_2}...x_{i_n}) \mid i_1 \in A_1, i_2 \in A_2, \ldots, i_{n-1} \in A_{n-1}, i_n \in A_n\}.$$

Then the base ring associated to transversal polymatroid presented by $\widetilde{\mathcal{A}}$, $K[\widetilde{\mathcal{A}}]$, is a Gorenstein ring if and only if there exists $\widetilde{N} \subset N$ such that

$$\mathbf{R}_+C_{\widetilde{A}} = \bigcap_{a \in \widetilde{N}} H_a^+$$

and H_a^+ with $a \in \widetilde{N}$ are just the facets of the cone $\mathbf{R}_+C_{\widetilde{A}}$.

5 The description of some transversal polymatroids with Gorenstein base ring in dimensions 3 and 4.

Dimension 3.

We consider the collection of sets $\mathcal{A} = \{\{1,2\},\{2,3\},\{3,1\}\}$. The base ring associated to transversal polymatroid presented by $\mathcal{A}$ is

$$R = K[\mathcal{A}] = K[x_1^2x_2, x_2^2x_1, x_2^2x_3, x_3^2x_2, x_1^2x_3, x_3^2x_1, x_1x_2x_3].$$

From [9], R is a normal ring.

We can see $R = K[Q]$, where

$Q = \mathbb{N}\{(2,1,0),(1,2,0),(0,2,1),(0,1,2),(1,0,2),(2,0,1),(1,1,1)\}$.

Our aim is to describe the facets of $C = \mathbb{R}_+Q$.

It is easy to see that C has 6 facets, with the support planes given by the equations:

$$H_{\{1\}} : -x_1 + 2x_2 + 2x_3 = 0,$$

$$H_{\{2\}} : 2x_1 - x_2 + 2x_3 = 0,$$

$$H_{\{3\}} : 2x_1 + 2x_2 - x_3 = 0,$$

$$H_{\{1,2\}} : x_3 = 0,$$

$$H_{\{2,3\}} : x_1 = 0,$$

$$H_{\{3,1\}} : x_2 = 0.$$

In fact, $C = H^+_{\{1\}} \cap H^+_{\{2\}} \cap H^+_{\{3\}} \cap H^+_{\{1,2\}} \cap H^+_{\{2,3\}} \cap H^+_{\{3,1\}}$. Since $(1,1,1)$ is the only point in $ri(\mathbb{R}_+Q)$, then, by Danilov-Stanley theorem, R is a Gorenstein ring and $\omega_R = R(-(1,1,1))$.

In order to obtain all the Gorenstein polymatroids of dimension 3, we remove sequentially some facets of C. For instance, if we remove the facet supported by $H_{\{2\}}$, we obtain a new cone $C' = H^+_{\{1\}} \cap H^+_{\{3\}} \cap H^+_{\{1,2\}} \cap H^+_{\{2,3\}} \cap H^+_{\{3,1\}}$. It is easy to note that $C' = \mathbb{R}_+Q'$, where $Q' = Q + \mathbb{N}\{(0,3,0)\}$. Q' is a saturated semigroup, and moreover, $K[Q'] = K[\mathcal{A}']$, where $\mathcal{A}' = \{\{1,2\},\{1,2,3\},\{2,3\}\}$. The Danilov-Stanley theorem assures us that $R' = K[Q'] = K[\mathcal{A}']$ is still Gorenstein with $\omega_{R'} = R'(-(1,1,1))$. (Remark. If we remove the facet supported by $H_{\{3\}}$ or $H_{\{1\}}$, instead of the facet supported by $H_{\{2\}}$ we obtain a new set $\mathcal{A}'$ which is only a permutation of $1,2,3$.)

Suppose that we remove from C' the facet supported by $H_{\{3\}}$. We obtain a new cone $C'' = H^+_{\{1\}} \cap H^+_{\{1,2\}} \cap H^+_{\{2,3\}} \cap H^+_{\{3,1\}}$. It is easy to see that $C'' = \mathbb{R}_+Q''$, where $Q'' = Q' + \mathbb{N}\{(0,0,3)\}$. Q'' is a saturated semigroup, and moreover, $K[Q''] = K[\mathcal{A}'']$, where $\mathcal{A}'' = \{\{1,2,3\},\{1,2,3\},\{2,3\}\}$. The Danilov-Stanley theorem implies that $R'' = K[Q''] = K[\mathcal{A}'']$ is Gorenstein and $\omega_{R''} = R''(-(1,1,1))$. Finally, we remove from C'' the facet supported by $H_{\{1\}}$. We obtain the cone $C''' = H^+_{\{1,2\}} \cap H^+_{\{2,3\}} \cap H^+_{\{3,1\}}$ which is a cone over $Q''' = Q'' + \mathbb{N}\{(3,0,0)\}$. Q''' is the saturated semigroup associated to the ring $R''' = K[\mathcal{A}''']$, where $\mathcal{A}''' = \{\{1,2,3\},\{1,2,3\},\{1,2,3\}\}$. Also, $\omega_{R'''} = R'''(-(1,1,1))$.

Thus the base ring associated to the transversal polymatroids presented by $\mathcal{A}$, $\mathcal{A}'$, $\mathcal{A}''$, $\mathcal{A}'''$ are Gorenstein rings and for $\mathcal{A}_{\mathbf{1}} = \{\{1,2\},\{2,3\}\}$ the base ring presented by $\mathcal{A}_{\mathbf{1}}$ is the Segre product $k[t_{11}, t_{12}] * k[t_{21}, t_{22}]$, thus is a Gorenstein ring. All of them have dimension 3.

The computations made so far make us believe that all polymatroids with Gorenstein base ring in dimension 3 are the ones classified above.

Dimension 4.

We consider the collection of sets $\mathcal{A} = \{\{1,2\},\{2,3\},\{3,4\},\{4,1\}\}$. The base ring associated to the transversal polymatroid presented by $\mathcal{A}$ is

$$R = K[\mathcal{A}] = K[x_{i_1}x_{i_2}x_{i_3}x_{i_4} \mid i_1 \in \{1,2\}, i_2 \in \{2,3\}, i_3 \in \{3,4\}, i_4 \in \{4,1\}].$$

From [9], R is a normal ring.
We can see $R = K[Q]$, where

$$Q = \mathbb{N}\{log(x_{i_1}x_{i_2}x_{i_3}x_{i_4}) \mid i_1 \in \{1,2\}, i_2 \in \{2,3\}, i_3 \in \{3,4\}, i_4 \in \{4,1\}\}.$$

Our aim is to describe the facets of $C = \mathbb{R}_+Q$. Using *Normaliz* we obtain 12 facets of the cone $C = \mathbb{R}_+Q$:

$$H_{\{1\}} : -x_1 + x_2 + x_3 + x_4 = 0,$$

$$H_{\{2\}} : x_1 - x_2 + x_3 + x_4 = 0,$$

$$H_{\{3\}} : x_1 + x_2 - x_3 + x_4 = 0,$$

$$H_{\{4\}} : x_1 + x_2 + x_3 - x_4 = 0,$$

$$H_{\{1,2\}} : -x_1 - x_2 + 3x_3 + 3x_4 = 0,$$

$$H_{\{2,3\}} : 3x_1 - x_2 - x_3 + 3x_4 = 0,$$

$$H_{\{3,4\}} : 3x_1 + 3x_2 - x_3 - x_4 = 0,$$

$$H_{\{1,4\}} : -x_1 + 3x_2 + 3x_3 - x_4 = 0,$$

$$H_{\{1,2,3\}} : x_4 = 0,$$

$$H_{\{2,3,4\}} : x_1 = 0,$$

$$H_{\{1,3,4\}} : x_2 = 0,$$

$$H_{\{1,2,4\}} : x_3 = 0.$$

It is easy to see that $C = H^+_{\{1\}} \cap H^+_{\{2\}} \cap H^+_{\{3\}} \cap H^+_{\{4\}} \cap H^+_{\{1,2\}} \cap H^+_{\{2,3\}} \cap H^+_{\{3,4\}} \cap H^+_{\{1,4\}} \cap H^+_{\{1,2,3\}} \cap H^+_{\{2,3,4\}} \cap H^+_{\{1,3,4\}} \cap H^+_{\{1,2,4\}}$. Since $(1,1,1,1)$ is the only point in $ri(\mathbb{R}_+Q)$, then, by Danilov-Stanley theorem, R is a Gorenstein ring and $\omega_R = R(-(1,1,1,1))$.

Now we want to proceed as in the case of dimension 3 to get a large class of transversal polymatroids with Gorenstein base ring. Using *Normaliz*, we can give a complete description, modulo a permutation, of the transversal polymatroids with Gorenstein base ring when we start with

$$\mathcal{A} = \{\{1,2\},\{2,3\},\{3,4\},\{4,1\}\}.$$

For $\mathcal{A}_{\mathbf{1}} = \{\{1,2,3\},\{2,3\},\{3,4\},\{4,1\}\}$, the associated cone is:
$C_1 = H^+_{\{1\}} \cap H^+_{\{2\}} \cap H^+_{\{4\}} \cap H^+_{\{1,2\}} \cap H^+_{\{2,3\}} \cap H^+_{\{1,4\}} \cap H^+_{\{1,2,3\}} \cap H^+_{\{2,3,4\}} \cap H^+_{\{1,3,4\}} \cap H^+_{\{1,2,4\}}$.

For $\mathcal{A}_{\mathbf{2}} = \{\{1,2,3,4\},\{2,3\},\{3,4\},\{4,1\}\}$, the associated cone is:
$C_2 = H^+_{\{1\}} \cap H^+_{\{2\}} \cap H^+_{\{1,2\}} \cap H^+_{\{2,3\}} \cap H^+_{\{1,4\}} \cap H^+_{\{1,2,3\}} \cap H^+_{\{2,3,4\}} \cap H^+_{\{1,3,4\}} \cap H^+_{\{1,2,4\}}$.

For $\mathcal{A}_{\mathbf{3}} = \{\{1,2,3,4\},\{2,3,4\},\{3,4\},\{4,1\}\}$, the associated cone is:
$C_3 = H^+_{\{1\}} \cap H^+_{\{2\}} \cap H^+_{\{1,2\}} \cap H^+_{\{2,3\}} \cap H^+_{\{1,2,3\}} \cap H^+_{\{2,3,4\}} \cap H^+_{\{1,3,4\}} \cap H^+_{\{1,2,4\}}$.

For $\mathcal{A}_{\mathbf{4}} = \{\{1,2,3,4\},\{1,2,3,4\},\{3,4\},\{4,1\}\}$, the associated cone is:
$C_4 = H^+_{\{2\}} \cap H^+_{\{1,2\}} \cap H^+_{\{2,3\}} \cap H^+_{\{1,2,3\}} \cap H_{\{2,3,4\}} \cap H_{\{1,3,4\}} \cap H_{\{1,2,4\}}$.

For $\mathcal{A}_{\mathbf{5}} = \{\{1,2,3,4\},\{1,2,3,4\},\{1,3,4\},\{4,1\}\}$, the associated cone is:
$C_5 = H^+_{\{2\}} \cap H^+_{\{2,3\}} \cap H^+_{\{1,2,3\}} \cap H^+_{\{2,3,4\}} \cap H^+_{\{1,3,4\}} \cap H^+_{\{1,2,4\}}$.

For $\mathcal{A}_{\mathbf{6}} = \{\{1,2,3,4\},\{1,2,3,4\},\{1,2,3,4\},\{4,1\}\}$, the associated cone is:
$C_6 = H^+_{\{2,3\}} \cap H^+_{\{1,2,3\}} \cap H^+_{\{2,3,4\}} \cap H^+_{\{1,3,4\}} \cap H^+_{\{1,2,4\}}$.

For $\mathcal{A}_{\mathbf{7}} = \{\{1,2,3,4\},\{1,2,3,4\},\{1,2,3,4\},\{1,2,3,4\}\}$, the associated cone is: $C_7 = H^+_{\{1,2,3\}} \cap H^+_{\{2,3,4\}} \cap H^+_{\{1,3,4\}} \cap H^+_{\{1,2,4\}}$.

For $\mathcal{A}_{\mathbf{8}} = \{\{1,2,3\},\{1,2,3\},\{3,4\},\{4,1\}\}$, the associated cone is: $C_8 = H^+_{\{2\}} \cap H^+_{\{4\}} \cap H^+_{\{1,2\}} \cap H^+_{\{2,3\}} \cap H^+_{\{1,2,3\}} \cap H^+_{\{2,3,4\}} \cap H^+_{\{1,3,4\}} \cap H^+_{\{1,2,4\}}$.

For $\mathcal{A}_{\mathbf{9}} = \{\{1,2,3\},\{1,2,3\},\{1,3,4\},\{4,1\}\}$, the associated cone is: $C_9 = H^+_{\{2\}} \cap H^+_{\{4\}} \cap H^+_{\{2,3\}} \cap H^+_{\{1,2,3\}} \cap H^+_{\{2,3,4\}} \cap H^+_{\{1,3,4\}} \cap H^+_{\{1,2,4\}}$.

For $\mathcal{A}_{\mathbf{10}} = \{\{1,2,3\},\{1,2,3\},\{1,2,3,4\},\{4,1\}\}$, the associated cone is: $C_{10} = H^+_{\{4\}} \cap H^+_{\{2,3\}} \cap H^+_{\{1,2,3\}} \cap H^+_{\{2,3,4\}} \cap H^+_{\{1,3,4\}} \cap H^+_{\{1,2,4\}}$.

For $\mathcal{A}_{\mathbf{11}} = \{\{1,2,3\},\{1,2,3\},\{1,3,4\},\{1,3,4\}\}$, the associated cone is: $C_{11} = H^+_{\{2\}} \cap H^+_{\{4\}} \cap H^+_{\{1,2,3\}} \cap H^+_{\{2,3,4\}} \cap H^+_{\{1,3,4\}} \cap H^+_{\{1,2,4\}}$.

For $\mathcal{A}_{\mathbf{12}} = \{\{1,2,3\},\{1,2,3\},\{1,3,4\},\{1,3,4\}\}$, the associated cone is: $C_{12} = H^+_{\{4\}} \cap H^+_{\{1,2,3\}} \cap H^+_{\{2,3,4\}} \cap H^+_{\{1,3,4\}} \cap H^+_{\{1,2,4\}}$.

The next four examples of transversal polymatroids with Gorenstein base ring are different from those already described.

For $\mathcal{A}_{\mathbf{13}} = \{\{1,2,3\},\{2,3,4\}\}$, the Hilbert series of base ring $K[\mathcal{A}_{\mathbf{13}}]$ is:

$$H_{K[\mathcal{A}_{\mathbf{13}}]}(t) = \frac{1+4t+t^2}{(1-t)^4}.$$

For $\mathcal{A}_{\mathbf{14}} = \{\{1,2,3,4\},\{2,3,4\}\}$, the Hilbert series of base ring $K[\mathcal{A}_{\mathbf{14}}]$ is:

$$H_{K[\mathcal{A}_{\mathbf{14}}]}(t) = \frac{1+5t+t^2}{(1-t)^4}.$$

For $\mathcal{A}_{\mathbf{15}} = \{\{1,2,3,4\},\{1,2,3,4\}\}$, the Hilbert series of base ring $K[\mathcal{A}_{\mathbf{15}}]$ is:

$$H_{K[\mathcal{A}_{\mathbf{15}}]}(t) = \frac{1+6t+t^2}{(1-t)^4}.$$

For $\mathcal{A}_{\mathbf{16}} = \{\{1,2\},\{2,3\},\{3,4\}\}$, the Hilbert series of base ring $K[\mathcal{A}_{\mathbf{16}}]$ is:

$$H_{K[\mathcal{A}_{\mathbf{16}}]}(t) = \frac{1+4t+t^2}{(1-t)^4}.$$

It seems that also, in the dimension 4, our examples cover all transversal polymatroids with Gorenstein base ring.

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